Research Article

Commutators of Singular Integral Operators and Lipschitz Functions on Amalgam Spaces

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In this paper, we obtain the boundedness of commutators generated by singular integral operators and Lipschitz functions on amalgam spaces and weighted amalgam spaces.

1. Introduction

Let b be a locally integrable function on $\mathbb{R}^n$ and T_Ω be a Calderón-Zygmund singular integral operator, then the commutator $[b, T_\Omega]$ generated by b and T_Ω is defined as

$$[b, T_\Omega]f(x) = b(x)T_\Omega(f)(x) - T_\Omega(bf)(x). \quad (1)$$

Coifman et al. [1] proved that $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \text{BMO}$. Then, Janson [2] obtained that $[b, T_\Omega]$ is bounded from L^p to L^q if and only if $b \in \text{Lip}_\beta$, where $1 < p < q < \infty$, $0 < \beta < 1$, and $(1/q) = (1/p) - (\beta/n)$. Recently, Wang [3] showed that $[b, T_\Omega]$ is bounded on weighted amalgam spaces when $b \in \text{BMO}$. On the contrary, in [4], the boundedness for the commutator generated by the singular integral operator and the weighted Lipschitz function on Lebesgue spaces is obtained. The aim of this paper is to establish the boundedness of the commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator and the weighted Lipschitz function on amalgam spaces.

Amalgam spaces were first introduced by Wiener [5, 6]. He considered the special cases (L^1, l^2) , (L^2, l^∞) and (L^∞, l^1) , (L^1, l^∞) . A systematic study on general amalgam spaces (L^p, l^q) goes back to the work of Holland [7]. He defined the spaces (L^p, l^q) ($1 \leq p, q \leq \infty$) to be the sets of all locally p -th power summable functions f on $\mathbb{R}$ such that

$$\|f\|_{(L^p, l^q)} := \left(\sum_{n=-\infty}^{\infty} \left(\int_n^{n+1} |f(x)|^p dx \right)^{q/p} \right)^{1/q} < \infty, \quad (2)$$

with the usual convention applied when p or q is infinity.

Stewart [8] gave the definition of spaces (L^p, l^q) over a locally compact Abelian group G . By the structure theorem, we may write $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a group which contains a compact open subgroup H (if G_1 is compact, we take $H = G_1$; if G_1 is discrete and infinite, we take $H = \{0\}$; otherwise, H is arbitrary but fixed). Let $I = [0, 1)$ and $I_\alpha = \alpha + I$, where $\alpha = (n_1, \dots, n_a, t)$, with $n_l \in \mathbb{Z}$ ($l = 1, \dots, a$) and $t \in G_1$ (the collection of t 's being a transversal of G_1 in H , i.e., $G_1 = \cup_t (t + H)$ is a coset decomposition of G_1). Then, in terms of the disjoint union $G = \cup_{\alpha \in J} I_\alpha$, where $J = \{\alpha: \alpha = (n_1, \dots, n_a, t), n_l \in \mathbb{Z} \text{ for } l = 1, \dots, a \text{ and } t \in G_1\}$, the amalgam space of L^p and l^q on G was defined by

$$(L^p, l^q)(G) = \{f: \|f\|_{p,q} < \infty\}, \quad (3)$$

where

$$\begin{aligned} \|f\|_{p,q} &= \sum_{\alpha \in J} \left(\|f\|_{L^p(I_\alpha)}^q \right)^{1/q}, \quad 0 < p \leq \infty, 0 < q < \infty, \\ \|f\|_{p,\infty} &= \sup_{\alpha \in J} \|f\|_{L^p(I_\alpha)}, \quad 0 < p \leq \infty. \end{aligned} \quad (4)$$

We refer the reader to the work of Fournier and Steward [9] for more information about these spaces.

Fofana introduced in [10] the amalgam spaces $(L^p, L^q)^\alpha(\mathbb{R}^n)$ as follows:

$$\|f\|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} = \sup_{r>0} \| |B(y, r)|^{(1/\alpha)-(1/p)-(1/q)} f \chi_{B(y, r)} \|_{L^q(\mathbb{R}^n)} \|_{L^p(\mathbb{R}^n)}, \quad (6)$$

here $B(y, r) = \{x \in \mathbb{R}^n : |x - y| < r\}$ and $|B(y, r)|$ is its Lebesgue measure. In [10], the author showed that the space $(L^p, L^q)^\alpha(\mathbb{R}^n)$ is nontrivial if and only if $p \leq \alpha \leq q$; thus, we will always assume $p \leq \alpha \leq q$ throughout the present article.

We recall the following interesting facts about the amalgam spaces $(L^p, L^q)^\alpha(\mathbb{R}^n)$:

- (1) For $1 \leq p \leq \alpha \leq q \leq \infty$, we can easily see that $(L^p, L^q)^\alpha(\mathbb{R}^n) \subseteq (L^p, L^q)(\mathbb{R}^n)$, where $(L^p, L^q)(\mathbb{R}^n)$ is Wiener's amalgam space (see [7])
- (2) If $1 \leq p < \alpha$ and $q = \infty$, then $(L^p, L^q)^\alpha(\mathbb{R}^n)$ is just the classical Morrey space $L^{p, 1-(p/\alpha)}(\mathbb{R}^n)$
- (3) If $p = \alpha$ and $q = \infty$, then $(L^p, L^q)^\alpha(\mathbb{R}^n)$ is just the usual Lebesgue space $L^p(\mathbb{R}^n)$

The first main result of this paper is as follows.

Theorem 1. Let $1 < p < q < \infty$, $0 < \gamma < 1$, $(1/q) = (1/p) - (\gamma/n)$, and $1/\beta = (1/\alpha) - (\gamma/n)$. Assume that $p < \alpha < s < \infty$ and $q < \beta < s < \infty$. If $b \in \text{Lip}_\gamma$, then $[b, T_\Omega]$ is bounded from $(L^p, L^s)^\alpha(\mathbb{R}^n)$ to $(L^q, L^s)^\beta(\mathbb{R}^n)$.

We also need the definition of the weighted amalgam spaces.

Definition 1 (see [3]). Let $1 \leq p \leq \alpha \leq q \leq \infty$ and w and μ be two weights on $\mathbb{R}^n$. Then, the weighted amalgam spaces $(L^p, L^q)^\alpha(w; \mu)$ are defined as

$$(L^p, L^q)^\alpha(w; \mu) = \{f \in L^p_{\text{loc}}(w) : \|f\|_{(L^p, L^q)^\alpha(w; \mu)} < \infty\}, \quad (7)$$

where

$$\|f\|_{(L^p, L^q)^\alpha(w; \mu)} = \sup_{r>0} \|w(B(y, r))^{(1/\alpha)-(1/p)-(1/q)} f \chi_{B(y, r)}\|_{L^q_w} \|_{L^p_\mu}, \quad (8)$$

with the usual modification when $q = \infty$. Moreover, the spaces defined above are Banach spaces.

Definition 2 (see [3]). Let $1 \leq p \leq \alpha \leq q \leq \infty$ and u , v , and μ be three weights on $\mathbb{R}^n$. Then, the weighted amalgam spaces $(L^p, L^q)^\alpha(v, u; \mu)$ are defined as

$$(L^p, L^q)^\alpha(v, u; \mu) = \{f \in L^p_{\text{loc}}(v) : \|f\|_{(L^p, L^q)^\alpha(v, u; \mu)} < \infty\}, \quad (9)$$

where

$$(L^p, L^q)^\alpha(\mathbb{R}^n) = \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} < \infty\}, \quad 1 \leq p, q, \alpha \leq \infty, \quad (5)$$

where

$$\|f\|_{(L^p, L^q)^\alpha(v, u; \mu)} = \sup_{r>0} \|u(B(y, r))^{(1/\alpha)-(1/p)-(1/q)} f \chi_{B(y, r)}\|_{L^q_v} \|_{L^p_\mu}, \quad (10)$$

with the usual modification when $q = \infty$. Moreover, the spaces defined above are Banach spaces.

The second main result is the following:

Theorem 2. Let $1 < p < q < \infty$, $0 < \beta < 1$, $p < \alpha_1 < s < \infty$, $q < \alpha_2 < s < \infty$, $(1/q) = (1/p) - (\beta/n)$, and $(1/\alpha_2) = (1/\alpha_1) - (\beta/n)$. Suppose that $u \in A_1$, v is a weight, and $b \in \text{Lip}_{\beta, u}$, then $[b, T_\Omega]$ is bounded from $(L^p, L^s)^{\alpha_1}(u, u^{1-q}; v)$ to $(L^q, L^s)^{\alpha_2}(u^{1-q}; v)$.

Throughout the paper, we will assume that the Calderón-Zygmund singular integral operator T_Ω is defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy, \quad (11)$$

where Ω is a homogeneous function of degree zero, $\int_{\mathcal{S}^{n-1}} \Omega(x') d\sigma(x') = 0$ and $\Omega \in C^\infty(\mathcal{S}^{n-1})$.

1.1. Some Preliminaries and Notations. We denote by Δ_h^k the k^{th} difference operator. That is,

$$\begin{aligned} \Delta_h^1 f(x) &= \Delta_h f(x) = f(x+h) - f(x), \\ \Delta_h^{k+1} f(x) &= \Delta_h^k f(x+h) - \Delta_h^k f(x), \quad k \geq 1. \end{aligned} \quad (12)$$

For a cube Q and a locally integrable function f , we write

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx. \quad (13)$$

For $\beta > 0$, the Lipschitz space Λ_β consists of functions f such that

$$\|f\|_{\Lambda_\beta} = \sup_{x, h \in \mathbb{R}^n, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty. \quad (14)$$

For $\beta > 0$ and $f \in L^1_{\text{loc}}$, we say $f \in \text{Lip}_\beta$ if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\beta, \quad (15)$$

for any $x, y \in \mathbb{R}^n$. The minimal constant C mentioned above is called the Lip_β norm of f , denoted by $\|f\|_{\text{Lip}_\beta}$.

Remark 1. When $0 < \beta < 1$, $\Lambda_\beta = \text{Lip}_\beta$; when $\beta \geq 1$, $\text{Lip}_\beta \subsetneq \Lambda_\beta$.

Lemma 1 (see [11]). If $0 < \beta < 1$ and $1 \leq p < \infty$, then

$$\begin{aligned} \|f\|_{\text{Lip}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+(\beta/n)}} \int_Q |f(x) - f_Q| dx \\ &\approx \sup_Q \frac{1}{|Q|^{(\beta/n)}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}. \end{aligned} \quad (16)$$

Lemma 2 (see [11]). If $Q^* \subseteq Q$, then $|f_{Q^*} - f_Q| \leq C \|f\|_{\text{Lip}_\beta} |Q|^{\beta/n}$.

From Lemma 1, we see that

$$|f(y) - f_Q| \leq C \|f\|_{\text{Lip}_\beta} |Q|^{\beta/n}, \quad \text{a.e. } y \in Q. \quad (17)$$

The following result was established in [2].

Theorem 3 (see [2]). Let $0 < \beta < 1$ and $1 \leq p < \infty$. Then, the following statements are equivalent:

- (a) $b \in \text{Lip}_\beta$
- (b) $[b, T_\Omega]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $(1/q) = (1/p) - (\beta/n)$

A weight w is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$, $(1/p) + (1/p') = 1$, if there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (18)$$

for any ball $B \subset \mathbb{R}^n$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B w(x) dx \leq C w(x), \quad \text{a.e. } x \in B, \quad (19)$$

for any ball $B \subset \mathbb{R}^n$ and $A_\infty = \cup_{p \geq 1} A_p$ (see [12]).

Let $0 < \beta < 1$ and w be a weight function, then the weighted Lipschitz space $\text{Lip}_{\beta,w}$ is defined by

$$\text{Lip}_{\beta,w} = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\text{Lip}_{\beta,w}} < \infty \right\}, \quad (20)$$

where

$$\|f\|_{\text{Lip}_{\beta,w}} = \sup_B \frac{1}{w(B)^{1+(\beta/n)}} \int_B |f(y) - f_B| dy. \quad (21)$$

Lemma 3 (see [4, 12]). If $0 < \beta < 1$, $u \in A_1$, $b \in \text{Lip}_{\beta,w}$ and $1 \leq p \leq \infty$, then we have

$$\|b\|_{\text{Lip}_{\beta,w}} \approx \sup_Q u(Q)^{-(\beta/n)} \left(u(Q)^{-1} \int_Q |b(x) - b_B|^p u(x)^{1-p} dx \right)^{1/p}. \quad (22)$$

Lemma 4 (see [4, 12]). If $0 < \beta < 1$, $u \in A_1$, $b \in \text{Lip}_{\beta,w}$ and Q is a cube, then there exists a constant $C > 0$, independent of b and Q , such that

$$\sup_{x \in Q} |b(x) - b_Q| \leq C \|b\|_{\text{Lip}_{\beta,w}} u(Q)^{1+(\beta/n)} |Q|^{-1}. \quad (23)$$

Lemma 5 (see [4, 12]). If $0 < \beta < 1$, $u \in A_1$, $b \in \text{Lip}_{\beta,w}$ and Q is a cube, then there exists a constant $C > 0$, independent of b and Q , such that

$$|b_{2^k Q} - b_Q| \leq C k u(x) u(2^k Q)^{\beta/n} \|b\|_{\text{Lip}_{\beta,w}}, \quad \text{for all } x \in Q. \quad (24)$$

Theorem 4 (see [4]). Let $u \in A_1$, $(1/q) = (1/p) - (\beta/n)$ for $0 < \beta < 1$ and $1 < p < q < \infty$. Then, $b \in \text{Lip}_{\beta,w}$ if and only if $[b, T_\Omega]$ is bounded from $L^p(u)$ to $L^q(u^{1-q})$.

1.2. Proofs of the Main Results

Proof of Theorem 1. Let $1 < p < q < \infty$, $0 < \gamma < 1$, $p < \alpha < s$, $q < \beta < s$, $(1/q) = (1/p) - (\gamma/n)$, $(1/\beta) = (1/\alpha) - (\gamma/n)$, and $f \in (L^p, L^s)^\alpha(\mathbb{R}^n)$. Fix $y \in \mathbb{R}^n$ and $r > 0$ and set $B = B(y, r)$ and $2B = B(y, 2r)$. Let $f_1 = f \chi_{2B}$ and $f_2 = f - f_1$. We write

$$\begin{aligned} &|B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \|[b, T_\Omega] f \chi_{B(y, r)}\|_{L^q} \\ &\leq |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \|[b, T_\Omega] f_1 \chi_{B(y, r)}\|_{L^q} \\ &\quad + |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \|[b, T_\Omega] f_2 \chi_{B(y, r)}\|_{L^q} \\ &=: J_1(y, r) + J_2(y, r). \end{aligned} \quad (25)$$

We will estimate $J_1(y, r)$ and $J_2(y, r)$ separately. Using Theorem 3, we have

$$\begin{aligned} J_1(y, r) &\leq |B(y, r)|^{1/\beta - (1/q) - (1/s)} \|[b, T_\Omega] f_1\|_{L^q} \\ &\leq C |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \|f \chi_{2B}\|_{L^p} \\ &= C |B(y, 2r)|^{(1/\alpha) - (1/p) - (1/s)} \|f \chi_{2B}\|_{L^p} \frac{|B(y, r)|^{(1/\beta) - (1/q) - (1/s)}}{|B(y, 2r)|^{(1/\alpha) - (1/p) - (1/s)}}. \end{aligned} \quad (26)$$

Since $(1/q) = (1/p) - (\gamma/n)$ and $(1/\beta) = (1/\alpha) - (\gamma/n)$, we have $(1/\alpha) - (1/p) - (1/s) = (1/\beta) - (1/q) - (1/s) < 0$. Consequently,

$$J_1(y, r) \leq C |B(y, 2r)|^{(1/\alpha) - (1/p) - (1/s)} \|f \chi_{2B}\|_{L^p}. \quad (27)$$

Now we treat $J_2(y, r)$. It is obvious that $|x - z| \approx |y - z|$, when $x \in B(y, r)$ and $z \in B(y, 2r)^c$. We have

$$\begin{aligned} |[b, T_\Omega] f_2(x)| &\leq |(b(x) - b_B) T_\Omega f_2(x)| \\ &\quad + |T_\Omega((b - b_B) f_2)(x)|. \end{aligned} \quad (28)$$

Decomposing $\mathbb{R}^n$ into a geometrically increasing sequence of concentric balls, we have

$$\begin{aligned}
|T_{\Omega} f_2(x)| &\leq C \int_{\mathbb{R}^n} \frac{|f_2(z)|}{|x-z|^n} dz \leq C \int_{B(y, 2r)^c} \frac{|f(z)|}{|y-z|^n} dz \\
&= C \sum_{j=1}^{\infty} \int_{B(y, 2^{j+1}r) \setminus B(y, 2^j r)} \frac{|f(z)|}{|y-z|^n} dz \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz
\end{aligned} \tag{29}$$

$$\begin{aligned}
|T_{\Omega}((b - b_B)f_2)(x)| &= \left| \int_{\mathbb{R}^n} (b(z) - b_B) f_2(z) \frac{\Omega(x-z)}{|x-z|^n} dz \right| \\
&\leq C \int_{(2B)^c} |b(z) - b_B| |f(z)| \frac{1}{|x-z|^n} dz \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |b(z) - b_B| |f(z)| dz.
\end{aligned}$$

From these estimates, it follows that

$$\begin{aligned}
J_2(y, r) &= |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \left(\int_{B(y, r)} |[b, T_{\Omega}] f_2(x)|^q dx \right)^{1/q} \\
&\leq C |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz \times \left(\int_{B(y, r)} |b(x) - b_B|^q dx \right)^{1/q} \\
&\quad + C |B(y, r)|^{(1/\beta) - (1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |b(z) - b_B| |f(z)| dz \\
&=: J_3(y, r) + J_4(y, r).
\end{aligned} \tag{30}$$

Using Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
J_3(y, r) &\leq C \|b\|_{\text{Lip}_\gamma} |B(y, r)|^{(1/\beta) - (1/q) - (1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz |B(y, r)|^{1/p} \\
&\leq C \|b\|_{\text{Lip}_\gamma} \sum_{j=1}^{\infty} |B(y, r)|^{(1/\beta) - (1/q) - (1/s) + (1/p)} \left(\frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)|^p dz \right)^{1/p} \\
&= C \|b\|_{\text{Lip}_\gamma} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha) - (1/p) - (1/s)} \|f \chi_{B(y, 2^{j+1}r)}\|_{L^p} \frac{|B(y, r)|^{(1/\alpha) - (1/s)}}{|B(y, 2^{j+1}r)|^{(1/\alpha) - (1/s)}} \\
&= C \|b\|_{\text{Lip}_\gamma} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha) - (1/p) - (1/s)} \|f \chi_{B(y, 2^{j+1}r)}\|_{L^p} \left(\frac{1}{2^{(j+1)n}} \right)^{(1/\alpha) - (1/s)}.
\end{aligned} \tag{31}$$

As for $J_4(y, r)$, we get

$$\begin{aligned}
 J_4(y, r) &\leq C|B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \times \int_{B(y, 2^{j+1}r)} (|b(z) - b_{2^{j+1}B}| + |b_{2^{j+1}B} - b_B|) |f(z)| dz \\
 &= C|B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |b(z) - b_{2^{j+1}B}| |f(z)| dz \\
 &\quad + C|B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |b_{2^{j+1}B} - b_B| |f(z)| dz \\
 &=: J_5(y, r) + J_6(y, r).
 \end{aligned} \tag{32}$$

It follows from Hölder's inequality and Lemma 1 that

$$\begin{aligned}
 J_5(y, r) &\leq C|B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \left(\int_{B(y, 2^{j+1}r)} |f(z)|^p dz \right)^{1/p} \times \left(\int_{B(y, 2^{j+1}r)} |b(z) - b_{2^{j+1}B}|^{p'} dz \right)^{1/p'} \\
 &\leq C\|b\|_{\text{Lip}_Y} |B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} |B(y, 2^{j+1}r)|^{(\gamma/n)-(1/p)} \\
 &= C\|b\|_{\text{Lip}_Y} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \frac{|B(y, r)|^{(1/\beta)-(1/s)}}{|B(y, 2^{j+1}r)|^{(1/\beta)-(1/s)}} \\
 &= C\|b\|_{\text{Lip}_Y} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \left(\frac{1}{2^{(j+1)n}} \right)^{(1/\beta)-(1/s)}.
 \end{aligned} \tag{33}$$

For the last term $J_6(y, r)$, we use Hölder's inequality and Lemma 2 to get

$$\begin{aligned}
 J_6(y, r) &\leq C|B(y, r)|^{(1/\beta)-(1/s)} \sum_{j=1}^{\infty} \|b\|_{\text{Lip}_Y} |B(y, 2^{j+1}r)|^{\gamma/n} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz \\
 &\leq C\|b\|_{\text{Lip}_Y} \sum_{j=1}^{\infty} |B(y, r)|^{(1/\beta)-(1/s)} \left(\int_{B(y, 2^{j+1}r)} |f(z)|^p dz \right)^{1/p} \cdot |B(y, 2^{j+1}r)|^{(\gamma/n)-(1/p)} \\
 &= C\|b\|_{\text{Lip}_Y} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \frac{|B(y, r)|^{(1/\beta)-(1/s)}}{|B(y, 2^{j+1}r)|^{(1/\beta)-(1/s)}} \\
 &= C\|b\|_{\text{Lip}_Y} \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \left(\frac{1}{2^{(j+1)n}} \right)^{(1/\beta)-(1/s)}.
 \end{aligned} \tag{34}$$

Summarizing the above estimates, we conclude that

$$\begin{aligned} J_2(y, r) &\leq C \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \left(\frac{1}{2^{(j+1)n}}\right)^{(1/\alpha)-(1/s)} \\ &\quad + C \sum_{j=1}^{\infty} |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p} \left(\frac{1}{2^{(j+1)n}}\right)^{(1/\beta)-(1/s)}. \end{aligned} \quad (35)$$

Taking the L^s -norm and using Minkowski's inequality, we get

$$\begin{aligned} &\| |B(y, r)|^{(1/\beta)-(1/q)-(1/s)} [b, T_\Omega] f\chi_{B(y, r)} \|_{L^q} \|_{L^s} \\ &\leq \|J_1\|_{L^s} + \|J_2\|_{L^s} \\ &\leq C \| |B(y, 2r)|^{(1/\alpha)-(1/p)-(1/s)} f\chi_{2B} \|_{L^p} \|_{L^s} \\ &\quad + C \sum_{j=1}^{\infty} \| |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} f\chi_{B(y, 2^{j+1}r)} \|_{L^p} \|_{L^s} \left(\frac{1}{2^{(j+1)n}}\right)^{(1/\alpha)-(1/s)} \\ &\quad + C \sum_{j=1}^{\infty} \| |B(y, 2^{j+1}r)|^{(1/\alpha)-(1/p)-(1/s)} f\chi_{B(y, 2^{j+1}r)} \|_{L^p} \|_{L^s} \left(\frac{1}{2^{(j+1)n}}\right)^{(1/\beta)-(1/s)} \\ &\leq C \|f\|_{(L^p, L^s)^\alpha(\mathbb{R}^n)}. \end{aligned} \quad (36)$$

Thus, by taking the supremum over all $r > 0$, the proof is completed. $\square$

Proof of Theorem 2. Let $1 < p < q < \infty$, $0 < \beta < 1$, $p < \alpha_1 < s$, $q < \alpha_2 < s$, $(1/q) = (1/p) - (\beta/n)$, $(1/\alpha_2) = (1/\alpha_1) - (\beta/n)$,

and $f \in (L^p, L^s)^{\alpha_1}(u, u^{1-q}; \nu)$. Fix $y \in \mathbb{R}^n$ and $r > 0$ and set $B = B(y, r)$ and $2B = B(y, 2r)$. Let $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. We write

$$\begin{aligned} &u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \| [b, T_\Omega] f\chi_{B(y, r)} \|_{L^q(u^{1-q})} \\ &\leq u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \| [b, T_\Omega] f_1\chi_{B(y, r)} \|_{L^q(u^{1-q})} \\ &\quad + u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \| [b, T_\Omega] f_2\chi_{B(y, r)} \|_{L^q(u^{1-q})} \\ &=: K_1(y, r) + K_2(y, r). \end{aligned} \quad (37)$$

Now we will estimate $K_1(y, r)$ and $K_2(y, r)$ separately. Using Theorem 4, we have

$$\begin{aligned} K_1(y, r) &\leq u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \| [b, T_\Omega] f_1 \|_{L^q(u^{1-q})} \\ &\leq C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \| f\chi_{2B} \|_{L^p(u)} \\ &= C u^{1-q}(B(y, 2r))^{(1/\alpha_1)-(1/p)-(1/s)} \| f\chi_{2B} \|_{L^p(u)} \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2r))^{(1/\alpha_1)-(1/p)-(1/s)}}. \end{aligned} \quad (38)$$

Noticing that $u^{1-q} \in A_\infty$ implies that u^{1-q} is a doubling measure and that $(1/q) = (1/p) - (\beta/n)$ and $(1/\alpha_2) = (1/\alpha_1) - (\beta/n)$, and it follows that

$$\frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2r))^{(1/\alpha_1)-(1/q)-(1/s)}} \leq C. \quad (39)$$

So we have

$$\begin{aligned} K_2(y, r) &= u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \left\| [b, T_\Omega] f_2 \chi_{B(y, r)} \right\|_{L^q(u^{1-q})} \\ &\leq C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz \\ &\quad \times \left(\int_{B(y, r)} |b(x) - b_B|^q u^{1-q}(x) dx \right)^{1/q} + C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |b(z) - b_B| |f(z)| dz \\ &=: K_3(y, r) + K_4(y, r). \end{aligned} \quad (41)$$

Notice that $u \in A_1$ implies $u \in A_q$ for all $q > 1$. Using Hölder's inequality and Lemma 4, we have

$$\begin{aligned} K_3(y, r) &\leq C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz \times \|b\|_{\text{Lip}_{\beta, u}} u(B(y, r))^{\beta/n} \frac{u(B(y, r))}{|B(y, r)|} \\ &\leq C \|b\|_{\text{Lip}_{\beta, u}} u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \times \left(\int_{B(y, 2^{j+1}r)} |f(z)|^p u(z) dz \right)^{1/p} \left(\int_{B(y, 2^{j+1}r)} u(z)^{-p'/p} dz \right)^{1/p'} \\ &\quad \times u(B(y, r))^{\beta/n} \frac{u(B(y, r))}{|B(y, r)|} \\ &\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \left\| f \chi_{B(y, 2^{j+1}r)} \right\|_{L^p(u)} \times u(B(y, 2^{j+1}r))^{-(1/p)} u(B(y, r))^{\beta/n} \cdot \text{ess inf}_{x \in B} u(x). \end{aligned} \quad (42)$$

From the fact that

$$u(B(y, r))^{\beta/n} \leq u(B(y, 2^{j+1}r))^{\beta/n}, \quad (43)$$

and $(1/q) = (1/p) - (\beta/n)$, we get

$$\begin{aligned} K_3(y, r) &\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \left\| f \chi_{B(y, 2^{j+1}r)} \right\|_{L^p(u)} \times u(B(y, 2^{j+1}r))^{-(1/q)} \text{ess inf}_{x \in B} u(x) \\ &= C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \left\| f \chi_{B(y, 2^{j+1}r)} \right\|_{L^p(u)} \\ &\quad \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} u(B(y, 2^{j+1}r))^{-(1/q)} \times u^{1-q}(B(y, r))^{1/q} \text{ess inf}_{x \in B} u(x). \end{aligned} \quad (44)$$

Using the estimate

$$\begin{aligned}
 u^{1-q}(B(y, r))^{1/q} \operatorname{ess\,inf}_{x \in B} u(x) &= \left(\int_{B(y, r)} u^{1-q}(x) dx \right)^{1/q} \operatorname{ess\,inf}_{x \in B} u(x) \\
 &\leq \left(\operatorname{ess\,inf}_{x \in B} u^q(x) \int_{B(y, r)} u^{1-q}(x) dx \right)^{1/q} \\
 &\leq \left(\int_{B(y, r)} u(x) dx \right)^{1/q} = u(B(y, r))^{1/q},
 \end{aligned} \tag{45}$$

we conclude that

$$K_3(y, r) \leq C \|b\|_{\operatorname{LiP}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} \left(\frac{u(B(y, r))}{u(B(y, 2^{j+1}r))} \right)^{1/q}. \tag{46}$$

Since $u^{1-q}, u \in A_{\infty}$, there exists a constant $\delta > 0$ such that

$$\begin{aligned}
 \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} &\leq C \frac{|B(y, r)|^{\delta((1/\alpha_2)-(1/q)-(1/s))}}{|B(y, 2^{j+1}r)|^{\delta((1/\alpha_2)-(1/q)-(1/s))}} \\
 &\leq C \left(\frac{1}{2^{(j+1)n}} \right)^{\delta((1/\alpha_2)-(1/q)-(1/s))} \\
 \left(\frac{u(B(y, r))}{u(B(y, 2^{j+1}r))} \right)^{1/q} &\leq C \frac{|B(y, r)|^{\delta/q}}{|B(y, 2^{j+1}r)|^{\delta/q}} \leq C \left(\frac{1}{2^{(j+1)n}} \right)^{\delta/q}.
 \end{aligned} \tag{47}$$

Hence, we have

$$K_3(y, r) \leq C \|b\|_{\operatorname{LiP}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \left(\frac{1}{2^{(j+1)n}} \right)^{\delta((1/\alpha_2)-(1/q)-(1/s))}. \tag{48}$$

As for $K_4(y, r)$, we have

$$\begin{aligned}
 K_4(y, r) &\leq C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \times \int_{B(y, 2^{j+1}r)} |b(z) - b_{2^{j+1}B}| |f(z)| dz \\
 &\quad + C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \times \int_{B(y, 2^{j+1}r)} |f(z)| dz \cdot |b_B - b_{2^{j+1}B}| \\
 &=: K_5(y, r) + K_6(y, r).
 \end{aligned} \tag{49}$$

It follows from Hölder's inequality, the fact that $u \in A_1$, and Lemma 4, then we have

$$\begin{aligned}
K_5(y, r) &\leq C u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} \frac{1}{|B(y, 2^{j+1}r)|} \int_{B(y, 2^{j+1}r)} |f(z)| dz \times \|b\|_{\text{Lip}_{\beta, u}} u(B(y, 2^{j+1}r))^{\beta/n} \frac{u(B(y, 2^{j+1}r))}{|B(y, 2^{j+1}r)|} \\
&\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times u(B(y, 2^{j+1}r))^{-(1/q)} \frac{u(B(y, 2^{j+1}r))}{|B(y, 2^{j+1}r)|} \\
&\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_1)-(1/p)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} \\
&\quad \times u^{1-q}(B(y, r))^{1/q} \cdot \text{ess inf}_{x \in 2^{j+1}B} u(x) u(B(y, 2^{j+1}r))^{-(1/q)} \\
&\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} \left(\frac{u(B(y, r))}{u(B(y, 2^{j+1}r))} \right)^{1/q} \\
&\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \left(\frac{1}{2^{(j+1)n}} \right)^{\delta((1/\alpha_2)-(1/s))}.
\end{aligned} \tag{50}$$

For the last term, applying Lemma 5 and the A_1 property, we have

$$\begin{aligned}
K_6(y, r) &\leq C \|b\|_{\text{Lip}_{\beta, u}} u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/s)} \sum_{j=1}^{\infty} (j+1) u(B(y, 2^{j+1}r))^{\beta/n} \times \text{ess inf}_{x \in B} u(x) \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} u(B(y, 2^{j+1}r))^{-(1/p)} \\
&= C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} (j+1) u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \\
&\quad \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} u^{1-q}(B(y, r))^{1/q} \times \text{ess inf}_{x \in B} u(x) \cdot u(B(y, 2^{j+1}r))^{-(1/q)}.
\end{aligned} \tag{51}$$

By an argument similar to that used in estimating K_5 , we get

$$\begin{aligned}
K_6(y, r) &\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} (j+1) u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \\
&\quad \times \frac{u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}}{u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}} \left(\frac{u(B(y, r))}{u(B(y, 2^{j+1}r))} \right)^{1/q} \\
&\leq C \|b\|_{\text{Lip}_{\beta, u}} \sum_{j=1}^{\infty} (j+1) u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)} \|f\chi_{B(y, 2^{j+1}r)}\|_{L^p(u)} \times \left(\frac{1}{2^{(j+1)n}} \right)^{\delta((1/\alpha_2)-(1/s))}.
\end{aligned} \tag{52}$$

Taking the $L^s(\nu)$ -norm and using Minkowski's inequality, we get

$$\begin{aligned}
 & \|u^{1-q}(B(y, r))^{(1/\alpha_2)-(1/q)-(1/s)}\| [b, T_\Omega] f \chi_{B(y, r)} \|_{L^q(u^{1-q})} \|_{L^s(\nu)} \\
 & \leq \|K_1\|_{L^s(\nu)} + \|K_2\|_{L^s(\nu)} \\
 & \leq C \|u^{1-q}(B(y, 2r))^{(1/\alpha_1)-(1/p)-(1/s)}\| f \chi_{2B} \|_{L^p(u)} \|_{L^s(\nu)} \\
 & \quad + C \sum_{j=1}^{\infty} \|u^{1-q}(B(y, 2^{j+1}r))^{(1/\alpha_1)-(1/p)-(1/s)}\| f \chi_{B(y, 2^{j+1}r)} \|_{L^p(u)} \|_{L^s(\nu)} \\
 & \quad \times (j+1) \left(\frac{1}{2^{(j+1)n}} \right)^{\delta((1/\alpha_2)-(1/s))} \\
 & \leq C \|f\|_{(L^p, L^s)^{\alpha_1}(u, u^{1-q}; \nu)}.
 \end{aligned} \tag{53}$$

Finally, by taking the supremum over all $r > 0$, the proof is completed. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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