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Research Article

Some Dynamic Inequalities of Hilbert's Type

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This paper is concerned with deriving some new dynamic Hilbert-type inequalities on time scales. The basic idea in proving the results is using some algebraic inequalities, Hölder's inequality and Jensen's inequality, on time scales. As a special case of our results, we will obtain some integrals and their corresponding discrete inequalities of Hilbert's type.

1. Introduction

It is evident that the Hilbert-type inequalities outplay a major role in mathematics, for pattern complex analysis, numerical analysis, and qualitative theory of differential equations and their implementations. In recent years, there were a lot of various refinements, generalizations, extensions, and applications of Hilbert's inequality which have seemed in the literature. Hilbert's discrete inequality and its integral formula ([1], Theorem 316) have been generalized in many trends (for example, see [2–6]). Lately, Pachpatte [7] proved new inequalities similar to those of Hilbert's inequality, namely, he proved that if $h, l \geq 1$, $A_m = \sum_{s=1}^{m} a_s \geq 0$, and $B_n = \sum_{t=1}^{n} b_t \geq 0$, then

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{A_i^h B_j^l}{i+j} \leq C^*(h,l,k,r)\left(\sum_{i=1}^{k}(k+1-i)\left(A_i^{h-1}a_i\right)^2\right)^{1/2} \times\left(\sum_{j=1}^{r}(r+1-j)\left(B_j^{l-1}b_j\right)^2\right)^{1/2}, \tag{1}$$

where

$$C^*(h,l,k,r) = \frac{1}{2}hl\sqrt{kr}. \tag{2}$$

An integral analogue of (1) is given in the following result. Let h, $l \geq 1$, $F(x) = \int_0^x f(\tau)\mathrm{d}\tau \geq 0$, and $G(y) = \int_0^y g(\nu)\mathrm{d}\nu \geq 0$, for $x, \tau \in (0,a)$ and $y, \nu \in (0,b)$. Then,

$$\int_0^a\int_0^b \frac{F^h(x)G^l(y)}{x+y}\mathrm{d}x\,\mathrm{d}y \leq D(h,l,a,b) \cdot\left(\int_0^a (a-x)\left[F^{h-1}(x)f(x)\right]^2\mathrm{d}x\right)^{1/2} \times\left(\int_0^b (b-y)\left[G^{l-1}(y)g(y)\right]^2\mathrm{d}y\right)^{1/2}, \tag{3}$$

where

$$D(h,l,a,b) = \frac{1}{2}hl\sqrt{ab}. \tag{4}$$

In 2001, Kim [8] gave some generalizations of (1) and (3) by introducing a parameter $\alpha > 0$ as

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{A_i^h B_j^l}{\left(i^\alpha+j^\alpha\right)^{1/\alpha}}\le C(h,l,k,r,\alpha)\left(\sum_{i=1}^{k}(k-i+1)\left(A_i^{h-1}a_i\right)^2\right)^{1/2}\times\left(\sum_{j=1}^{r}(r-j+1)\left(B_j^{l-1}b_j\right)^2\right)^{1/2}, \tag{5}$$

where $h,\ l\ge 1$, $A_m=\sum_{p=1}^{m}a_p\ge 0$, $B_n=\sum_{q=1}^{n}b_q\ge 0$, and

$$C(h,l,k,r,\alpha)=\left(\frac{1}{2}\right)^{1/\alpha}hl\sqrt{kr}. \tag{6}$$

An integral analogue of (5) is given in the following result. Let $p,\ q\ge 1$, $\alpha>0$, $F(t)=\int_0^t f(\tau)\mathrm{d}\tau\ge 0$, and $G(s)=\int_0^s g(\nu)\mathrm{d}\nu\ge 0$, for $t,\tau\in(0,a)$, $s,\nu\in(0,b)$. Then,

$$\int_0^a\int_0^b\frac{F^h(x)G^l(y)}{\left(x^\alpha+y^\alpha\right)^{1/\alpha}}\mathrm{d}x\,\mathrm{d}y\le D(h,l,a,b)\cdot\left(\int_0^a(a-x)\left[F^{h-1}(x)f(x)\right]^2\mathrm{d}x\right)^{1/2}\times\left(\int_0^b(b-y)\left[G^{l-1}(y)g(y)\right]^2\mathrm{d}y\right)^{1/2}, \tag{7}$$

where

$$D(h,l,a,b)=\left(\frac{1}{2}\right)^{1/\alpha}hl\sqrt{ab}. \tag{8}$$

In 2009, Yang [9] gave another generalization of (1) and (3) by introducing parameter $\alpha>1$ and $\gamma>1$ as follows. Let $h,\ l\ge 1$, $A_m=\sum_{p=1}^{m}a_p\ge 0$, and $B_n=\sum_{q=1}^{n}b_q\ge 0$. Then,

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{A_i^h B_j^l}{\gamma i^{((\alpha-1)(\alpha+\gamma))/\alpha\gamma}+\alpha j^{((\alpha-1)(\alpha+\gamma))/\alpha\gamma}}\le C(h,l,k,r,\alpha,\gamma)\left(\sum_{i=1}^{k}(k-i+1)\left(A_i^{h-1}a_i\right)^\alpha\right)^{1/\alpha}\times\left(\sum_{j=1}^{r}(r-j+1)\left(B_j^{l-1}b_j\right)^\gamma\right)^{1/\gamma}, \tag{9}$$

where

$$C(h,l,k,r,\alpha,\gamma)=\frac{hl}{\alpha+\gamma}k^{(\alpha-1)/\alpha}r^{(\gamma-1)/\gamma}. \tag{10}$$

An integral analogue of (9) is given as follows. If $h,\ l\ge 1$, $\alpha>1$, $\gamma>1$, $F(x)=\int_0^x f(\tau)\mathrm{d}\tau\ge 0$, and $G(y)=\int_0^y g(\nu)\mathrm{d}\nu\ge 0$, for $x,\tau\in(0,a)$ and $y,\nu\in(0,b)$, then

$$\int_a^0\int_b^0\frac{F^h(x)G^l(y)}{\gamma x^{((\alpha-1)(\alpha+\gamma))/\alpha\gamma}+\alpha y^{((\alpha-1)(\alpha+\gamma))/\alpha\gamma}}\mathrm{d}x\,\mathrm{d}y\le D(h,l,a,b,\alpha,\gamma)\left(\int_0^a(a-x)\left[F^{h-1}(x)f(x)\right]^\alpha\mathrm{d}x\right)^{1/\alpha}\times\left(\int_0^b(b-y)\left[G^{l-1}(y)g(y)\right]^\gamma\mathrm{d}y\right)^{1/\gamma}, \tag{11}$$

where

$$D(h,l,a,b,\alpha,\gamma)=\frac{pq}{\alpha+\gamma}a^{(\alpha-1)/\alpha}b^{(\gamma-1)/\gamma}. \tag{12}$$

In [10], the authors deduced several generalizations of inequalities (1) and (3) on time scales, namely, they proved that if h and $l\ge 1$ are real numbers, $A(x_1)=\int_w^{x_1}a(\tau_1)\Delta\tau_1$, $B(y_1)=\int_w^{y_1}b(\tau_1)\Delta\tau_1$, and $\eta>1$, $\eta_*>1$ with $\eta^{-1}+\eta_*^{-1}=1$, then

$$\int_w^{x_2}\int_w^{y_2}\frac{A^h(x_1)B^l(y_1)}{\eta_*(x_1-w)^{\eta-1}+\eta(y_1-w)^{\eta_*-1}}\Delta y_1\Delta x_1\le M(\eta,\eta_*)\left(\int_w^{x_2}(\sigma(x_2)-x_1)\left[a(x_1)A^{h-1}(\sigma(x_1))\right]^\eta\Delta x_1\right)^{\eta^{-1}}\times\left(\int_w^{y_2}(\sigma(y_2)-y_1)\left[b(y_1)B^{l-1}(\sigma(y_1))\right]^{\eta_*}\Delta y_1\right)^{\eta_*^{-1}}, \tag{13}$$

where

$$M(\eta,\eta_*)=\frac{hl}{\eta\eta_*}(x_2-w)^{(\eta-1)/\eta}(y_2-w)^{(\eta_*-1)/\eta_*}. \tag{14}$$

In [11], the authors gave some extensions of inequalities (5) and (7) on time scales. Minutely, they proved that if $\gamma>0$ and h and $l\ge 1$ are real numbers, $A(s)=\int_{t_0}^s a(\tau)\Delta\tau$, $B(t)=\int_{t_0}^t b(\tau)\Delta\tau$, and $\eta>1$, $\eta_*>1$ with $\eta^{-1}+\eta_*^{-1}=1$, then

$$\int_{t_0}^x\int_{t_0}^y\frac{A^h(\sigma(s))B^l(\sigma(t))}{\left((\sigma(s)-t_0)^\gamma+(\sigma(t)-t_0)^\gamma\right)^{2/\eta\gamma}}\Delta s\Delta t\le C(h,l,\eta,\gamma)\left(\int_{t_0}^x(\sigma(x)-\sigma(s))\left[a(s)A^{h-1}(\sigma(s))\right]^{\eta_*}\Delta s\right)^{\eta_*^{-1}}\times\left(\int_{t_0}^y(\sigma(y)-\sigma(t))\left[b(t)B^{l-1}(\sigma(t))\right]^{\eta_*}\Delta t\right)^{\eta_*^{-1}}, \tag{15}$$

where

$$C(h,l,\eta,\gamma)=hl\left(\frac{1}{2}\right)^{2/\eta\gamma}(x-t_0)^{1/\eta}(y-t_0)^{1/\eta}. \tag{16}$$

Following this trend and to develop the study of dynamic inequalities on time scales, we will prove some new inequalities of Hilbert's type on time scales, namely, we prove time scale versions of inequalities (9) and (11) on time scale $\mathbb{T}$. These inequalities can be considered as extensions and generalizations of some Hilbert-type inequalities proved in [10]. We also derive some inequalities on time scale as special cases.

2. Definitions and Basic Results

In this division, we will present some fundamental concepts and effects on time scales which will be beneficial for deducing our main results. The following definitions and theorems are referred from [12, 13].

Time scale $\mathbb{T}$ is defined as a nonempty arbitrary locked subplot of real numbers $\mathbb{R}$. We define the forward jump operator $\sigma\colon \mathbb{T} \longrightarrow \mathbb{T}$ as

$$\sigma(\tau) = \inf\{\theta \in \mathbb{T}\colon \theta > \tau\} \tag{17}$$

and the backward jump operator $\rho\colon \mathbb{T} \longrightarrow \mathbb{T}$ as

$$\rho(\tau) = \sup\{\theta \in \mathbb{T}\colon \theta < \tau\}. \tag{18}$$

From the above two definitions, it can be stated that a point $\tau \in \mathbb{T}$ with $\inf \mathbb{T} < \tau < \sup \mathbb{T}$ is called right-scattered if $\sigma(\tau) > \tau$, right-dense if $\sigma(\tau) = \tau$, left-scattered if $\rho(\tau) < \tau$, and left-dense if $\rho(\tau) = \tau$. If $\mathbb{T}$ has left-scattered maximum s_m, then $\mathbb{T}^k = \mathbb{T} - \{s_m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. Finally, the graininess function $\mu\colon \mathbb{T} \longrightarrow [0, \infty)$ for any $\tau \in \mathbb{T}$ is defined by

$$\mu(\tau) = \sigma(\tau) - \tau. \tag{19}$$

For a function $\chi\colon \mathbb{T} \longrightarrow \mathbb{R}$, the delta derivative of χ at $\tau \in \mathbb{T}^k$ is defined as for each $\varepsilon > 0$, there is a neighborhood U of τ such that

$$\left|[\chi(\sigma(\tau)) - \chi(\theta)] - \chi^{\Delta}(\tau)[\sigma(\tau) - \theta]\right| \le \varepsilon|\sigma(\tau) - \theta|, \quad \text{for every } \theta \in U. \tag{20}$$

Moreover, χ is called delta differentiable on $\mathbb{T}^k$ if it is delta differentiable at every $\tau \in \mathbb{T}^k$.

A function $\chi\colon \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous (rd−continuous) as long as it is continuous at all right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The classes of real rd−continuous functions on an interval I will be denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. For θ, $t \in \mathbb{T}$, the Cauchy integral of χ^{Δ} is defined as

$$\int_{\theta}^{t} \chi^{\Delta}(\tau)\Delta\tau = \chi(t) - \chi(\theta). \tag{21}$$

Note that

(a) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} \sigma(\tau) &= \tau, \\ \mu(\tau) &= 0, \\ \chi^{\Delta}(\tau) &= \chi'(\tau), \\ \int_{\theta}^{t} \chi^{\Delta}(\tau)\Delta &= \int_{\theta}^{t} \chi(\tau)\mathrm{d}\tau. \end{aligned} \tag{22}$$

(b) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} \sigma(\tau) &= \tau + 1, \\ \mu(\tau) &= 1, \\ \chi^{\Delta}(\tau) &= \Delta\chi(\tau), \\ \int_{\theta}^{t} \chi^{\Delta}(\tau)\Delta\tau &= \sum_{\tau=s}^{t-1} \chi(\tau). \end{aligned} \tag{23}$$

In what follows, we will present Hölder's inequality, Jensen's inequality, and the power rules for integration on time scales.

Theorem 1 (Hölder's inequality (see [14, 15])). *Let $u, v \in \mathbb{T}$. For $\zeta, \chi \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, we have*

$$\int_{u}^{v} |\zeta(\theta)\chi(\theta)|\Delta\theta \le \left[\int_{u}^{v} |\zeta(\theta)|^{\eta}\Delta\theta\right]^{\eta^{-1}} \left[\int_{u}^{v} |\chi(\theta)|^{\eta_*}\Delta\theta\right]^{\eta_*^{-1}}, \tag{24}$$

where $\eta > 1$ and $\eta_* > 1$ with $\eta^{-1} + \eta_*^{-1} = 1$.

Theorem 2 (Jensen's inequality (see [14, 16])). *Suppose that $\zeta \in C_{\mathrm{rd}}([u, v]_{\mathbb{T}}, (w, z))$ and $\eta \in C_{\mathrm{rd}}([u, v]_{\mathbb{T}}, \mathbb{R})$ are nonnegative with*

$$\int_{u}^{v} \eta(s)\Delta s > 0. \tag{25}$$

If $\Phi \in C_{rd}((w, z), \mathbb{R})$ is convex, then

$$\Phi\left(\frac{\int_{u}^{v} \eta(s)\zeta(s)\Delta s}{\int_{u}^{v} \zeta(s)\Delta s}\right) \le \frac{\int_{u}^{v} \zeta(s)\Phi(\eta(s))\Delta s}{\int_{u}^{v} \zeta(s)\Delta s}. \tag{26}$$

Lemma 3 (see [17]). *Let $u, s \in \mathbb{T}$, and $\zeta \in C_{rd}([u, s]_{\mathbb{T}}, \mathbb{R})$ be nonnegative. If $\alpha \ge 1$, then*

$$\left(\int_{u}^{s} \zeta(\tau)\Delta\tau\right)^{\alpha} \le \alpha \int_{u}^{s} \zeta(\eta)\left(\int_{u}^{v} \zeta(\tau)\Delta\tau\right)^{\alpha-1} \Delta v. \tag{27}$$

Now, we will present the formula that will reduce double integrals to single integrals which is the desired in [18].

Lemma 4. *Let $\chi\colon \mathbb{T} \longrightarrow \mathbb{R}$ and $u, s, \theta \in \mathbb{T}$. Then,*

$$\int_{u}^{s} \int_{u}^{\theta} \chi(\tau)\Delta\tau\,\Delta\theta = \int_{u}^{s} [s - \sigma(\theta)]\chi(\tau)\Delta t, \quad \text{for } s \in \mathbb{T}, \tag{28}$$

assuming the integrals considered exist.

Lemma 5 (see [19]). *Let $r > 0$, $\mu_q > 0$, and $\sum_{q=1}^{m} \mu_q = \Omega_m$. Then,*

$$\left(\prod_{q=1}^{m} s_q^{\mu_q}\right)^{1/\Omega_m} \le \left(\frac{1}{\Omega_m} \sum_{q=1}^{m} \mu_q s_q^{r}\right)^{1/r}. \tag{29}$$

3. Main Results

In this division, we will prove our main results. Throughout this section, we will assume that all functions are nonnegative and the integrals considered are assumed to exist. Also, we will assume that h and $l \geq 1$ be real numbers and $\eta > 1$ and $\eta_* > 1$ with $\eta^{-1} + \eta_*^{-1} = 1$.

Theorem 6. *Let s, θ, and $t_0 \in \mathbb{T}$ and $f \in C_{rd}([t_0, x]_{\mathbb{T}}, \mathbb{R}^+)$ and $g \in C_{\mathrm{rd}}([t_0, y]_{\mathbb{T}}, \mathbb{R}^+)$. Suppose that $F(s)$ and $G(\theta)$ are defined as*

$$F(s) = \int_{t_0}^{s} f(\xi)\Delta\xi, \qquad G(\theta) = \int_{t_0}^{\theta} g(\xi)\Delta\xi. \tag{30}$$

Then, for $s \in [t_0, x]_{\mathbb{T}}$ and $\theta \in [t_0, y]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_{t_0}^{x}\int_{t_0}^{y} &\frac{F^h(s)G^l(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \\ &\leq C(h,l,\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)\left[F^{h-1}(\sigma(s))f(s)\right]^{\eta}\Delta s\right)^{1/\eta} \\ &\times\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left[G^{l-1}(\sigma(\theta))g(\theta)\right]^{\eta_*}\Delta\theta\right)^{1/\eta_*},\end{aligned} \tag{31}$$

where

$$C(h,l,\eta,\eta_*) = \frac{hl}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}. \tag{32}$$

Proof. By using inequality (27) (see Lemma 3), we see that

$$F^h(s) \leq h\int_{t_0}^{s} F^{h-1}(\xi)f(\xi)\Delta\xi, \tag{33}$$

$$G^l(\theta) \leq l\int_{t_0}^{\theta} G^{l-1}(\xi)g(\xi)\Delta\xi. \tag{34}$$

Then, we have

$$F^h(s)G^l(\theta) \leq hl\left(\int_{t_0}^{s} F^{h-1}(\xi)f(\xi)\Delta\xi\right)\left(\int_{t_0}^{\theta} G(\xi)g(\xi)\Delta\xi^{l-1}\right). \tag{35}$$

Applying Hölder's inequality (1) on $\int_{t_0}^{s} F^{h-1}(\xi)f(\xi)\Delta\xi$ with indices η and $\eta/(\eta-1)$, we find that

$$\int_{t_0}^{s} F^{h-1}(\xi)f(\xi)\Delta\xi \leq (s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s}\left[F^{h-1}(\xi)f(\xi)\right]^{\eta}\Delta\xi\right)^{1/\eta}, \tag{36}$$

and on the integral $\int_{t_0}^{\theta} G^{l-1}(\xi)g(\xi)\Delta\xi$ with indices and $\eta_*/(\eta_*-1)$, we find that

$$\int_{t_0}^{\theta} G^{l-1}(\xi)g(\xi)\Delta\xi \leq (\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{\theta}\left[G^{l-1}(\xi)g(\xi)\right]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{37}$$

From (36) and (37), we get

$$F^h(s)G^l(\theta) \leq hl(s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{s}\left[F^{h-1}(\xi)f(\xi)\right]^{\eta}\Delta\xi\right)^{1/\eta} \times\left(\int_{t_0}^{\theta}\left[G^{l-1}(\xi)g(\xi)\right]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{38}$$

Using inequality (29) of power means, we observe that

$$\left(s_1^{\omega_1}s_2^{\omega_2}\right)^{r/(\omega_1+\omega_2)} \leq \frac{1}{\omega_1+\omega_2}\left(\omega_1 s_1^r + \omega_2 s_2^r\right). \tag{39}$$

Now, by setting $s_1 = (s-t_0)^{\eta-1}$, $s_2 = (\theta-t_0)^{\eta_*-1}$, $\omega_1 = 1/\eta$, $\omega_2 = 1/\eta_*$, and $r = \omega_1 + \omega_2$ in (39), we get

$$\begin{aligned}&(s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*} \\ &\leq \frac{\eta\eta_*}{\eta+\eta_*}\left(\frac{(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}}{\eta} + \frac{(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}{\eta_*}\right).\end{aligned} \tag{40}$$

Substituting (40) into (38) yields

$$\begin{aligned}F^h(s)G^l(t) \leq \frac{hl\eta\eta_*}{\eta+\eta_*}&\left(\frac{(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}}{\eta} + \frac{(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}{\eta_*}\right) \\ &\times\left(\int_{t_0}^{s}\left[F^{h-1}(\xi)f(\xi)\right]^{\eta}\Delta\xi\right)^{1/\eta}\left(\int_{t_0}^{\theta}\left[G^{l-1}(\xi)g(\xi)\right]^{\eta_*}\Delta\xi\right)^{1/\eta_*}.\end{aligned} \tag{41}$$

Dividing both sides of (41) by the last factor $\eta_*(s-t_0)^{[(\eta-1)(\eta+\eta_*)]/\eta\eta_*} + \eta(\theta-t_0)^{[(\eta_*-1)(\eta+\eta_*)]/\eta\eta_*}$, we obtain

$$\frac{F^h(s)G^l(t)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}} \le \frac{hl}{\eta+\eta_*}\left(\int_{t_0}^{s}\left[F^{h-1}(\xi)f(\xi)\right]^{\eta}\Delta\xi\right)^{1/\eta} \times\left(\int_{t_0}^{\theta}\left[G^{l-1}(\xi)g(\xi)\right]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{42}$$

Integrating the above relation and applying Hölder's inequality (1), we have

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{F^h(s)G^l(t)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le \frac{hl}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{x}\left(\int_{t_0}^{s}\left[F^{h-1}(\xi)f(\xi)\right]^{\eta}\Delta\xi\right)\Delta s\right)^{1/\eta} \times\left(\int_{t_0}^{y}\left(\int_{t_0}^{\theta}\left[G^{l-1}(\xi)g(\xi)\right]^{\eta_*}\Delta\xi\right)\Delta\theta\right)^{1/\eta_*}. \tag{43}$$

Applying Lemma 4 on (43) and using the fact that $\sigma(n)\ge n$, we conclude that

$$\begin{aligned}\int_{t_0}^{x}\int_{t_0}^{y}&\frac{F^h(s)G^l(t)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta\\ &\le \frac{hl}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{x}(\sigma(x)-s)\left[F^{h-1}(\sigma(s))f(s)\right]^{\eta}\Delta s\right)^{1/\eta}\\ &\quad\times\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left[G^{l-1}(\sigma(\theta))g(\theta)\right]^{\eta_*}\Delta\theta\right)^{1/\eta_*}\\ &= C(h,l,\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)\left[F^{h-1}(\sigma(s))f(s)\right]^{\eta}\Delta s\right)^{1/\eta}\\ &\quad\times\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left[G^{l-1}(\sigma(\theta))g(\theta)\right]^{\eta_*}\Delta\theta\right)^{1/\eta_*},\end{aligned} \tag{44}$$

which proves (31). This completes the proof.

Remark 1. Letting $1/\eta+1/\eta_*=1$ in (31), we get Theorem 3.1 due to Saker et al. ([11], Theorem 3.1).

By using relations (22) and putting $\mathbb{T}=\mathbb{R}$ and $t_0=0$ in Theorem 6, we get the following conclusion.

Corollary 7. *Assume that $f(\xi)$ and $g(\xi)$ are two nonnegative functions, and define*

$$F(s)=\int_0^s f(\xi)\mathrm{d}\xi,\qquad G(q)=\int_0^{\theta} g(\xi)\mathrm{d}\xi. \tag{45}$$

Then, for $s\in(0,x)$ and $\theta\in(0,y)$, we have

$$\int_0^x\int_0^y\frac{F^h(s)G^l(t)}{\eta_*(s)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\mathrm{d}s\,\mathrm{d}\theta \le C^*(h,l,\eta,\eta_*)\left(\int_0^x(x-s)\left[F^{h-1}(s)f(s)\right]^{\eta}\mathrm{d}s\right)^{1/\eta} \times\left(\int_0^y(y-\theta)\left[G^{l-1}(\theta)g(\theta)\right]^{\eta_*}\mathrm{d}\theta\right)^{1/\eta_*}, \tag{46}$$

where

$$C^*(h,l,\eta,\eta_*) = \frac{hl}{\eta+\eta_*}(x)^{(\eta-1)/\eta}(y)^{(\eta_*-1)/\eta_*}, \tag{47}$$

which was proved by Yang ([9], Theorem 3.1).

By using relations (23) and putting $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$ in Theorem 6, we get the following conclusion.

Corollary 8. *Assume that $\{a_i\}$ and $\{b_j\}$ are two nonnegative sequences of real numbers, and define*

$$A_i = \sum_{s=1}^{i} a_s, \qquad B_j = \sum_{t=1}^{j} b_t. \tag{48}$$

Then,

$$\sum_{i=1}^{k}\sum_{j=1}^{r} \frac{A_i^h B_j^l}{\eta_*(i)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(j)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}} \le C^*(h,l,\eta,\eta_*)\left(\sum_{i=1}^{k}(k-i+1)\left(A_i^{h-1}a_i\right)^{\eta}\right)^{1/\eta} \times\left(\sum_{j=1}^{r}(r-j+1)\left(B_j^{l-1}b_j\right)^{\eta_*}\right)^{1/\eta_*}, \tag{49}$$

where

$$C^*(h,l,\eta,\eta_*) = \frac{hl}{\eta+\eta_*}(k)^{(\eta-1)/\eta}(r)^{(\eta_*-1)/\eta_*}, \tag{50}$$

which was proved by Yang ([9], Theorem 2.1).

Remark 2. In Theorem 6, setting $h = l = 1$, we have

$$\int_{t_0}^{x}\int_{t_0}^{y} \frac{F(s)G(t)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le C_0(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)[f(s)]^{\eta}\Delta s\right)^{1/\eta}\left(\int_{t_0}^{y}(\sigma(y)-\theta)[g(\theta)]^{\eta_*}\Delta\theta\right)^{1/\eta_*}, \tag{51}$$

where

$$C_0(\eta,\eta_*) = \frac{1}{\eta+\eta_*}(s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}. \tag{52}$$

Remark 3. In Remark 2, if $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{R}$, and $t_0 = 0$, then we get Remarks 2 and 5, respectively, due to Yang [9].

In the following theorems, we give a further generalization of (51) obtained in Remark 2. Before we give our result, we assume that there exist two functions Φ and Ψ which are real-valued, nonnegative, convex, and submultiplicative functions defined on $[0,\infty)$. A function χ is a submultiplicative if $\chi(st) \le \chi(s)\,\chi(t)$ for $s,\ t \ge 0$.

Theorem 9. *Let s, θ, and $t_0 \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([t_0,x]_{\mathbb{T}},\mathbb{R}^+)$, $g \in C_{\mathrm{rd}}([t_0,y]_{\mathbb{T}},\mathbb{R}^+)$, and $h(\tau)$ and $l(\xi)$ be two positive functions defined for $\tau \in [t_0,x]_{\mathbb{T}}$ and $\xi \in [t_0,y]_{\mathbb{T}}$. Suppose that $F(s)$ and $G(\theta)$ are as defined in Theorem 6, and let*

$$H(s) = \int_{t_0}^{s} h(\tau)\Delta\tau, \qquad L(\theta) = \int_{t_0}^{\theta} l(\xi)\Delta\xi. \tag{53}$$

Then, for $s \in [t_0,x]_{\mathbb{T}}$ and $\theta \in [t_0,y]_{\mathbb{T}}$, we have

$$\int_{t_0}^{x}\int_{t_0}^{y} \frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le D(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\eta}\Delta s\right)^{1/\eta} \times\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left(l(\theta)\Psi\left[\frac{g(\theta)}{l(\theta)}\right]\right)^{\eta_*}\Delta\theta\right)^{1/\eta_*}, \tag{54}$$

where

$$D(\eta,\eta_*) = \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\frac{\Phi(H(s))}{H(s)}\right)^{\eta/(\eta-1)}\Delta s\right)^{(\eta-1)/\eta} \cdot\left(\int_{t_0}^{y}\left(\frac{\Psi(L(\theta))}{L(\theta)}\right)^{\eta_*/(\eta_*-1)}\Delta\theta\right)^{(\eta_*-1)/\eta_*}. \tag{55}$$

Proof. According to Theorem 2 and the definition of function Φ, it is clear that

$$\Phi(F(s)) = \Phi\left(\frac{H(s)\int_{t_0}^{s} h(\tau)(f(\tau)/h(\tau))\Delta\tau}{\int_{t_0}^{s} h(\tau)\Delta\tau}\right) \le \Phi(H(s))\Phi\left(\frac{\int_{t_0}^{s} h(\tau)(f(\tau)/h(\tau))\Delta\tau}{\int_{t_0}^{s} h(\tau)\Delta\tau}\right) \le \frac{\Phi(H(s))}{H(s)}\int_{t_0}^{s} h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\Delta\tau. \tag{56}$$

By applying Hölder's inequality (1) on (56), we find that

$$\Phi(F(s)) \le \frac{\Phi(H(s))}{H(s)}(s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s}\left(h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\right)^{\eta}\Delta\tau\right)^{1/\eta}. \tag{57}$$

Analogously,

$$\Psi(G(\theta)) \le \frac{\Psi(L(\theta))}{L(\theta)}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{t}\left(l(\xi)\Psi\left[\frac{g(\xi)}{l(\xi)}\right]\right)^{\eta_*}\Delta\xi\right)^{1/\eta*}. \tag{58}$$

Thus, from (57) and (58), it can be acquired that

$$\Phi(F(s))\Psi(G(\theta)) \le (s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\frac{\Phi(H(s))}{H(s)}\left(\int_{t_0}^{s}\left(h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\right)^{\eta}\Delta\tau\right)^{1/\eta}\right)$$
$$\times\left(\frac{\Psi(L(\theta))}{L(\theta)}\left(\int_{t_0}^{t}\left(l(\xi)\Psi\left[\frac{g(\xi)}{l(\xi)}\right]\right)^{\eta_*}\Delta\xi\right)^{1/\eta_*}\right). \tag{59}$$

Applying (39) on the term $(s-t_0)^{(\eta-1)/\eta}\times(\theta-t_0)^{(\eta_*-1)/\eta_*}$, we get

$$\Phi(F(s))\Psi(G(\theta)) \le \frac{\eta\eta_*}{\eta+\eta_*}\left(\frac{(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}}{\eta}+\frac{(t-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}{\eta_*}\right)$$
$$\times\left(\frac{\Phi(H(s))}{H(s)}\left(\int_{t_0}^{s}\left(h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\right)^{\eta}\Delta\tau\right)^{1/\eta}\right) \tag{60}$$
$$\times\left(\frac{\Psi(L(\theta))}{L(\theta)}\left(\int_{t_0}^{t}\left(l(\xi)\Psi\left[\frac{g(\xi)}{l(\xi)}\right]\right)^{\eta_*}\Delta\xi\right)^{1/\eta_*}\right).$$

From (60), we observe that

$$\frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(t-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}$$
$$\le\frac{1}{\eta+\eta_*}\left(\frac{\Phi(H(s))}{H(s)}\left\{\int_{t_0}^{s}\left(h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\right)^{\eta}\Delta\tau\right\}^{1/\eta}\right)$$
$$\times\left(\frac{\Psi(L(\theta))}{L(\theta)}\left\{\int_{t_0}^{t}\left(l(\xi)\Psi\left[\frac{g(\xi)}{l(\xi)}\right]\right)^{\eta_*}\Delta\xi\right\}^{1/\eta_*}\right). \tag{61}$$

Integrating the above relation and using Hölder's inequality (1) again with indices η, $\eta/(\eta-1)$ and η_*, $\eta_*/(\eta_*-1)$, we find that

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta$$
$$\le\frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\frac{\Phi(H(s))}{H(s)}\right)^{\eta/(\eta-1)}\Delta s\right)^{(\eta-1)/\eta}\left(\int_{t_0}^{x}\int_{t_0}^{s}\left(h(\tau)\Phi\left[\frac{f(\tau)}{h(\tau)}\right]\right)^{\eta}\Delta\tau\Delta s\right)^{1/\eta} \tag{62}$$
$$\times\left(\int_{t_0}^{y}\left(\frac{\Psi(L(\theta))}{L(\theta)}\right)^{\eta_*/(\eta_*-1)}\Delta\theta\right)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{y}\int_{t_0}^{\theta}\left(l(\xi)\Psi\left[\frac{g(\xi)}{l(\xi)}\right]\right)^{\eta_*}\Delta\xi\Delta\theta\right)^{1/\eta_*}.$$

Applying Lemma 4 on (62) and using $\sigma(n) \geq n$, we get

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \\
&\leq \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\frac{\Phi(H(s))}{H(s)}\right)^{\eta/(\eta-1)}\Delta s\right)^{(\eta-1)/\eta}\left(\int_{t_0}^{y}\left(\frac{\Psi(L(\theta))}{L(\theta)}\right)^{\eta_*/(\eta_*-1)}\Delta\theta\right)^{(\eta_*-1)/\eta_*} \\
&\quad\times\left(\int_{t_0}^{x}(x-\sigma(s))\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\eta}\Delta s\right)^{1/\eta}\left(\int_{t_0}^{y}(y-\sigma(\theta))\left(l(\theta)\Psi\left[\frac{g(\theta)}{l(\theta)}\right]\right)^{\eta_*}\Delta\theta\right)^{1/\eta_*} \\
&\leq D(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\eta}\Delta s\right)^{1/\eta}\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left(l(\theta)\Psi\left[\frac{g(\theta)}{l(\theta)}\right]\right)^{\eta_*}\Delta\theta\right)^{1/\eta_*},
\end{aligned} \tag{63}$$

which is (54). This completes the proof. □

Remark 4. Letting $1/\eta + 1/\eta_* = 1$ in (54), then we get Theorem 3.2 due to Saker et al. [11].

By using relations (22) and putting $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$ in Theorem 9, we get the following conclusion.

Corollary 10. *Assume that f(s) and g(θ) are two nonnegative functions and h(s) and l(θ) are two positive functions, and let*

$$\begin{aligned}
F(s) &= \int_0^s f(\tau)\mathrm{d}\tau, \\
G(\theta) &= \int_0^\theta g(\tau)\mathrm{d}\tau, \\
H(s) &= \int_0^s h(\tau)\mathrm{d}\tau, \\
L(\theta) &= \int_0^\theta l(\tau)\mathrm{d}\tau.
\end{aligned} \tag{64}$$

Then, for $s \in (0, x)$ and $\theta \in (0, y)$, we have

$$\begin{aligned}
&\int_0^x\int_0^y\frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\mathrm{d}s\mathrm{d}t \\
&\leq D^*(\eta,\eta_*)\left(\int_0^x(x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\eta}\mathrm{d}s\right)^{1/\eta} \\
&\quad\times\left(\int_0^y(y-\theta)\left(l(\theta)\Psi\left[\frac{g(\theta)}{l(\theta)}\right]\right)^{\eta_*}\mathrm{d}\theta\right)^{1/\eta_*},
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
D^{**}(\eta,\eta_*) &= \frac{1}{\eta+\eta_*}\left(\int_0^x\left(\frac{\Phi(H(s))}{H(s)}\right)^{\eta/(\eta-1)}\mathrm{d}s\right)^{(\eta-1)/\eta} \\
&\quad\cdot\left(\int_0^y\left(\frac{\Psi(L(\theta))}{L(\theta)}\right)^{\eta_*/(\eta_*-1)}\mathrm{d}\theta\right)^{(\eta_*-1)/\eta_*}.
\end{aligned} \tag{66}$$

It is clear that we can have the same inequality in [9], Theorem 3.2.

By using relations (23) and putting $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$ in Theorem 9, we get the following conclusion.

Corollary 11. *Assume that $\{a_i\}$ and $\{b_j\}$ are two nonnegative sequences of real numbers and $\{h_i\}$ and $\{l_j\}$ are positive sequences, and define*

$$\begin{aligned}
A_i &= \sum_{p=1}^{i} a_p, \\
B_j &= \sum_{q=1}^{j} b_q, \\
H_i &= \sum_{p=1}^{i} h_p, \\
L_j &= \sum_{q=1}^{j} l_q.
\end{aligned} \tag{67}$$

Then,

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{\Phi(A_i)\Psi(B_j)}{\eta_*(i)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(j)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}$$
$$\le D^{**}(\eta,\eta_*)\left(\sum_{i=1}^{k}(k-i+1)\left(h_i\Phi\left[\frac{a_i}{h_i}\right]\right)^{\eta}\right)^{1/\eta}\times\left(\sum_{j=1}^{r}(r-j+1)\left(l_j\Psi\left[\frac{b_j}{l_j}\right]\right)^{\eta_*}\right)^{1/\eta_*}, \tag{68}$$

where

$$D^{**}(\eta,\eta_*)=\frac{1}{\eta+\eta_*}\left(\sum_{i=1}^{k}\left(\frac{\Phi(H_i)}{H_i}\right)^{\eta/(\eta-1)}\right)^{(\eta-1)/\eta}\cdot\left(\sum_{j=1}^{r}\left(\frac{\Psi(L_j)}{L_j}\right)^{\eta_*/(\eta_*-1)}\right)^{(\eta_*-1)/\eta_*}. \tag{69}$$

Remark 5. From inequality (39), we obtain

$$s_1^{\omega_1}s_2^{\omega_2}\le\frac{1}{\omega_1+\omega_2}\left(\omega_1 s_1^{\omega_1+\omega_2}+\omega_2 s_2^{\omega_1+\omega_2}\right),\quad \text{for } \omega_1>0,\ \omega_2>0. \tag{70}$$

If we apply (70) on (31) in Theorem 6 and (54) in Theorem 9, then we get the following, respectively, inequalities:

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{F^h(s)G^l(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta$$
$$\le\frac{\eta\eta_* D_0(h,l,\eta,\eta_*)}{\eta+\eta_*}\left\{\frac{1}{\eta}\left(\int_{t_0}^{x}(\sigma(x)-s)\left[F^{h-1}(\sigma(s))f(s)\right]^{\eta}\Delta s\right)^{(\eta+\eta_*)/\eta\eta_*}+\frac{1}{\eta_*}\left(\int_{t_0}^{y}(\sigma(y)-\theta)\left[G^{l-1}(\sigma(\theta))g(\theta)\right]^{\eta_*}\Delta\theta\right)^{(\eta+\eta_*)/\eta\eta_*}\right\}, \tag{71}$$

where

$$D_0(h,l,\eta,\eta_*)=\frac{hl}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}. \tag{72}$$

Also,

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta$$
$$\le\frac{\eta\eta_* D_0^*(h,l,\eta,\eta_*)}{\eta+\eta_*}\left\{\frac{1}{\eta}\left[\int_{t_0}^{x}(\sigma(x)-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\eta}\Delta s\right]^{(\eta+\eta_*)/\eta\eta_*}+\frac{1}{\eta_*}\left[\int_{t_0}^{y}(\sigma(y)-\theta)\left(l(\theta)\Psi\left[\frac{g(\theta)}{l(\theta)}\right]\right)^{\eta_*}\Delta\theta\right]^{(\eta+\eta_*)/\eta\eta_*}\right\}, \tag{73}$$

where

$$D_0^*(h,l,\eta,\eta_*)=\frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\frac{\Phi(H(s))}{H(s)}\right)^{\eta/(\eta-1)}\Delta s\right)^{(\eta-1)/\eta}\cdot\left(\int_{t_0}^{y}\left(\frac{\Psi(L(\theta))}{L(\theta)}\right)^{\eta_*/(\eta_*-1)}\Delta\theta\right)^{(\eta_*-1)/\eta_*}. \tag{74}$$

Remark 6. In Remark 5, if $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=\mathbb{R}$, and $t_0=0$, then we get Remarks 3 and 5, respectively, due to Yang [9].

The following theorems deal with slight variants of inequality (54) given in Theorem 9.

Theorem 12. *Let s, θ, and $t_0\in\mathbb{T}$, $f\in C_{\mathrm{rd}}([t_0,x]_{\mathbb{T}},\mathbb{R}^+)$, and $g\in C_{\mathrm{rd}}([t_0,y]_{\mathbb{T}},\mathbb{R}^+)$. Define*

$$F(s)=\frac{1}{s-t_0}\int_{t_0}^{s}f(\xi)\Delta\xi,\qquad G(\theta)=\frac{1}{\theta-t_0}\int_{t_0}^{\theta}g(\xi)\Delta\xi. \tag{75}$$

Then, for $s\in[t_0,x]_{\mathbb{T}}$ and $\theta\in[t_0,y]_{\mathbb{T}}$, we have

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta$$
$$\le E(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)[\Phi(f(s))]^{\eta}\Delta s\right)^{1/\eta}\times\left(\int_{t_0}^{y}(\sigma(y)-\theta)[\Psi(g(\theta))]^{\eta_*}\Delta\theta\right)^{1/\eta_*}, \tag{76}$$

where

$$E(\eta,\eta_*)=\frac{1}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}. \tag{77}$$

Proof. By assumption and using Jensen's inequality (26), we see that

$$\Phi(F(s))=\Phi\left(\frac{1}{s-t_0}\int_{t_0}^{s}f(\xi)\Delta\xi\right)\le\frac{1}{s-t_0}\int_{t_0}^{s}\Phi[f(\xi)]\Delta\xi. \tag{78}$$

By applying inequality (1) on (78) with indices η, $\eta/(\eta-1)$, we have

$$\Phi(F(s))\le\frac{1}{s-t_0}(s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s}(\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}. \tag{79}$$

This implies that

$$\Phi(F(s))(s-t_0) \le (s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}. \tag{80}$$

Analogously,

$$\Psi(G(\theta))(\theta-t_0) \le (\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{81}$$

From (80) and (81), we get

$$\begin{aligned}\Phi(F(s))\Psi(G(\theta))&(s-t_0)(\theta-t_0)\\ &\le (s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}\\ &\quad\times\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)^{1/\eta_*}.\end{aligned} \tag{82}$$

Applying elementary inequality (39) on the term $(s-t_0)^{(\eta-1)/\eta}\times(\theta-t_0)^{(\eta_*-1)/\eta_*}$, where $s_1=(s-t_0)^{\eta-1}$, $s_2=(t-t_0)^{\eta_*-1}$, $\omega_1=1/\eta$, $\omega_2=1/\eta_*$, and $r=\omega_1+\omega_2$, we get

$$\begin{aligned}\Phi(F(s))\Psi(G(\theta))&(s-t_0)(\theta-t_0)\\ &\le \frac{\eta\eta_*}{\eta+\eta_*}\left(\frac{(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}}{\eta}+\frac{(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}{\eta_*}\right)\\ &\quad\times\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)^{1/\eta_*}.\end{aligned} \tag{83}$$

From (83), we have

$$\begin{aligned}&\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\\ &\le \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)^{1/\eta_*}.\end{aligned} \tag{84}$$

Taking delta integrating on both sides of (84), first over s from t_0 to x and then over θ from t_0 to y, we find that

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta\\ &\le \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)^{1/\eta}\Delta s\right)\\ &\quad\times\left(\int_{t_0}^{y}\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)^{1/\eta_*}\Delta\theta\right).\end{aligned} \tag{85}$$

By applying inequality (1) on (85) with indices η, $\eta/(\eta-1)$ and η_*, $\eta_*/(\eta_*-1)$, we get

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta\\ &\le \frac{1}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{x}\left(\int_{t_0}^{s} (\Phi[f(\xi)])^{\eta}\Delta\xi\right)\Delta s\right)^{1/\eta}\\ &\quad\times\left(\int_{t_0}^{y}\left(\int_{t_0}^{t} (\Psi[g(\xi)])^{\eta_*}\Delta\xi\right)\Delta\theta\right)^{1/\eta_*}.\end{aligned} \tag{86}$$

Applying Lemma 4 on (86) and using the fact $\sigma(n)\ge n$, we find that

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta\\ &\le \frac{1}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{x}(x-\sigma(s))(\Phi[f(s)])^{\eta}\Delta s\right)^{1/\eta}\\ &\quad\times\left(\int_{t_0}^{y}(y-\sigma(\theta))(\Psi[g(\theta)])^{\eta_*}\Delta\theta\right)^{1/\eta_*},\\ &\le E(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)(\Phi[f(s)])^{\eta}\Delta s\right)^{1/\eta}\left(\int_{t_0}^{y}(\sigma(y)-t)(\Psi[g(\theta)])^{\eta_*}\Delta\theta\right)^{1/\eta_*}.\end{aligned} \tag{87}$$

The proof is complete.

Remark 7. Letting $1/\eta + 1/\eta_* = 1$ in (76), then we get Theorem 3.3 due to Saker et al. [11].

By using relations (22) and putting $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$ in Theorem 12, we get the following conclusion.

Corollary 13. *Assume that f(s) and $g(t)$ are nonnegative functions, and define*

$$F(s) = \frac{1}{s}\int_0^s f(\xi)\mathrm{d}\xi,$$
$$G(\theta) = \frac{1}{\theta}\int_0^\theta g(\xi)\mathrm{d}\xi. \tag{88}$$

Then, for $s \in (0, x)$ and $\theta \in (0, y)$, we have

$$\int_0^x\int_0^y \frac{s\theta\Phi(F(s))\Psi(G(t))}{\eta_*(s)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(\theta)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\mathrm{d}s\,\mathrm{d}\theta$$
$$\le E^*(\eta,\eta_*)\left(\int_0^x (x-s)(\Phi[f(s)])^\eta \mathrm{d}s\right)^{1/\eta}$$
$$\times\left(\int_0^y (y-\theta)(\Psi[g(\theta)])^{\eta_*}\mathrm{d}\theta\right)^{1/\eta_*}, \tag{89}$$

where

$$E^*(\eta,\eta_*) = \frac{1}{\eta+\eta_*}(x)^{(\eta-1)/\eta}(y)^{(\eta_*-1)/\eta_*}, \tag{90}$$

which is the same inequality in [9], Theorem 3.3.

By using relations (23) and putting $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$ in Theorem 12, we get the following conclusion.

Corollary 14. *Assume that $\{a_i\}$ and $\{b_j\}$ are two nonnegative sequences of real numbers, and define*

$$A_i = \frac{1}{i}\sum_{p=1}^{i} a_p,$$
$$B_j = \frac{1}{j}\sum_{q=1}^{j} b_q. \tag{91}$$

Then,

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{ij\Phi(A_m)\Psi(B_n)}{\eta_*(i)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(j)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}$$
$$\le E^{**}(\eta,\eta_*)\left(\sum_{i=1}^{k}(k-i+1)[\Phi(a_i)]^\eta\right)^{1/\eta}$$
$$\cdot\left(\sum_{j=1}^{r}(r-j+1)\left[\Psi(b_j)\right]^{\eta_*}\right)^{1/\eta_*}, \tag{92}$$

where

$$E^{**}(\eta,\eta_*) = \frac{1}{\eta+\eta_*}(k)^{(\eta-1)/\eta}(r)^{(\eta_*-1)/\eta_*}, \tag{93}$$

which is the same inequality in [9], Theorem 2.3.

Theorem 15. *Let s, θ, and $t_0 \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([t_0,x]_{\mathbb{T}},\mathbb{R}^+)$, $g \in C_{\mathrm{rd}}([t_0,y]_{\mathbb{T}},\mathbb{R}^+)$, and h(ξ) and l(ξ) be two positive functions defined for $\xi \in [t_0,x]_{\mathbb{T}}$ and $\xi \in [t_0,y]_{\mathbb{T}}$ and H and L be as defined in Theorem 9, and let*

$$F(s) = \frac{1}{H(s)}\int_{t_0}^{s} h(\xi)f(\xi)\Delta\xi,$$
$$G(\theta) = \frac{1}{L(\theta)}\int_{t_0}^{\theta} l(\xi)g(\xi)\Delta\xi. \tag{94}$$

Then, for $s \in [t_0,y]_{\mathbb{T}}$ and $\theta \in [t_0,x]_{\mathbb{T}}$, we have

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta$$
$$\le W(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)(h(s)\Phi[f(s)])^\eta\Delta s\right)^{1/\eta}$$
$$\times\left(\int_{t_0}^{y}(\sigma(y)-\theta)(l(\theta)\Psi[g(\theta)])^{\eta_*}\Delta\theta\right)^{1/\eta_*}, \tag{95}$$

where

$$W(\eta,\eta_*) = \frac{1}{\eta+\eta_*}(x-t_0)^{(\eta-1)/\eta}(y-t_0)^{(\eta_*-1)/\eta_*}. \tag{96}$$

Proof. Using the hypotheses of Theorem 15 and Jensen's inequality, we find that

$$\Phi(F(s)) = \Phi\left(\frac{1}{H(s)}\int_{t_0}^{s} h(\xi)f(\xi)\Delta\xi\right)$$
$$\le \frac{1}{H(s)}\int_{t_0}^{s} h(\xi)\Phi(f(\xi))\Delta\xi. \tag{97}$$

By applying inequality (1) on (97) with indices η, $\eta/(\eta-1)$, we have

$$\Phi(F(s)) \le \frac{1}{H(s)}(s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^\eta\Delta\xi\right)^{1/\eta}. \tag{98}$$

From (98), we get

$$\Phi(F(s))H(s) \le (s-t_0)^{(\eta-1)/\eta}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^\eta\Delta\xi\right)^{1/\eta}. \tag{99}$$

Analogously,

$$\Psi(G(\theta))L(\theta) \le (\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{100}$$

From (99) and (100), we find that

$$\Phi(F(s))\Psi(G(\theta))H(s)L(\theta) \le (s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^{\eta}\Delta\xi\right)^{1/\eta} \times\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{101}$$

Applying elementary inequality (39), we get

$$\Phi(F(s))\Psi(G(\theta))H(s)L(\theta) \le \frac{\eta\eta_*}{\eta+\eta_*}\left(\frac{(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}}{\eta}+\frac{(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}{\eta_*}\right) \cdot\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^{\eta}\Delta\xi\right)^{1/\eta}\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{102}$$

This implies that

$$\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}} \le \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^{\eta}\Delta\xi\right)^{1/\eta}\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)^{1/\eta_*}. \tag{103}$$

Taking delta integrating on both sides of (103), first over s from t_0 to x and then over θ from t_0 to y, we obtain

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le \frac{1}{\eta+\eta_*}\left(\int_{t_0}^{x}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^{\eta}\Delta\xi\right)^{1/\eta}\Delta s\right) \times\left(\int_{t_0}^{y}\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)^{1/\eta_*}\Delta\theta\right). \tag{104}$$

By applying inequality (1) on (104) with indices η, $\eta/(\eta-1)$ and η_*, $\eta_*/(\eta_*-1)$, we get

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le \frac{1}{\eta+\eta_*}(s-t_0)^{(\eta-1)/\eta}(\theta-t_0)^{(\eta_*-1)/\eta_*}\left(\int_{t_0}^{x}\left(\int_{t_0}^{s}[h(\xi)\Phi(f(\xi))]^{\eta}\Delta\xi\right)\Delta s\right)^{1/\eta} \times\left(\int_{t_0}^{y}\left(\int_{t_0}^{\theta}[l(\xi)\Psi(g(\xi))]^{\eta_*}\Delta\xi\right)\Delta\theta\right)^{1/\eta_*}. \tag{105}$$

Applying Lemma 4 and using $\sigma(n)\ge n$, we get

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s-t_0)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta-t_0)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\Delta s\,\Delta\theta \le W(\eta,\eta_*)\left(\int_{t_0}^{x}(\sigma(x)-s)(h(s)\Phi[f(s)])^{\eta}\Delta s\right)^{1/\eta} \times\left(\int_{t_0}^{y}(\sigma(y)-\theta)(l(\theta)\Psi[g(\theta)])^{\eta_*}\Delta\theta\right)^{1/\eta_*}. \tag{106}$$

This completes the proof. □

Remark 8. Letting $1/\eta+1/\eta_*=1$ in (95), then we get Theorem 3.4 due to Saker et al. [11].

By using relations (22) and putting $\mathbb{T}=\mathbb{R}$ and $t_0=0$ in Theorem 15, we get the following conclusion.

Corollary 16. *Assume that f(s) and $g(\theta)$ are two nonnegative functions and h(s) and l(θ) are two positive functions, and define*

$$H(s)=\int_0^s h(\xi)\mathrm{d}\xi,\quad L(\theta)=\int_0^\theta l(\xi)\mathrm{d}\xi,\quad F(s)=\frac{1}{H(s)}\int_0^s f(\xi)h(\xi)\mathrm{d}\xi,\quad G(\theta)=\frac{1}{L(\theta)}\int_0^\theta g(\xi)l(\xi)\mathrm{d}\xi. \tag{107}$$

Then, for $s\in(0,x)$ and $\theta\in(0,y)$, we have

$$\int_0^x\int_0^y\frac{\Phi(F(s))\Psi(G(\theta))H(s)L(\theta)}{\eta_*(s)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*}+\eta(\theta)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}\mathrm{d}s\,\mathrm{d}\theta \le W^*(\eta,\eta_*)\left(\int_0^x(x-s)[h(s)\Phi(f(s))]^{\eta}\mathrm{d}s\right)^{1/\eta} \times\left(\int_0^y(y-\theta)[l(\theta)\Psi(g(\theta))]^{\eta_*}\mathrm{d}\theta\right)^{1/\eta_*}, \tag{108}$$

where

$$W^*(\eta,\eta_*)=\frac{1}{\eta+\eta_*}(x)^{((\eta-1)/\eta)}(y)^{((\eta_*-1)/\eta_*)}. \tag{109}$$

It is clear that it is the same inequality in [9], Theorem 3.4.

By using relations (23) and putting $\mathbb{T}=\mathbb{Z}$ and $t_0=0$ in Theorem 15, we have the following conclusion.

Corollary 17. *Assume that $\{a_i\}$ and $\{b_j\}$ are nonnegative sequences and $\{h_i\}$ and $\{l_j\}$ are positive sequences, and define*

$$H_i = \sum_{p=1}^{i} h_p,$$
$$L_j = \sum_{q=1}^{j} l_q,$$
$$A_i = \frac{1}{H_i} \sum_{p=1}^{i} h_p a_p, \tag{110}$$
$$B_j = \frac{1}{L_j} \sum_{q=1}^{j} l_q b_q.$$

Then,

$$\sum_{i=1}^{k} \sum_{j=1}^{r} \frac{H_i L_j \Phi(A_i) \Psi(B_j)}{\eta_* (i)^{((\eta-1)(\eta+\eta_*))/\eta\eta_*} + \eta (j)^{((\eta_*-1)(\eta+\eta_*))/\eta\eta_*}}$$
$$\leq W^{**}(\eta, \eta_*) \left(\sum_{i=1}^{k} (k-i+1) [h_i \Phi(a_i)]^{\eta} \right)^{1/\eta}$$
$$\times \left(\sum_{j=1}^{r} (r-j+1) [l_j \Psi(b_j)]^{\eta_*} \right)^{1/\eta_*}, \tag{111}$$

where

$$W^{**}(\eta, \eta_*) = \frac{1}{\eta + \eta_*} (k)^{(\eta-1)/\eta} (r)^{(\eta_*-1)/\eta_*}, \tag{112}$$

which is the same inequality in [9], Theorem 2.4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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