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Research Article

On Local Weak Solutions for Fractional in Time SOBOLEV-Type Inequalities

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We consider two fractional in time nonlinear Sobolev-type inequalities involving potential terms, where the fractional derivatives are defined in the sense of Caputo. For both problems, we study the existence and nonexistence of nontrivial local weak solutions. Namely, we show that there exists a critical exponent according to which we have existence or nonexistence.

1. Introduction and Main Results

We consider the fractional in time Sobolev-type inequalities

$$\begin{cases} \partial_t^\sigma + \Delta_\eta(t,z) + \Delta_\eta(t,z)\left(1+|z|^2\right)^{\rho/2}|\eta(t,z)|^p \le 0, (t,z) \in (0,\infty)\times\mathbb{R}^N, \\ \eta(0,z) = \eta_0(z), z \in \mathbb{R}^N \end{cases} \quad (1)$$

and

$$\begin{cases} \partial_t^{\sigma+1} + \Delta_\eta(t,z) + \Delta_\eta(t,z)\left(1+|z|^2\right)^{\rho/2}|\eta(t,z)|^p \le 0, (t,z) \in (0,\infty)\times\mathbb{R}^N, \\ \eta(0,z) = \eta_0(z), \partial_t\eta(0,z) = \eta_1(z), z \in \mathbb{R}^N, \end{cases} \quad (2)$$

where $N \ge 1$, $p > 1$, and $\rho > -2$. Here, $\sigma \in (0,1)$ and $\partial_t^\lambda, \lambda \in \{\sigma, \sigma+1\}$ is the derivative of fractional order λ in the sense of Caputo. Namely, we are concerned with the existence and nonexistence of nontrivial local weak solutions to problems (1) and (2). We shall establish that there exists a critical exponent $p_c > 1$ that depends on N and ρ such that if $p \in (1, p_c)$, then problems (1) and (2) admit no nontrivial local weak solutions (i.e., we have an instantaneous blow-up), while if $p \in (p_c, \infty)$, then the considered problems admit local solutions for some initial values. In the proofs of our results, we use the test function method with some integral estimates. For more details about the test function method and its applications to partial differential equations, we refer to [1–3] and the references therein.

The absence of solutions (complete blow-up phenomenon) was observed in [4] for the following elliptic inequality with a singular potential term

$$-\Delta_\eta \ge \frac{|\eta|^2}{|z|^2} in \mathscr{D}'(g\Omega \setminus \{0\}), \eta \ge 0 a.e., \quad (3)$$

where Ω is a smooth bounded domain in R^N containing 0. In the same reference, an instantaneous blow-up result was obtained for the parabolic analogue of (3), namely

$$\partial_t\eta - \Delta_\eta \ge \frac{|\eta|^2}{|z|^2} in \mathscr{D}'((0,T)\times\Omega\{0\}), \eta \ge 0 a.e., \quad (4)$$

Notice that the method in [4] is based on comparison principles. In [5], using the test function method and avoiding the maximum principle, instantaneous blow-up results were obtained for certain classes of elliptic and parabolic inequalities including as special cases (3) and (4). For more results on instantaneous blow-up for nonlinear evolution equations, we refer to [6–9] and the references therein.

The investigation of instantaneous blow-up for linear Sobolev-type equations was first considered in [10]. Namely, the following problem was studied

$$\partial_t(\eta_{zz}+\eta)=\eta_{zz}, \eta(0,z)=\eta_0(z), \eta(t,0)=\eta(t,L), L>0. \quad (5)$$

In the limit case $\sigma=1$ and $\rho=0$, (1) and (2) (with equalities instead of inequalities), reduce, respectively, to

$$\begin{cases} \partial_t+\Delta_\eta(t,z)+\Delta_\eta(t,z)+|\eta(t,z)|^p=0, (t,z)\in(0,\infty)\times\mathbb{R}^N, \\ \eta(0,z)=\eta_0(z), z\in\mathbb{R}^N \end{cases} \quad (6)$$

$$\begin{cases} \partial_{tt}+\Delta_\eta(t,z)+\Delta_\eta(t,z)+|\eta(t,z)|^p=0, (t,z)\in(0,\infty)\times\mathbb{R}^N, \\ \eta(0,z)=\eta_0(z), \partial_t\eta(0,z)=\eta_1(z), z\in\mathbb{R}^N. \end{cases} \quad (7)$$

The nonexistence of local weak solutions to (6) and (7) was considered in [11] by the use of the test function method. When $N\in\{1,2\}$, it was proved that for all $p>1$, (6) and (7) admit no nontrivial local weak solutions. If $N\geq 3$, it was shown that if $1<p\leq N/N-2$, then (6) and (7) admit no nontrivial local weak solutions, while if $P>N/N-2$, then local solutions exist. Our aim in this paper is to study the instantaneous blow-up for the fractional in time versions of (6) and (7) with the potential term $V(z)=(1+|z|^2)^{\rho/2}$.

For existence results for stationary problems involving potential terms, see for example [12] and the references therein.

Notice that the study of fractional in time Sobolev-type equations was first considered in [13], where the nonexistence of global weak solutions was investigated.

Before mentioning our main results, let us define the meaning of solutions to (1) and (2).

Let $T\in(0,\infty)$. Given $\kappa>0$ and $\omega\in C([0,T])$, $T>0$, we define the fractional integral operators

$$(\mathcal{J}_0^\kappa\omega)(t)=\frac{1}{\Gamma(\kappa)}\int_0^t(t-\varsigma)^{\kappa-1}\omega(\varsigma)d\varsigma, 0<t\leq T \quad (8)$$

and

$$(\mathcal{J}_T^\kappa\omega)(t)=\frac{1}{\Gamma(\kappa)}\int_t^T(\varsigma-t)^{\kappa-1}\omega(\varsigma)d\varsigma, 0<t\leq T. \quad (9)$$

Given $\omega_i\in C([0,T])$, $i=1,2$, one has (see e.g., [14, 15])

$$\int_0^T(\mathcal{J}_0^\kappa\omega_1)(t)\omega_2(t)dt=\int_0^T(\mathcal{J}_T^\kappa\omega_2)(t)\omega_1(t)dt. \quad (10)$$

For $\sigma\in(0,1)$, the derivatives of fractional orders σ and $\sigma+1$ in the Caputo sense are defined, respectively, by

$$\begin{aligned} (\partial_t^\sigma\omega)(t)&=\left(\mathcal{J}_0^{1-\sigma}\omega'\right)(t), 0<t\leq T, \omega\in C^1([0,T]) \\ \left(\partial_t^{\sigma+1}\omega\right)(t)&=\left(\mathcal{J}_0^{1-\sigma}\omega''\right)(t), 0<t\leq T, \omega\in C^2([0,T]). \end{aligned} \quad (11)$$

Using the above notions and property (10), we define local weak solutions to problem (1) as follows.

Definition 1. Let $\eta_0\in L^1_{loc}(\mathbb{R}^N)$. We say that η is a local weak solution to (1), if there exists $\eta\in L^p_{loc}(\Lambda_{T,N})$ and satisfies

$$\begin{aligned} \int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt-\int_{\mathscr{R}^N}\eta_0(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz \\ \leq-\int_{\Lambda T,N}\eta\Delta\xi dzdt+\int_{\Lambda T,N}\eta\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]dzdt, \end{aligned} \quad (12)$$

for all $\xi\in C\infty(\Lambda_{T,N})$, $\xi\geq 0$, with $supp_z(\xi)\subset\subset R^N$. Here, $\Lambda_{T,N}$ is the product set $[0,T]\times R^N$. Local weak solutions to problem (2) are defined as follows.

Definition 2. Let $\eta_0,\eta_1\in L^1{}_{loc}(\mathrm{R}^N)$. We say that η is a local weak solution to (2), if there exists $T\in(0,\infty)$ such that $\eta\in L^p_{loc}(\Lambda_{T,N})$ and satisfies

$$\begin{aligned} \int_{\Lambda_{T,N}}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt+\int_{\mathbb{R}^N}\eta 0(z)\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]dz \\ -\int_{\mathbb{R}^N}\eta_1(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz\leq-\int_{\Lambda_{T,N}}\eta\Delta\xi dzdt \\ -\int_{\Lambda^{T,N}}\eta\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]dzdt, \end{aligned} \quad (13)$$

for all $\xi\in C\infty(\Lambda_{T,N})$, $\xi\geq 0$, with $\mathrm{supp}_z(\xi)\subset\subset R^N$ and $\partial t\,(JT1-\sigma\xi)(T,\cdot)\equiv 0$.

Now, we state our results. We first define the (critical) exponent

$$p_c=\begin{cases} \infty & if\ N\in\{1,2\}, \\ \dfrac{N+\rho}{N-2} & if\ N\geq 3. \end{cases} \quad (14)$$

Theorem 3. *Let $N\geq 1$, $\rho>-2$, and $\sigma\in(0,1)$.*

(i) If $\eta_0\in L^1(R^N)$ and $1<p<p_c$, then problem (1) admits no nontrivial local weak solution

(ii) If $N\geq 3$ and $p>p_c$, then problem (1) admits local solutions for some $\eta_0>0$

Theorem 4. *Let $N \geq 1$, $\rho > -2$, and $\sigma \in (0, 1)$.*

(i) *If $\eta_0, \eta_1 \in L^1(\mathbb{R}^N)$ and $1 < p < p_c$, then problem (2) admits no nontrivial local weak solution*

(ii) *If $N \geq 3$ and $p > p_c$, then problem (2) admits local solutions for some $\eta_0 > 0$ and $\eta 1 \equiv 0$*

The next section contains some preliminary estimates that will be useful in the proofs of our results. Section 3 is devoted to the proofs of Theorems 3 and 4.

2. Preliminaries

For $S, T \in (0,\infty)$, let

$$\alpha(t) = \left(1 - T^{-1}t\right)^m, t \in [0, T] \tag{15}$$

and

$$\beta(z) = F\left(\frac{|z|^2}{S^2}\right)^m, z \in \mathbb{R}^N, \tag{16}$$

where $m \gg 1$ (i.e., sufficiently large) is a natural number and $F \in C^\infty(R_+)$ satisfies

$$0 \leq F \leq 1, F_{|[0,1]} \equiv 1, F_{|[2,\infty)} \equiv 0. \tag{17}$$

Let us introduce the function

$$\xi(t, z) = \alpha(t)\beta(z), \quad (t, z) \in \Lambda_{T,N}. \tag{18}$$

Clearly, one has $\xi \in C^\infty(\Lambda_{T,N}), \xi \geq 0$ *and* $\sup p_z(\xi) \subset \subset \mathbb{R}^N$.

Lemma 5. *The function α defined by (15) satisfies the following properties:*

$$\left(\mathcal{J}_T^k\alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m}, t \in [0, T], \tag{19}$$

$$\alpha_t\left(\mathcal{J}_T^k\alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m)} T^{-m}(T - t)^{\kappa+m-1}, t \in [0, T], \tag{20}$$

$$\partial_{tt}\left(\mathcal{J}_T^k\alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m-2}, t \in [0, T], \tag{21}$$

where $\kappa \in (0, 1)$.

Proof. One has

$$\left(\mathcal{J}_T^k\alpha\right)(t) = \frac{1}{\Gamma(\kappa)} T^{-m}\int_t^T (\varsigma - t)^{\kappa-1}(T - \varsigma)^m d\varsigma. \tag{22}$$

Taking $\tau = T - \varsigma/T - t$, the above integral reduces to

$$\begin{aligned}\left(\mathcal{J}_T^k\alpha\right)(t) &= \frac{1}{\Gamma(\kappa)} T^{-m}(T - t)^{\kappa+m}\int_0^1 (1 - \tau)^{\kappa-1}\tau^m d\tau \\ &= \frac{1}{\Gamma(\kappa)} T^{-m}(T - t)^{\kappa+m} \frac{\Gamma(\kappa)\Gamma(m + 1)}{\Gamma(\kappa + m + 1)}, \\ &= \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m},\end{aligned} \tag{23}$$

which proves (19). Next, (20) and (21) follow by differentiating (19).

Lemma 6. *For sufficiently large S, one has*

$$\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz \leq CS^{N-2p+p/p-1}. \tag{24}$$

Here, $C > 0$ is a constant (independent on S).

Proof. Using(16) and (17), one obtains

$$\begin{aligned}&\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz \\ &\quad = \int_{S^l<|z|<\sqrt{2S^l}} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz.\end{aligned} \tag{25}$$

On the other hand, an elementary calculation shows that

$$|\Delta\beta(z)| \leq CS^{-2}F\left(\frac{|z|^2}{S^2}\right)^{m-2}, S < |z| < \sqrt{2S}. \tag{26}$$

Here and below, $C > 0$ is a constant independent on S, whose value may change from line to line. Hence, one deduces that

$$\begin{aligned}&\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)}|\Delta\beta(z)|^{p/p-1} \\ &\quad \leq CS^{-2p/p-1}\int_{S<|z|<\sqrt{2S}} \left(1 + |z|^2\right)^{-\rho/2(p-1)}F\left(\frac{|z|^2}{S^2}\right)^{m-2p/p-1}dz \\ &\quad \leq CS^{-2p/p-1}\int_{S<|z|<\sqrt{2S}} \left(1 + |z|^2\right)^{-\rho/2(p-1)}dz \\ &\quad \leq CS^{-2p/p-1}\int_{r=S}^{\sqrt{2S}} \left(1 + r^2\right)^{-\rho/2(p-1)}r^{N-1}dr \\ &\quad \leq CS^{-2p/p-1}\left(1 + S^2\right)^{-\rho/2(p-1)}S^{N-1}S = CS^{N-2p+\rho/p-1},\end{aligned} \tag{27}$$

which proves the desired result.

The following result is obvious.

Lemma 7. *For $T > 0$, one has*

$$\int_0^T \alpha(t)dt = \frac{T}{m+1}. \tag{28}$$

Using (20), one obtains easily the following result.

Lemma 8. *For $T > 0$, one has*

$$\int_0^T \alpha(t)^{-1/p-1} |\alpha_t(\mathcal{J}_T^\kappa \alpha)(t)| dt = CT^{1+(\kappa-1)p/p-1}, \tag{29}$$

where $\kappa \in (0, 1)$.

Using (21), the following result follows.

Lemma 9. *For $T > 0$, one has*

$$\int_0^T \alpha(t)^{-1/p-1} |\partial_{tt}(\mathcal{J}_T^\kappa \alpha)(t)| dt = CT^{1+(\kappa-2)p/p-1}, \tag{30}$$

where $\kappa \in (0, 1)$.

3. Proofs of the Main Results

Proof of Theorem 10.

(i) Suppose that $\eta \in L^p_{loc}(\Lambda_{T,N})$ is a nontrivial local wessak solution to (1) for some fixed $T \in (0,\infty)$. Then, using (12) with ξ is the function defined by (18), one obtains

$$\begin{aligned}
&\int_{\Lambda T,N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \xi dz dt - \int_{\mathbb{R}^N} \eta_0(z) \Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right) dz \\
&\quad \le \int_{\Lambda T,N} |\eta||\Delta\xi| dz dt + \int_{\Delta T,N} |\eta| \left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right| dz dt.
\end{aligned} \tag{31}$$

Next, using ε-Young inequality with $0 < \varepsilon < 1/2$, one obtains

$$\begin{aligned}
\int_{\Lambda T,N} |\eta||\Delta\xi| dz dt &\le \varepsilon \int_{\Lambda T,N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \xi dz dt + C \\
&\quad + \int_{\Delta T,N} \xi^{-1/P-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\xi|^{p/p-1} dz dt.
\end{aligned} \tag{32}$$

Similarly, one has

$$\begin{aligned}
\int_{\Lambda T,N} |\eta| \Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right] dz dt &\le \varepsilon \int_{\Lambda T,N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \xi dz dt \\
&\quad + C \int_{\Delta T,N} \xi^{-1/P-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta \\
&\quad \cdot \left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]|^{p/p-1} dz dt.
\end{aligned} \tag{33}$$

It follows from (31), (32), and (33) that

$$\begin{aligned}
&(1-2\varepsilon) \int_{\Lambda T,N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \xi dz dt + \int_{\mathbb{R}^N} \eta_0(z) \Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right) dz \\
&\quad \le C\Bigg(\int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\xi|^{p/p-1} dz dt \\
&\qquad + \int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} \left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1} dz dt\Bigg).
\end{aligned} \tag{34}$$

On the other hand, by (18), one has

$$\begin{aligned}
&\int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\xi|^{p/p-1} dz dt = \left(\int_0^T \alpha(t) dt\right) \\
&\quad \cdot \left(\int_{\mathbb{R}^N} \beta(z)^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz\right).
\end{aligned} \tag{35}$$

Hence, using Lemmas 6 and 7, for sufficiently large S, one obtains

$$\int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\xi|^{p/p-1} dz dt \le CTS^{N-2p+\rho/p-1}. \tag{36}$$

Again, by (18), one has

$$\begin{aligned}
&\int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} \left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1} dz dt \\
&\quad = \left(\int_0^T \alpha(t)^{-1/p-1} \left|\partial_t\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(t)\right| dt\right) \\
&\quad \cdot \left(\int_{\mathbb{R}^N} \beta(z)^{-1/p-1} \left(1+|z|^2\right)^{-p/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz\right).
\end{aligned} \tag{37}$$

Therefore, using Lemma 6 and Lemma 8 with $\kappa = 1-\sigma$, for sufficiently large S, one obtains

$$\begin{aligned}
&\int_{\Lambda T,N} \xi^{-1/p-1} \left(1+|z|^2\right)^{-\rho/2(p-1)} \left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1} dz dt \\
&\quad \le CT^{1-\sigma p/p-1} S^{N-2p+\rho/p-1}.
\end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\eta_0(z)| \left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right| dz = \left|\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right| \\
&\quad \cdot \int_{\mathbb{R}^N} |\eta_0(z)||\Delta\beta(z)| dz.
\end{aligned} \tag{39}$$

Hence, using (19) (with $\kappa = 1-\sigma$) and (26), one deduces that

$$\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz \le CT^{1-\sigma}S^{-2}\int_{\mathbb{R}^N}|\eta_0(z)|\left|F\left(\frac{|z|^2}{S^2}\right)\right|^{m-2}dz. \tag{40}$$

Notice that since $\eta_0 \in L^1(\mathbb{R}^N)$, by (17), one deduces from the above estimate that

$$\lim_{S\longrightarrow\infty}\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz = 0. \tag{41}$$

Next, it follows from (34), (36), (38), and (40) that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}&\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)F\left(\frac{|z|^2}{S^2}\right)^m dzdt\\ &\le C\left(TS^{N-2p+\rho/p-1}+T^{1-\sigma p/p-1}S^{N-2-+\rho/p-1}\right.\\ &\quad\left.+\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz\right).\end{aligned} \tag{42}$$

Notice that since $1<p<p_c$, then $N-2p+p/p-1<0$. Hence, passing to the infimum limit as $S\longrightarrow\infty$ in (42), using (17), (41), and Fatou's lemma, one deduces that

$$\int_0^T\int_{\mathbb{R}^N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)dzdt = 0, \tag{43}$$

which contradicts the fact that η is nontrivial.

(ii) Let

$$\frac{\rho+2}{p-1}\le a<N-2 \tag{44}$$

$$\varkappa = [a(N-a-2)]^{1/p-1}. \tag{45}$$

Notice that since $p>p_c$, the set of values of a satisfying (44) is nonempty. Consider the function

$$w(z)=\varkappa\left(1+|z|^2\right)^{-a/2}, z\in\mathbb{R}^N. \tag{46}$$

An elementary calculation shows that

$$\Delta w(z)=\varkappa a\left(1+|z|^2\right)^{-a/2-2}\left[N+(N-a-2)|z|^2\right]. \tag{47}$$

Hence, using (44), (45), and (46), one obtains

$$\begin{aligned}-\Delta w(z) &= \left(1+|z|^2\right)^{\rho/2}|w(z)|^p = \varkappa a\left(1+|z|^2\right)^{-a/2-2}\\ &\quad\cdot\left[N+N-a-2|z|^2\right]-\varkappa^p\left(1+|z|^2\right)^{\rho/2-ap/2}\\ &> \varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-1}-\varkappa^p\left(1+|z|^2\right)^{\rho/2-ap/2}\\ &= \varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-1}\\ &\quad\cdot\left[1-\frac{\varkappa^{p-1}}{a(N-a-2)}\left(1+|z|^2\right)^{\rho/2-ap/2+a/2+1}\right]\\ &= \varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-2}\\ &\quad\cdot\left[1-\left(1+|z|^2\right)^{\rho+2/2-a(p-1)/2}\right]>0.\end{aligned} \tag{48}$$

This shows that

$$\eta(t,z)=w(z), t\ge 0, z\in\mathbb{R}^N, \tag{49}$$

is a global solution (so local solution) to (1) with $\eta_0 = w>0$.

Proof of Theorem 11.

(i) Suppose that $\eta\in L_{loc}(\Lambda_{T,N})$ is a nontrivial local weak solution to (2) for some fixed $T\in(0,\infty)$. Then, using (13) with ξ is the function defined by (18), and one obtains

$$\begin{aligned}\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt &\le \int_{\Lambda T,N}|\eta||\Delta\xi|dzdt\\ &\quad+\int_{\Lambda T,N}|\eta|\left|\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|dzdt\\ &\quad+\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]\right|dz\\ &\quad+\int_{\mathbb{R}^N}|\eta_1(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right|dz.\end{aligned} \tag{50}$$

Notice that by (20) (with $\kappa=1-\sigma$), one has $\partial_t(\mathcal{J}_T^{1-\sigma}\xi)(T,\cdot)\equiv 0$. Hence, by Definition 2, the choice of the test function ξ defined by (18) is allowed. Next, following the same arguments used in the proof of part (i) of Theorem 3, by the use of ε-Young inequality, one obtains

$$\int_{\Delta T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt\le C\sum_{j=1}^5 A_j(S), \tag{51}$$

where

$$\begin{aligned}
A_1(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\xi|^{p/p-1}dzdt,\\
A_2(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1}dzdt,\\
A_3(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}\left|\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1}dzdt,\\
A_4(S) &= \int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]\right|dz,\\
A_5(S) &= \int_{\mathbb{R}^N}|\eta_1(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz.
\end{aligned} \tag{52}$$

Notice that by (18), one has

$$\begin{aligned}
A_3(S) = {} & \left(\int_0^T \alpha(t)^{-1/p-1}\left|\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(t)\right|dt\right)\\
& \cdot\left(\int_{\mathbb{R}^N}\beta(z)^{-1/p-1}\left(1+|z|^2\right)^{-p/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz\right).
\end{aligned} \tag{53}$$

Hence, using Lemma 6 and Lemma 9 with $\kappa = 1-\sigma$, for sufficiently large S, one deduces that

$$A_3(S) \le CT^{-(\sigma p+1)/p-1}S^{N-2p+\rho/p-1}. \tag{54}$$

Furthermore, by (19) and (20) (with $\kappa = 1-\sigma$), one has

$$\begin{aligned}
A_4(S) &= \left|\partial_t\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right|\int_{\mathbb{R}^N}|\eta_0(z)||\Delta\beta(z)|dz\\
&= CT^{-\sigma}\int_{\mathbb{R}^N}|\eta_0(z)||\Delta\beta(z)|dz
\end{aligned} \tag{55}$$

$$\begin{aligned}
A_5(S) &= \left|\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right|\int_{\mathbb{R}^N}|\eta_1(z)||\Delta\beta(z)|dz\\
&= CT^{1-\sigma}\int_{\mathbb{R}^N}|\eta_1(z)||\Delta\beta(z)|dz.
\end{aligned} \tag{56}$$

Next, using (36), (38), (51), (54), (55), and (56), for sufficiently large S, one obtains

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)F\left(\frac{|z|^2}{S^2}\right)^m dzdt\\
&\quad\le C\left[\left(T+T^{1-\sigma p/p-1}+T^{-(\sigma p+1)/p-1}\right)S^{N-2p+\rho/p-1}\right.\\
&\quad\quad\left.+\int_{\mathbb{R}^N}\left(T^{-\sigma}|\eta_0(z)|+T^{1-\sigma}|\eta_1(z)|\right)|\Delta\beta(z)dz|\right]
\end{aligned} \tag{57}$$

Notice that by (26), since $\eta_0,\eta_1 \in L^1(\mathbb{R}^N)$, one has

$$\lim_{S\to\infty}\int_{\mathbb{R}^N}\left(T^{-\sigma}|\eta_0(z)|+T^{1-\sigma}|\eta_1(z)|\right)|\Delta\beta(z)|dz = 0. \tag{58}$$

Therefore, passing to the infimum limit as $S \longrightarrow \infty$ in (57) and using Fatou's lemma, since $1 < p < p_c$, it holds that

$$\int_0^T\int_{\mathbb{R}^N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)dzdt = 0, \tag{59}$$

which contradicts the fact that η is nontrivial.

(ii) From the proof of part (ii) of Theorem 3, one deduces that the function $\eta(t,z) = w(z)$, where w is defined by (46), is a global solution (so local solution) to (2) with $\eta_0 = w > 0$ and $\eta_1 \equiv 0$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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