Some Dynamic Inequalities of Hilbert’s Type

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This paper is concerned with deriving some new dynamic Hilbert-type inequalities on time scales. The basic idea in proving the results is using some algebraic inequalities, Hölder’s inequality and Jensen’s inequality, on time scales. As a special case of our results, we will obtain some integrals and their corresponding discrete inequalities of Hilbert’s type.

1. Introduction

It is evident that the Hilbert-type inequalities outplay a major role in mathematics, for pattern complex analysis, numerical analysis, and qualitative theory of differential equations and their implementations. In recent years, there were a lot of various refinements, generalizations, extensions, and applications of Hilbert’s inequality which have seemed in the literature. Hilbert’s discrete inequality and its integral formula ([1], Theorem 316) have been generalized in many trends (for example, see [2–6]). Lately, Pachpatte [7] proved new inequalities similar to those of Hilbert’s inequality, namely, he proved that if \( h, l \geq 1 \), \( A_m = \sum_{i=1}^{m} a_i \geq 0 \), and \( B_n = \sum_{i=1}^{n} b_i \geq 0 \), then

\[
\sum_{i=1}^{k} \sum_{j=1}^{l} \frac{A_i^h B_j^l}{i+j} \leq C^* (h, l, k, r) \left( \sum_{i=1}^{k} (k+1-i) (A_i^{h-1} a_i)^2 \right)^{1/2} \times \left( \sum_{j=1}^{l} (r+1-j) (B_j^{l-1} b_j)^2 \right)^{1/2},
\]

where

\[
C^* (h, l, k, r) = \frac{1}{2} hl^{1/2} k r.
\]  

An integral analogue of (1) is given in the following result. Let \( h, l \geq 1 \), \( F(x) = \int_0^x f(t) dt \geq 0 \), and \( G(y) = \int_0^y g(t) dt \geq 0 \), for \( x, r \in (0,a) \) and \( y, v \in (0,b) \). Then,

\[
\int_0^a \int_0^b \frac{F(x) G(y)}{x+y} dx \, dy \leq D(h, l, a, b)
\] \[\cdot \left( \int_0^a (a-x) \left[ F^{h-1} (x) f(x) \right]^2 dx \right)^{1/2}
\] \[\times \left( \int_0^b (b-y) \left[ G^{l-1} (y) g(y) \right]^2 dy \right)^{1/2},
\]

where

\[
D(h, l, a, b) = \frac{1}{2} hl \sqrt{ab}.
\]  

In 2001, Kim [8] gave some generalizations of (1) and (3) by introducing a parameter \( \alpha > 0 \) as
\[ \sum_{i=1}^{k} \sum_{j=1}^{r} A_{i}^{h} B_{j}^{l} \leq C(h, l, k, r, a) \left( \sum_{i=1}^{k} (k - i + 1)(A_{i}^{h-1}a_{i})^{\alpha} \right)^{1/\alpha} \]
\[ \times \left( \sum_{j=1}^{r} (r - j + 1)(B_{j}^{l-1}b_{j})^{\gamma} \right)^{1/\gamma}, \]
where \( h, l \geq 1 \), \( A_m = \sum_{p=1}^{m} a_p \geq 0 \), \( B_n = \sum_{q=1}^{n} b_q \geq 0 \), and
\[ C(h, l, k, r, a) = \left( \frac{1}{2} \right)^{1/\alpha} hl \sqrt{kr}. \]

An integral analogue of (5) is given in the following result. Let \( p, q \geq 1 \), \( a > 0 \), \( F(t) = \int_{0}^{t} f(r)dr \geq 0 \), and \( G(s) = \int_{0}^{s} g(\nu)d\nu \geq 0 \), for \( t, \tau \in (0, a) \), \( s, \nu \in (0, b) \). Then,
\[ \int_{0}^{a} \int_{0}^{b} F(h) G(y) \left( \frac{a}{x + y} \right)^{1/\alpha} dx dy \leq D(h, l, a, b) \]
\[ \cdot \left( \int_{0}^{a} (a - x) \left( F^{h-1}(x) f(x) \right)^{2/\alpha} dx \right)^{1/2} \]
\[ \times \left( \int_{0}^{b} (b - y) \left( G^{l-1}(y) g(y) \right)^{2/\gamma} dy \right)^{1/2}, \]
where
\[ D(h, l, a, b) = \left( \frac{1}{2} \right)^{1/\alpha} hl \sqrt{ab}. \]

In 2009, Yang [9] gave another generalization of (1) and (3) by introducing parameter \( \alpha > 1 \) and \( \gamma > 1 \) as follows. Let \( h, l \geq 1 \), \( A_m = \sum_{p=1}^{m} a_p \geq 0 \), \( B_n = \sum_{q=1}^{n} b_q \geq 0 \). Then,
\[ \sum_{i=1}^{k} \sum_{j=1}^{r} A_{i}^{h} B_{j}^{l} \leq C(h, l, k, r, a, \gamma) \left( \sum_{i=1}^{k} (k - i + 1)(A_{i}^{h-1}a_{i})^{\alpha} \right)^{1/\alpha} \]
\[ \times \left( \sum_{j=1}^{r} (r - j + 1)(B_{j}^{l-1}b_{j})^{\gamma} \right)^{1/\gamma}, \]
where
\[ C(h, l, k, r, a, \gamma) = \frac{hl}{\alpha + \gamma r} \left( \frac{1}{2} \right)^{1/\alpha} (a-1)/\gamma r^{1/\gamma}. \]

An integral analogue of (9) is given as follows. If \( h, l \geq 1 \), \( a > 1 \), \( \gamma > 1 \), \( F(x) = \int_{0}^{x} f(r)dr \geq 0 \), and \( G(y) = \int_{0}^{y} g(\nu)d\nu \geq 0 \), for \( x, r \in (0, a) \) and \( y, \nu \in (0, b) \), then
\[ \int_{0}^{a} \int_{0}^{b} F^{h}(x) G^{l}(y) \left( \frac{a}{x + y} \right)^{\alpha} dx dy \]
\[ \leq D(h, l, a, b, \alpha, \gamma) \left( \int_{0}^{a} (a - x) \left( F^{h-1}(x) f(x) \right)^{\alpha} dx \right)^{1/\alpha} \]
\[ \times \left( \int_{0}^{b} (b - y) \left( G^{l-1}(y) g(y) \right)^{\gamma} dy \right)^{1/\gamma}, \]
where
\[ D(h, l, a, b, \alpha, \gamma) = \frac{pq}{a + \gamma r} \left( \frac{1}{2} \right)^{1/\alpha} (a-1)/\gamma r^{1/\gamma}. \]

In [10], the authors deduced several generalizations of inequalities (1) and (3) on time scales, namely, they proved that if \( h \) and \( l \geq 1 \) are real numbers, \( A_{x_{1}} = \int_{x_{1}}^{x_{2}} a(t)\Delta x_{1} \), \( B_{y_{1}} = \int_{y_{1}}^{y_{2}} b(t)\Delta y_{1} \), and \( h > 1 \), \( s > 1 \) with \( h^{-1} + s^{-1} = 1 \), then
\[ \int_{x_{1}}^{x_{2}} \int_{w}^{w} A_{x_{1}} B_{y_{1}} \left( \frac{a}{x_{1} - w} \right)^{\alpha} + \eta \left( y_{1} - w \right) \Delta y_{1} \Delta x_{1} \]
\[ \leq M(\eta, s) \left( \int_{w}^{w} (\sigma(x_{1}) - x_{1}) \left[ a(x_{1}) A_{h-1}(\sigma(x_{1})) \right]^{\eta} \Delta x_{1} \right)^{1/\eta} \]
\[ \times \left( \int_{x_{1}}^{x_{2}} (\sigma(y_{1}) - y_{1}) \left[ b(y_{1}) B_{l-1}(\alpha(y_{1})) \right]^{\gamma} \Delta y_{1} \right)^{1/\gamma}, \]
where
\[ M(\eta, s) = \frac{hl}{\eta \eta_{s}} \left( x_{2} - w \right)^{1-\eta} / \left( y_{2} - w \right)^{1-\eta_{s}}. \]

In [11], the authors gave some extensions of inequalities (5) and (7) on time scales. Minutely, they proved that if \( \gamma > 0 \) and \( h \) and \( l \geq 1 \) are real numbers, \( A(s) = \int_{0}^{s} a(t)\Delta t \), \( B(t) = \int_{0}^{t} b(t)\Delta t \), and \( h > 1 \), \( s > 1 \) with \( h^{-1} + s^{-1} = 1 \), then
\[ \int_{0}^{s} \int_{t_{0}}^{t_{0}} A_{h}(\sigma(t)) B_{l}(\sigma(t)) \left( \frac{a}{\sigma(t) - t_{0}} \right)^{\alpha} + \eta \left( \sigma(t) - t_{0} \right) \Delta s \Delta t \]
\[ \leq C(h, l, \gamma, \eta) \left( \int_{t_{0}}^{t_{0}} (\sigma(s) - \sigma(s)) \left[ a(s) A_{h-1}(\sigma(s)) \right]^{\eta} \Delta s \right)^{1/\eta} \]
\[ \times \left( \int_{t_{0}}^{s} (\sigma(y) - \sigma(t)) \left[ b(t) B_{l-1}(\alpha(t)) \right]^{\gamma} \Delta t \right)^{1/\gamma}, \]
where
\[ C(h, l, \eta, \gamma) = \frac{h^{2}/\gamma}{\eta} \left( x - t_{0} \right)^{1/\eta} / \left( y - t_{0} \right)^{1/\gamma}. \]
Following this trend and to develop the study of dynamic inequalities on time scales, we will prove some new inequalities of Hilbert's type on time scales, namely, we prove time scale versions of inequalities (9) and (11) on time scale $\mathbb{T}$. These inequalities can be considered as extensions and generalizations of some Hilbert-type inequalities proved in [10]. We also derive some inequalities on time scale as special cases.

2. Definitions and Basic Results

In this division, we will present some fundamental concepts and effects on time scales which will be beneficial for deducing our main results. The following definitions and theorems are referred from [12, 13].

Time scale $\mathbb{T}$ is defined as a nonempty arbitrary locked subplot of real numbers $\mathbb{R}$. We define the forward jump operator $\sigma$: $\mathbb{T} \rightarrow \mathbb{T}$ as

$$\sigma(\tau) = \inf\{\theta \in \mathbb{T}: \theta > \tau\}$$

(17)

and the backward jump operator $\rho$: $\mathbb{T} \rightarrow \mathbb{T}$ as

$$\rho(\tau) = \sup\{\theta \in \mathbb{T}: \theta < \tau\}.$$  

(18)

From the above two definitions, it can be stated that a point $\tau \in \mathbb{T}$ with $\inf \mathbb{T} < \tau < \sup \mathbb{T}$ is called right-scattered if $\sigma(\tau) = \tau$, right-dense if $\sigma(\tau) < \tau$, left-scattered if $\rho(\tau) < \tau$, and left-dense if $\rho(\tau) = \tau$. If $\mathbb{T}$ has left-scattered maximum $\sup_{\mathbb{T}}$, then $\mathbb{T} = \mathbb{T} \cup \{s_{\min}\}$; otherwise, $\mathbb{T} = \mathbb{T}$. Finally, the graininess function $\mu$: $\mathbb{T} \rightarrow [0, \infty)$ for any $\tau \in \mathbb{T}$ is defined by

$$\mu(\tau) = \sigma(\tau) - \tau.$$

(19)

For a function $\chi$: $\mathbb{T} \rightarrow \mathbb{R}$, the delta derivative of $\chi$ at $\tau \in \mathbb{T}$ is defined as for each $\varepsilon > 0$, there is a neighborhood $U$ of $\tau$ such that

$$|\chi(\sigma(\tau) - \chi(\theta)) - \chi^\Delta(\tau)| - \varepsilon|\sigma(\tau) - \theta|.$$

(20)

Moreover, $\chi$ is called delta differentiable on $\mathbb{T}$ if $\chi$ is delta differentiable at every $\tau \in \mathbb{T}$.

A function $\chi$: $\mathbb{T} \rightarrow \mathbb{R}$ is called right-dense continuous (rd-continuous) as long as it is continuous at all right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The classes of real rd-continuous functions on an interval $I$ will be denoted by $C_{rd}(I, \mathbb{R})$. For $\theta$, $t \in \mathbb{T}$, the Cauchy integral of $\chi^\Delta$ is defined as

$$\int_\theta^t \chi^\Delta(\tau) \Delta \tau = \chi(t) - \chi(\theta).$$

(21)

Note that

(a) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\tau) = \tau,
\mu(\tau) = 0,
\chi^\Delta(\tau) = \chi'(\tau),$$

(22)

$$\int_\theta^t \chi^\Delta(\tau) \Delta \tau = \int_\theta^t \chi(\tau) \Delta \tau.$$

(b) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\tau) = \tau + 1,
\mu(\tau) = 1,
\chi^\Delta(\tau) = \Delta \chi(\tau),$$

(23)

$$\int_\theta^t \chi^\Delta(\tau) \Delta \tau = \sum_{\tau = \theta}^{\tau - 1} \chi(\tau).$$

In what follows, we will present Hölder's inequality, Jensen's inequality, and the power rules for integration on time scales.

Theorem 1 (Hölder’s inequality (see [14, 15])). Let $u, v \in \mathbb{T}$. For $\zeta, \chi \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_u^v |\zeta(\theta)\chi(\theta)| \Delta \theta \leq \left[\int_u^v |\zeta(\theta)|^\mu \Delta \theta\right]^{\eta^{-1}} \left[\int_u^v |\chi(\theta)|^\nu \Delta \theta\right]^{\eta_*^{-1}},$$

(24)

where $\eta > 1$ and $\eta_* > 1$ with $\eta^{-1} + \eta_*^{-1} = 1$.

Theorem 2 (Jensen’s inequality (see [14, 16])). Suppose that $\zeta \in C_{rd}(\{u, v\}, (w, z))$ and $\eta \in C_{rd}(\{u, v\}, \mathbb{R})$ are nonnegative with

$$\int_u^v \eta(\theta) \Delta \theta > 0.$$  

(25)

If $\Phi \in C_{rd}(\{w, z\}, \mathbb{R})$ is convex, then

$$\Phi\left(\int_u^v \eta(\zeta(\theta)) \Delta \theta\right) \leq \frac{\int_u^v \zeta(\eta) \Phi(\eta(\zeta) \Delta \theta)}{\int_u^v \zeta(\Delta \theta)}.$$  

(26)

Lemma 3 (see [17]). Let $u, s \in \mathbb{T}$, and $\zeta \in C_{rd}(\{u, s\}, \mathbb{R})$ be nonnegative. If $\alpha \geq 1$, then

$$\int_u^s \zeta(\tau) \Delta \tau \leq \alpha \int_u^s \zeta(\tau) \Delta \tau.$$

(27)

Now, we will present the formula that will reduce double integrals to single integrals which is the desired in [18].

Lemma 4. Let $\chi$: $\mathbb{T} \rightarrow \mathbb{R}$ and $u, s, \theta \in \mathbb{T}$. Then,

$$\int_u^s \int_u^\theta \chi(\tau) \Delta \tau \Delta \theta = \int_u^s (s - \sigma(\theta)) \chi(\tau) \Delta \tau,$$

(28)

assuming the integrals considered exist.

Lemma 5 (see [19]). Let $r > 0$, $\mu_i > 0$, and $\sum_{q=1}^m \mu_q = \Omega_m$. Then,

$$\left(\prod_{q=1}^m \mu_q^{\frac{1}{\Omega_q}}\right)^{1/r} \leq \left(\frac{1}{\Omega_m} \sum_{q=1}^m \mu_q^{\frac{1}{\Omega_q}}\right)^{1/r}.$$  

(29)
3. Main Results

In this division, we will prove our main results. Throughout this section, we will assume that all functions are nonnegative and the integrals considered are assumed to exist. Also, we will assume that \( h \) and \( f \geq 1 \) be real numbers and \( \eta > 1 \) and \( \eta_* > 1 \) with \( \eta^{-1} + \eta_*^{-1} = 1 \).

**Theorem 6.** Let \( s, \theta, \) and \( t_0 \in \mathbb{T} \) and \( f \in C_{\alpha}([t_0, x], \mathbb{R}^+) \) and \( g \in C_{\alpha}([t_0, y], \mathbb{R}^+) \). Suppose that \( F(s) \) and \( G(\theta) \) are defined as

\[
F(s) = \int_{t_0}^{s} f(\xi)\Delta \xi,
\]

\[
G(\theta) = \int_{t_0}^{\theta} g(\xi)\Delta \xi.
\]

Then, for \( s \in [t_0, x] \) and \( \theta \in [t_0, y] \), we have

\[
\int_{t_0}^{s} \int_{t_0}^{s} \frac{(s-t_0)^{(\eta^{-1} - 1)/\eta} \cdot (y-t_0)^{(\eta^{-1} - 1)/\eta_*)}{\eta + \eta_*} \cdot \Delta \theta \cdot \Delta \xi \leq C(h, l, \eta, \eta_*) \left( \int_{t_0}^{s} (\sigma(x) - s) \left[ F^{h-1}(\sigma(s)) f(s) \right]^{\gamma} \Delta s \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\sigma(y) - \theta) \left[ G^{l-1}(\sigma(\theta)) g(\theta) \right]^{\eta_0} \Delta \theta \right)^{1/\eta_0},
\]

where

\[
C(h, l, \eta, \eta_*) = \frac{hl}{\eta + \eta_*} \left( x - t_0 \right)^{(\eta^{-1} - 1)/\eta} \left( y - t_0 \right)^{(\eta^{-1} - 1)/\eta_*}.
\]

**Proof.** By using inequality (27) (see Lemma 3), we see that

\[
F^{h}(s) \leq h \int_{t_0}^{s} F^{h-1}(\xi) f(\xi)\Delta \xi,
\]

\[
G^{l}(\theta) \leq l \int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi.
\]

Then, we have

\[
F^{h}(s)G^{l}(\theta) \leq hl \left( \int_{t_0}^{s} F^{h-1}(\xi) f(\xi)\Delta \xi \right) \left( \int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\sigma(y) - \theta) \left[ G^{l-1}(\sigma(\theta)) g(\theta) \right]^{\eta_0} \Delta \theta \right)^{1/\eta_0}.
\]

Applying Hölder’s inequality (1) on \( \int_{t_0}^{s} F^{h-1}(\xi) f(\xi)\Delta \xi \) with indices \( \eta \) and \( \eta/(\eta-1) \), we find that

\[
\int_{t_0}^{s} F^{h-1}(\xi) f(\xi)\Delta \xi \leq (s - t_0)^{(\eta-1)/\eta} \left( \int_{t_0}^{s} [F^{h-1}(\xi) f(\xi)]^{\eta} \Delta \xi \right)^{1/\eta},
\]

and on the integral \( \int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi \) with indices and \( \eta_* \) with \( \eta_*/(\eta_*-1) \), we find that

\[
\int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi \leq (\theta - t_0)^{(\eta_*-1)/\eta_*} \left( \int_{t_0}^{\theta} [G^{l-1}(\xi) g(\xi)]^{\eta_*} \Delta \xi \right)^{1/\eta_*}.
\]

From (36) and (37), we get

\[
F^{h}(s)G^{l}(\theta) \leq hl \left( \int_{t_0}^{s} F^{h-1}(\xi) f(\xi)\Delta \xi \right) \left( \int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\sigma(y) - \theta) \left[ G^{l-1}(\sigma(\theta)) g(\theta) \right]^{\eta_0} \Delta \theta \right)^{1/\eta_0}.
\]

Using inequality (29) of power means, we observe that

\[
(s_{1}^{\omega_1} s_{2}^{\omega_2})^{r/(\omega_1 + \omega_2)} \leq \frac{1}{\omega_1 + \omega_2} \left( s_{1}^{\omega_1} + s_{2}^{\omega_2} \right).
\]

Now, by setting \( s_{1} = (s - t_0)^{\eta^{-1}} \), \( s_{2} = (\theta - t_0)^{\eta^{-1}} \), \( \omega_1 = 1/\eta \), \( \omega_2 = 1/\eta_* \), and \( r = \omega_1 + \omega_2 \) in (39), we get

\[
(s - t_0)^{(\eta^{-1} - 1)/\eta} \cdot (\theta - t_0)^{(\eta^{-1} - 1)/\eta_*} \leq \eta_* \frac{(s - t_0)^{(\eta^{-1} - 1)/\eta} \cdot (\theta - t_0)^{(\eta^{-1} - 1)/\eta_*}}{\eta + \eta_*}.
\]

Substituting (40) into (38) yields

\[
F^{h}(s)G^{l}(\theta) \leq hl \eta_* \left( \frac{(s - t_0)^{(\eta^{-1} - 1)/\eta} \cdot (\theta - t_0)^{(\eta^{-1} - 1)/\eta_*}}{\eta + \eta_*} + \left( \frac{(s - t_0)^{(\eta^{-1} - 1)/\eta} \cdot (\theta - t_0)^{(\eta^{-1} - 1)/\eta_*}}{\eta + \eta_*} \right) \right)^{1/\eta} \times \left( \int_{t_0}^{\theta} G^{l-1}(\xi) g(\xi)\Delta \xi \right)^{1/\eta_*}.
\]

Dividing both sides of (41) by the last factor \( \eta_* \), we obtain
Due to Saker et al. ([11], Theorem 3.1), which proves (31), this completes the proof.

Applying Lemma 4 on (43) and using the fact that \(\sigma(n) \geq n\), we conclude that

\[
\int_{t_0}^{x} \int_{t_0}^{y} \frac{F^h(s)G^l(t)}{\eta_s \left( (s-t_0)^{(\eta-1)(\eta_s,\eta_s)} + \eta \left( \theta - t_0 \right)^{(\eta-1)(\eta_s,\eta_s)} \right)^{\eta_s}} \Delta s \Delta \theta \\
\leq \frac{hl}{\eta + \eta_s} (x-t_0)^{(\eta-1)\eta_s} (y-t_0)^{(\eta-1)\eta_s} \left( \int_{t_0}^{x} \int_{t_0}^{y} \left( \left[ F^h(s) \eta_s \left( \sigma(s) \right) f(s) \right]^{\eta_s} \Delta s \right)^{\eta_s} \Delta \theta \right) \left( \int_{t_0}^{x} \int_{t_0}^{y} \left( \left[ G^l(t) \eta_s \left( \sigma(t) \right) g(t) \right]^{\eta_s} \Delta t \right)^{\eta_s} \Delta \theta \right)^{\eta_s} \right)^{\eta_s}.
\]

which proves (31). This completes the proof.

**Remark 1.** Letting \(1/\eta_s + 1/\eta_s = 1\) in (31), we get Theorem 3.1 due to Saker et al. ([11], Theorem 3.1).

By using relations (22) and putting \(\mathbb{T} = \mathbb{R}\) and \(t_0 = 0\) in Theorem 6, we get the following conclusion.

**Corollary 7.** Assume that \(f(\xi)\) and \(g(\xi)\) are two nonnegative functions, and define

\[
F(s) = \int_{0}^{s} f(\xi) d\xi, \quad \text{and} \quad G(q) = \int_{0}^{q} g(\xi) d\xi.
\]

Then, for \(s \in (0, x)\) and \(\theta \in (0, y)\), we have

\[
\int_{0}^{x} \int_{0}^{y} \frac{F^h(s)G^l(t)}{\eta_s \left( (s-t_0)^{(\eta-1)(\eta_s,\eta_s)} + \eta \left( \theta - t_0 \right)^{(\eta-1)(\eta_s,\eta_s)} \right)^{\eta_s}} ds \, d\theta \\
\leq C^* (h, l, \eta_s, \eta_s) \left( \int_{0}^{x} \left( x-s \left[ F^h(s) f(s) \right]^{\eta_s} ds \right)^{\frac{1}{\eta_s}} \right)^{\frac{1}{\eta_s}} \right)^{\frac{1}{\eta_s}}.
\]

where

\[
\int_{0}^{x} \int_{0}^{y} \frac{F^h(s)G^l(t)}{\eta_s \left( (s-t_0)^{(\eta-1)(\eta_s,\eta_s)} + \eta \left( \theta - t_0 \right)^{(\eta-1)(\eta_s,\eta_s)} \right)^{\eta_s}} ds \, d\theta \\
\leq C^* (h, l, \eta_s, \eta_s) \left( \int_{0}^{x} \left( x-s \left[ F^h(s) f(s) \right]^{\eta_s} ds \right)^{\frac{1}{\eta_s}} \right)^{\frac{1}{\eta_s}} \right)^{\frac{1}{\eta_s}}.
\]
Theorem 9. Let \( s, \theta, \) and \( t_0 \) be \( \in \mathbb{R} \), \( f \in C_{\alpha,q} ([t_0, x]_\mathbb{T}, \mathbb{R}^+), \) \( g \in C_{\beta,q} ([t_0, y]_\mathbb{T}, \mathbb{R}^+) \), and \( h(\tau) \) and \( l(\xi) \) be two positive functions defined for \( \tau \in [t_0, x]_\mathbb{T} \) and \( \xi \in [t_0, y]_\mathbb{T} \). Suppose that \( F(s) \) and \( G(\theta) \) are as defined in Theorem 6, and let
\[
H(s) = \int_{t_0}^{s} h(\tau) \Delta \tau,
\]
\[
L(\theta) = \int_{t_0}^{\theta} l(\xi) \Delta \xi.
\]
Then, for \( s \in [t_0, x]_\mathbb{T} \) and \( \theta \in [t_0, y]_\mathbb{T} \), we have
\[
\Phi(F(s)) \Psi(G(\theta)) \leq \Phi(H(s)) \Psi(L(\theta)) \leq \Phi(H(s)) \Psi(L(\theta)).
\]

Proof. According to Theorem 2 and the definition of function \( \Phi \), it is clear that
\[
\Phi(F(s)) = \Phi \left( \frac{H(s)}{H(s)} \int_{t_0}^{s} h(\tau) (f(\tau)/h(\tau)) \Delta \tau \right)
\]
\[
\leq \Phi(H(s)) \Phi \left( \int_{t_0}^{s} h(\tau) (f(\tau)/h(\tau)) \Delta \tau \right)
\]
\[
\leq \Phi(H(s)) \Phi \left( \int_{t_0}^{s} h(\tau) (f(\tau)/h(\tau)) \Delta \tau \right)
\]

By applying Hölder’s inequality (1) on (56), we find that
\[
\Phi(F(s)) \leq \Phi(H(s)) \left( s - t_0 \right)^{(q - 1)/q} \left( \int_{t_0}^{s} h(\tau) \left( f(\tau)/h(\tau) \right) \Delta \tau \right)^{\eta/\eta}. \tag{57}
\]

Analogously,
\[
\Psi(G(\theta)) \leq \Psi(L(\theta)) \left( s - t_0 \right)^{(q - 1)/q} \left( \int_{t_0}^{\theta} l(\xi) \left( g(\xi)/l(\xi) \right) \Delta \xi \right)^{1/q}. \tag{58}
\]

Thus, from (57) and (58), it can be acquired that
\[ \Phi(F(s))\Psi(G(\theta)) \leq (s - t_0)^{(\eta - 1)/\eta}(\theta - t_0)^{(\eta - 1)/\eta_\ast} \left( \frac{\Phi(H(s))}{H(s)} \left( \int_{t_0}^{s} \left( h(\tau)\Phi \left[ \frac{f(\tau)}{h(\tau)} \right] \Delta \tau \right)^{\eta} \Delta \tau \right)^{1/\eta} \right) \]

\[ \times \left( \frac{\Psi(L(\theta))}{L(\theta)} \left( \int_{t_0}^{t} \left( I(\xi)\Psi \left[ \frac{g(\xi)}{I(\xi)} \right] \Delta \xi \right)^{\eta} \Delta \xi \right)^{1/\eta_\ast} \right). \]  

Applying (39) on the term \((s - t_0)^{(\eta - 1)/\eta} \times (\theta - t_0)^{(\eta - 1)/\eta_\ast}\), we get

\[ \Phi(F(s))\Psi(G(\theta)) \leq \frac{\eta\eta_\ast}{\eta + \eta_\ast} \left( \frac{(s - t_0)^{(\eta - 1)/(\eta + \eta_\ast))}}{\eta} + \frac{(t - t_0)^{(\eta - 1)/(\eta + \eta_\ast))}}{\eta_\ast} \right) \]

\[ \times \left( \frac{\Phi(H(s))}{H(s)} \left( \int_{t_0}^{s} \left( h(\tau)\Phi \left[ \frac{f(\tau)}{h(\tau)} \right] \Delta \tau \right)^{\eta} \Delta \tau \right)^{1/\eta} \right) \]

\[ \times \left( \frac{\Psi(L(\theta))}{L(\theta)} \left( \int_{t_0}^{t} \left( I(\xi)\Psi \left[ \frac{g(\xi)}{I(\xi)} \right] \Delta \xi \right)^{\eta} \Delta \xi \right)^{1/\eta_\ast} \right). \]  

From (60), we observe that

\[ \frac{\Phi(F(s))\Psi(G(\theta))}{\eta_\ast (s - t_0)^{(\eta - 1)/(\eta + \eta_\ast))} + \eta(t - t_0)^{(\eta - 1)/(\eta + \eta_\ast))} \]

\[ \leq \frac{1}{\eta + \eta_\ast} \left( \int_{t_0}^{s} \left( h(\tau)\Phi \left[ \frac{f(\tau)}{h(\tau)} \right] \Delta \tau \right)^{\eta} \Delta \tau \right)^{1/\eta} \]

\[ \times \left( \frac{\Psi(L(\theta))}{L(\theta)} \left( \int_{t_0}^{t} \left( I(\xi)\Psi \left[ \frac{g(\xi)}{I(\xi)} \right] \Delta \xi \right)^{\eta} \Delta \xi \right)^{1/\eta_\ast} \right). \]

Integrating the above relation and using Hölder’s inequality (1) again with indices \(\eta, \eta/(\eta - 1)\) and \(\eta_\ast, \eta_\ast/(\eta_\ast - 1)\), we find that

\[ \int_{t_0}^{x} \int_{t_0}^{y} \Phi(F(s))\Psi(G(\theta)) \Delta s \Delta \theta \]

\[ \leq \frac{1}{\eta + \eta_\ast} \left( \int_{t_0}^{x} \left( \frac{\Phi(H(s))}{H(s)} \right)^{(\eta - 1)/\eta} \Delta s \right)^{(\eta - 1)/\eta} \left( \int_{t_0}^{x} \int_{t_0}^{y} \left( h(\tau)\Phi \left[ \frac{f(\tau)}{h(\tau)} \right] \Delta \tau \right)^{\eta} \Delta \tau \Delta s \right)^{1/\eta} \]

\[ \times \left( \int_{t_0}^{y} \left( \frac{\Psi(L(\theta))}{L(\theta)} \right)^{\eta_\ast/(\eta_\ast - 1)} \Delta \theta \right)^{(\eta_\ast - 1)/\eta_\ast} \left( \int_{t_0}^{y} \int_{t_0}^{\theta} \left( I(\xi)\Psi \left[ \frac{g(\xi)}{I(\xi)} \right] \Delta \xi \right)^{\eta} \Delta \xi \Delta \theta \right)^{1/\eta_\ast}. \]
Applying Lemma 4 on (62) and using \( \sigma(n) \geq n \), we get

\[
\int_{t_0}^{x} \int_{t_0}^{y} \frac{\Phi\left( F(s) \right) \Psi\left( G(\theta) \right)}{\eta_s(s - t_0)^{(\eta-1)\eta_s}} \Delta s \Delta \theta
\]

\[
\leq \frac{1}{\eta + \eta_*} \left( \int_{t_0}^{x} \left( \frac{\Phi\left( H(s) \right)}{H(s)} \right)^{\eta / (\eta - 1)} \Delta s \right)^{(\eta - 1) / \eta_*} \left( \int_{t_0}^{y} \left( \frac{\Psi\left( L(\theta) \right)}{L(\theta)} \right)^{\eta_* / (\eta - 1)} \Delta \theta \right)^{(\eta - 1) / \eta_*} 
\]

\[
\times \left( \int_{t_0}^{x} (x - \sigma(s)) \left( h(s) \Phi \left( \frac{f(s)}{h(s)} \right) \right)^{\eta / \eta_s} \Delta s \right)^{1 / \eta_s} \left( \int_{t_0}^{y} (y - \sigma(\theta)) \left( l(\theta) \Psi \left( \frac{g(\theta)}{l(\theta)} \right) \right)^{\eta_* / \eta_s} \Delta \theta \right)^{1 / \eta_s} 
\]

\[
\leq D(\eta, \eta_*) \left( \int_{t_0}^{x} \left( \frac{\Phi\left( H(s) \right)}{H(s)} \right)^{\eta / (\eta - 1)} ds \right)^{(\eta - 1) / \eta_*} \left( \int_{t_0}^{y} \left( \frac{\Psi\left( L(\theta) \right)}{L(\theta)} \right)^{\eta_* / (\eta - 1)} d\theta \right)^{(\eta - 1) / \eta_*}, 
\]

which is (54). This completes the proof. \( \square \)

**Remark 4.** Letting \( 1 / \eta + 1 / \eta_* = 1 \) in (54), then we get Theorem 3.2 due to Saker et al. [11].

By using relations (22) and putting \( \mathbb{T} = \mathbb{R} \) and \( t_0 = 0 \) in Theorem 9, we get the following conclusion.

**Corollary 10.** Assume that \( f(s) \) and \( g(\theta) \) are two nonnegative functions and \( h(s) \) and \( l(\theta) \) are two positive functions, and let

\[
F(s) = \int_{0}^{s} f(\tau) d\tau, \\
G(\theta) = \int_{0}^{\theta} g(\tau) d\tau, \\
H(s) = \int_{0}^{s} h(\tau) d\tau, \\
L(\theta) = \int_{0}^{\theta} l(\tau) d\tau. 
\]

Then, for \( s \in (0, x) \) and \( \theta \in (0, y) \), we have

\[
\int_{0}^{x} \int_{0}^{y} \frac{\Phi\left( F(s) \right) \Psi\left( G(\theta) \right)}{\eta_s(s - t_0)^{(\eta-1)\eta_s}} + \eta / (\eta - 1) \eta_s(s - t_0)^{(\eta-1)\eta_s}) + \eta / (\eta - 1) \eta_s(s - t_0)^{(\eta-1)\eta_s}) + dsdr
\]

\[
\leq D^* \left( \eta, \eta_* \right) \left( \int_{0}^{x} \frac{(x - s)}{\eta_s} \left( h(s) \Phi \left( \frac{f(s)}{h(s)} \right) \right)^{\eta / \eta_s} ds \right)^{1 / \eta_s}
\]

\[
\times \left( \int_{0}^{y} \left( l(\theta) \Psi \left( \frac{g(\theta)}{l(\theta)} \right) \right)^{\eta_* / \eta_s} d\theta \right)^{1 / \eta_s}, 
\]

where

\[
D^* \left( \eta, \eta_* \right) = \frac{1}{\eta + \eta_*} \left( \int_{0}^{x} \left( \frac{\Phi\left( H(s) \right)}{H(s)} \right)^{\eta / (\eta - 1)} ds \right)^{(\eta - 1) / \eta_*}
\]

\[
\times \left( \int_{0}^{y} \left( \frac{\Psi\left( L(\theta) \right)}{L(\theta)} \right)^{\eta_* / (\eta - 1)} d\theta \right)^{(\eta - 1) / \eta_*}. 
\]

**Corollary 11.** Assume that \( \{a_i\} \) and \( \{b_i\} \) are two nonnegative sequences of real numbers and \( \{h_i\} \) and \( \{l_i\} \) are positive sequences, and define

\[
A_i = \sum_{p=1}^{i} a_p, \\
B_i = \sum_{q=1}^{j} b_q, \\
H_i = \sum_{p=1}^{i} h_p, \\
L_i = \sum_{q=1}^{j} l_q. 
\]

Then,
\[
\sum_{i=1}^{k} \sum_{j=1}^{r} \Phi(A_i) \Psi(B_i) \left( \frac{h_i \Phi(a_i)}{h_i} \right)^{\eta/(\eta-1)} \leq D^{**}(\eta, \eta) \left( \sum_{i=1}^{k} (k-i+1) \left( \frac{h_i \Phi(a_i)}{h_i} \right)^{\eta/(\eta-1)} \right)^{1/\eta}
\times \left( \sum_{j=1}^{r} (r-j+1) \left( l_j \Psi(b_j) \right)^{\eta/(\eta-1)} \right)^{1/\eta},
\]

where

\[
D^{**}(\eta, \eta) = \frac{1}{\eta + \eta^*} \left( \sum_{i=1}^{k} \left( \frac{\Phi(H_i)}{H_i} \right)^{\eta/(\eta-1)} \right)^{1/\eta} \cdot \left( \sum_{j=1}^{r} \left( \frac{\Psi(L_j)}{L_j} \right)^{\eta/(\eta-1)} \right)^{1/\eta}.
\]

Remark 5. From inequality (39), we obtain

\[
s_1^2 + s_2^2 \leq \frac{1}{\omega_1 + \omega_2} \left( \omega_1 s_1^\omega_1 + \omega_2 s_2^\omega_2 \right), \quad \text{for } \omega_1 > 0, \omega_2 > 0.
\]

If we apply (70) on (31) in Theorem 6 and (54) in Theorem 9, then we get the following, respectively, inequalities:

\[
\int_{t_0}^{t} \int_{\eta, (s-t_0)}^{(n+1)-(n+1)} \frac{F(s)G(\theta)}{(s-t_0) \eta, (s-t_0) \eta} \Delta s \Delta \theta \leq \frac{\eta \eta D_0(h, l, \eta, \eta)}{\eta + \eta^*} \left\{ \frac{1}{\eta} \left( \int_{t_0}^{t} (\sigma(x) - s) F(s) f(s) \Delta s + \int_{t_0}^{t} (\sigma(y) - \theta) G(s) g(s) \Delta s \right)^{\eta/(\eta-1)} \right\},
\]

where

\[
D_0(h, l, \eta, \eta) = \frac{h l}{\eta + \eta^*} \left( x - t_0 \right)^{\eta/(\eta-1)} y - t_0 \right)^{\eta/(\eta-1)}.
\]

Also,

\[
\int_{t_0}^{t} \int_{\eta, (s-t_0)}^{(n+1)-(n+1)} \frac{\Phi(F(s)) \Psi(G(\theta))}{(s-t_0) \eta, (s-t_0) \eta} \Delta s \Delta \theta \leq \frac{\eta \eta D_2(h, l, \eta, \eta)}{\eta + \eta^*} \left\{ \frac{1}{\eta} \left( \int_{t_0}^{t} (\sigma(x) - s) \Phi(h) \frac{f(s)}{h(s)} \Delta s + \int_{t_0}^{t} (\sigma(y) - \theta) \Psi(g) \Delta \theta \right)^{\eta/(\eta-1)} \right\},
\]

where

\[
\Phi(F(s)) = \Phi \left( \frac{1}{s-t_0} \int_{t_0}^{t} f(\xi) \Delta \xi \right)
\]

By applying inequality (1) on (78) with indices \( \eta, \eta/(\eta-1) \), we have

\[
\Phi(F(s)) \leq \frac{1}{s-t_0} \left( s-t_0 \right)^{\eta/(\eta-1)} \left( \int_{t_0}^{t} \Phi(f(\xi)) \Delta \xi \right)^{1/\eta}.
\]

This implies that

\[
\Phi(F(s)) = \Phi \left( \frac{1}{s-t_0} \int_{t_0}^{t} f(\xi) \Delta \xi \right)
\]

Theorem 12. Let \( s, \theta, \text{ and } t_0 \in \mathbb{T}, f \in C_{rd}([t_0, x], \mathbb{R}^+), \) and \( g \in C_{rd}([t_0, y], \mathbb{R}^+). \) Define

\[
F(s) = \frac{1}{s-t_0} \int_{t_0}^{t} f(\xi) \Delta \xi,
\]

\[
G(\theta) = \frac{1}{\theta-t_0} \int_{t_0}^{\theta} g(\xi) \Delta \xi.
\]

Then, for \( s \in [t_0, x], \) and \( \theta \in [t_0, y], \) we have

\[
\int_{t_0}^{t} \int_{\eta, (s-t_0)}^{(n+1)-(n+1)} \frac{\Phi(F(s)) \Psi(G(\theta))}{(s-t_0) \eta, (s-t_0) \eta} \Delta s \Delta \theta \leq \frac{\eta \eta D_0(h, l, \eta, \eta)}{\eta + \eta^*} \left\{ \frac{1}{\eta} \left( \int_{t_0}^{t} (\sigma(x) - s) \Phi(h) \frac{f(s)}{h(s)} \Delta s + \int_{t_0}^{t} (\sigma(y) - \theta) \Psi(g) \Delta \theta \right)^{\eta/(\eta-1)} \right\},
\]

where

\[
E(\eta, \eta) = \frac{1}{\eta + \eta^*} \left( x - t_0 \right)^{\eta/(\eta-1)} y - t_0 \right)^{\eta/(\eta-1)}.
\]

Proof. By assumption and using Jensen’s inequality (26), we see that

\[
\Phi(F(s)) = \Phi \left( \frac{1}{s-t_0} \int_{t_0}^{t} f(\xi) \Delta \xi \right)
\]

By applying inequality (1) on (78) with indices \( \eta, \eta/(\eta-1) \), we have

\[
\Phi(F(s)) \leq \frac{1}{s-t_0} \left( s-t_0 \right)^{\eta/(\eta-1)} \left( \int_{t_0}^{t} \Phi(f(\xi)) \Delta \xi \right)^{1/\eta}.
\]
\[
\Phi(F(s))(s-t_0) \leq (s-t_0)^{(\eta-1)/\eta} \left( \int_{t_0}^{s} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(80)

Analogously,
\[
\Psi(G(\theta))(\theta-t_0) \leq (\theta-t_0)^{(\eta-1)/\eta} \left( \int_{t_0}^{\theta} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(81)

From (80) and (81), we get
\[
\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)
\leq (s-t_0)^{(\eta-1)/\eta} (\theta-t_0)^{(\eta-1)/\eta} \left( \int_{t_0}^{s} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{\theta} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(82)

Applying elementary inequality (39) on the term \((s-t_0)^{(\eta-1)/\eta} \times (\theta-t_0)^{(\eta-1)/\eta} \), where \(s_1 = (s-t_0)^{(\eta-1)/\eta} \), \(s_2 = (t-t_0)^{(\eta-1)/\eta} \), \(\omega_1 = 1/\eta \), \(\omega_2 = 1/\eta \), and \(r = \omega_1 + \omega_2 \), we get
\[
\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)
\leq \frac{\eta_1}{\eta + \eta_1} \left( \int_{t_0}^{s} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{\theta} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(83)

From (83), we have
\[
\frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_1 (s-t_0)^{(\eta-1)/\eta} + \eta_1 (\theta-t_0)^{(\eta-1)/\eta}} \leq \frac{1}{\eta + \eta_1} \left( \int_{t_0}^{s} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{\theta} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(84)

Taking delta integrating on both sides of (84), first over \(s\) from \(t_0\) to \(x\) and then over \(\theta\) from \(t_0\) to \(y\), we find that
\[
\int_{t_0}^{x} \int_{t_0}^{y} \frac{\Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)}{\eta_1 (s-t_0)^{(\eta-1)/\eta} + \eta_1 (\theta-t_0)^{(\eta-1)/\eta}} \Delta s \Delta \theta \leq \frac{1}{\eta + \eta_1} \left( \int_{t_0}^{x} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(85)

By applying inequality (1) on (85) with indices \(\eta, \eta/(\eta-1)\) and \(\eta_1, \eta_1/(\eta_1-1)\), we get
\[
\int_{t_0}^{x} \int_{t_0}^{y} \Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)
\leq \frac{1}{\eta + \eta_1} \left( \int_{t_0}^{x} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(86)

Applying Lemma 4 on (86) and using the fact \(\sigma(\eta) \geq \eta\), we find that
\[
\int_{t_0}^{x} \int_{t_0}^{y} \Phi(F(s))\Psi(G(\theta))(s-t_0)(\theta-t_0)
\leq \frac{1}{\eta + \eta_1} \left( \int_{t_0}^{x} (\Phi[f(x)])^\eta \Delta x \right)^{1/\eta} \times \left( \int_{t_0}^{y} (\Psi[g(x)])^\eta \Delta x \right)^{1/\eta}.
\]

(87)
The proof is complete.

Remark 7. Letting $1/\eta + 1/\eta_* = 1$ in (76), then we get Theorem 3.3 due to Saker et al. [11].

By using relations (22) and putting $T = \mathbb{R}$ and $t_0 = 0$ in Theorem 12, we get the following conclusion.

**Corollary 13.** Assume that $f(s)$ and $g(t)$ are nonnegative functions, and define

\[ F(s) = \frac{1}{s} \int_0^s f(\xi) d\xi, \quad G(\theta) = \frac{1}{\theta} \int_0^\theta g(\xi) d\xi. \]

Then, for $s \in (0, x)$ and $\theta \in (0, y)$, we have

\[
\int_0^y \int_0^{s(t)} s \Phi(F(s)) \Psi(G(\theta)) ds d\theta \\
\leq E^*(\eta, \eta_*) \left( \int_0^x (x - s) (\Phi[f(s)])^\eta ds \right)^{1/\eta} \\
\times \left( \int_0^y (y - \theta) (\Psi[g(\theta)])^\eta d\theta \right)^{1/\eta},
\]

where

\[
E^*(\eta, \eta_*) = \frac{1}{\eta + \eta_*} (x)^{(\eta - 1)/\eta} (y)^{(\eta - 1)/\eta},
\]

which is the same inequality in [9], Theorem 3.3.

By using relations (23) and putting $T = \mathbb{Z}$ and $t_0 = 0$ in Theorem 12, we get the following conclusion.

**Corollary 14.** Assume that $\{a_i\}$ and $\{b_j\}$ are two nonnegative sequences of real numbers, and define

\[ A_i = \frac{1}{i} \sum_{p=1}^i a_p, \quad B_j = \frac{1}{j} \sum_{q=1}^j b_q. \]

Then,

\[
\sum_{i=1}^k \sum_{j=1}^r i j \Phi(A_m) \Psi(B_n) \\
\leq E^{**}(\eta, \eta_*) \left( \sum_{i=1}^k (k - i + 1) (\Phi(a_i))^\eta \right)^{1/\eta} \\
\times \left( \sum_{j=1}^r (r - j + 1) (\Psi(b_j))^\eta \right)^{1/\eta},
\]

where

\[ E^{**}(\eta, \eta_*) = \frac{1}{\eta + \eta_*} (k)^{(\eta - 1)/\eta} (r)^{(\eta - 1)/\eta}. \]

which is the same inequality in [9], Theorem 2.3.

**Theorem 15.** Let $s, \theta$, and $t_0 \in T$, $f \in C_{rd}([t_0, x]_T, \mathbb{R}^+)$, $g \in C_{rd}([t_0, y]_T, \mathbb{R}^+)$, and $h(\xi)$ and $l(\xi)$ be two positive functions defined for $\xi \in [t_0, x]_T$ and $\xi \in [t_0, y]_T$ and $H$ and $L$ be as defined in Theorem 9, and let

\[
F(s) = \frac{1}{H(s)} \int_{t_0}^s h(\xi) f(\xi) d\xi, \quad G(\theta) = \frac{1}{L(\theta)} \int_{t_0}^\theta l(\xi) g(\xi) d\xi.
\]

Then, for $s \in [t_0, y]_T$ and $\theta \in [t_0, x]_T$, we have

\[
\int_{t_0}^y \int_{t_0}^{s(t)} \Phi(F(s)) \Psi(G(\theta)) H(s) L(\theta) ds d\theta \\
\leq W(\eta, \eta_*)(s - t_0)^{(\eta - 1)/\eta} (y - t_0)^{(\eta - 1)/\eta},
\]

where

\[ W(\eta, \eta_*) = \frac{1}{\eta + \eta_*} (x - t_0)^{(\eta - 1)/\eta} (y - t_0)^{(\eta - 1)/\eta}. \]

**Proof.** Using the hypotheses of Theorem 15 and Jensen’s inequality, we find that

\[
\Phi(F(s)) \leq \frac{1}{H(s)} \left( \int_{t_0}^s h(\xi) f(\xi) d\xi \right)^{1/\eta}.
\]

From (98), we get

\[
\Phi(F(s)) H(s) \leq (s - t_0)^{(\eta - 1)/\eta} \left( \int_{t_0}^s h(\xi) \Phi(f(\xi)) d\xi \right)^{1/\eta}.
\]

Analogously,

\[
\Psi(G(\theta)) L(\theta) \leq \left( \int_{t_0}^\theta l(\xi) \Psi(g(\xi)) d\xi \right)^{1/\eta}.
\]

From (99) and (100), we find that
Applying Lemma 4 and using $\sigma(n) \geq n$, we get

$$
\int_{t_0}^{x} \int_{t_0}^{y} \Phi(F(s))\Psi(G(\theta))H(s)\mathcal{L}(\theta) \, ds \, d\theta.
$$

This completes the proof.

Remark 8. Letting $1/\alpha + 1/\beta = 1$ in (95), then we get Theorem 3.4 due to Saker et al. [11].

By using relations (22) and putting $T = \mathbb{R}$ and $t_0 = 0$ in Theorem 15, we get the following conclusion.

**Corollary 16.** Assume that $f(s)$ and $g(\theta)$ are two nonnegative functions and $h(s)$ and $l(\theta)$ are two positive functions, and define

$$
H(s) = \int_{0}^{s} h(\xi) d\xi,
$$

$$
L(\theta) = \int_{0}^{\theta} l(\xi) d\xi,
$$

$$
F(s) = \frac{1}{H(s)} \int_{0}^{s} f(\xi) h(\xi) d\xi,
$$

$$
G(\theta) = \frac{1}{L(\theta)} \int_{0}^{\theta} g(\xi) l(\xi) d\xi.
$$

Then, for $s \in (0, x)$ and $\theta \in (0, y)$, we have

$$
\int_{0}^{x} \int_{0}^{y} \Phi(F(s))\Psi(G(\theta))H(s)\mathcal{L}(\theta) \, ds \, d\theta.
$$

where

$$
W^*(\eta, \eta_0) = \frac{1}{\eta + \eta_0}(\alpha(\eta/\alpha) \beta(\eta/\beta)).
$$

It is clear that it is the same inequality in [9], Theorem 3.4.

By using relations (23) and putting $T = \mathbb{Z}$ and $t_0 = 0$ in Theorem 15, we have the following conclusion.

**Corollary 17.** Assume that $|a_j|$ and $|b_j|$ are nonnegative sequences and $|h_j|$ and $|l_j|$ are positive sequences, and define
\[ H_i = \sum_{p=1}^{j} h_p, \]
\[ L_j = \sum_{q=1}^{j} l_q, \]
\[ A_j = \frac{1}{H_i} \sum_{p=1}^{i} h_p \alpha_p, \]
\[ B_j = \frac{1}{L_j} \sum_{q=1}^{j} l_q b_q. \]

Then,
\[ \sum_{i=1}^{k} \sum_{j=1}^{r} \frac{H_i L_j \Phi(A_i) \Psi(B_j)}{\eta_j(i)(\eta^{-1})(\eta^{-1})(\eta)} + \eta(j)(\eta^{-1})(\eta^{-1})/\eta_i, \]
\[ \leq W^{**}(\eta, \eta_i) \left( \sum_{i=1}^{k} (k - i + 1)[h_i \Phi(a_i)]^{\eta_i} \right)^{1/\eta_i}, \]
\[ \times \left( \sum_{j=1}^{r} (r - j + 1)[l_j \Psi(b_j)]^{\eta_j} \right)^{1/\eta_j}, \]

where
\[ W^{**}(\eta, \eta_i) = \frac{1}{\eta + \eta_i} (k)^{(q-1)/\eta} (r)^{(q-1)/\eta}, \]

which is the same inequality in [9], Theorem 2.4.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**