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*Research Article*

# Global Existence and Decay for a System of Two Singular Nonlinear Viscoelastic Equations with General Source and Localized Frictional Damping Terms

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The current paper deals with the proof of a global solution of a viscoelasticity singular one-dimensional system with localized frictional damping and general source terms, taking into consideration nonlocal boundary condition. Moreover, similar to that in Boulaaras' recent studies by constructing a Lyapunov functional and use it together with the perturbed energy method in order to prove a general decay result.

## 1. Introduction

The evolution problem with integral conditions is related with many branches of sciences ([1–6]). Cause of this, interest in it occurs naturally in inflation cosmology, nuclear physics, supersymmetric field theories, and quantum mechanics (see for example [2, 7]). Later, by the motivation of this work, some authors gave necessary and sufficient conditions for the hyperbolic equation with source term (see, e.g., [8–10]).

This manuscript is devoted to the study of the global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds + \mu(x)u_t = f_1(u,v), \text{ in } (0,L)\times(0,T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds + \mu(x)v_t = f_2(u,v), \text{ in } (0,L)\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in (0,L), \\ u(L,t) = v(L,t) = 0, \int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0, \end{cases} \tag{1}$$

where $L < \infty$, $T < \infty$, $\mu \in C^1((0,\alpha))$, $g_1(.), g_2(.)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $f_1(.,.), f_2(.,.)$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$ are given functions, which will be specified later.

The problems related with localized frictional damping have extensively studied by many teams as [11], where the authors obtained an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation:

$$\begin{aligned} u_{tt} - \Delta u + f(x,t,u) + \int_0^t g_1(t-s)\Delta u(s)ds + a(x)u_t \\ = 0, \text{ in } (0,L)\times(0,T), \end{aligned} \tag{2}$$

for damping term $a(x)u_t$ may be null for some part of the domain.

We used the techniques in [11]; we have proven in [8] the existence of a global solution using the potential well theory for the following viscoelastic system with nonlocal boundary condition and localized frictional damping

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds + a(x)u_t = |v|^{q+1}|u|^{p-1}u, \text{ in } (0,L)\times(0,T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds + a(x)v_t = |v|^{q+1}|u|^{p-1}v, \text{ in } (0,L)\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,\alpha), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0, \int_0^\alpha xu(x,t)dx = \int_0^\alpha xv(x,t)dx = 0. \end{cases} \tag{3}$$

Very recently, in ([9]), we study the following singular one-dimensional nonlinear equations that arise in generalized viscoelasticity with long-term memory:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds = f_1(u,v), \text{ in } (0,L)\times(0,T), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds = f_2(u,v), \text{ in } (0,L)\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in (0,L), \\ u(L,t) = v(L,t) = 0, \int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0. \end{cases} \tag{4}$$

In view of the articles mentioned above in ([8, 9, 11]), much less effort has been devoted to the existence of a global solution to the system of two singular nonlinear equations which arise in generalized viscoelasticity with localized frictional damping terms using the potential-well theory. Moreover, we prove a general decay result by constructing a Lyapunov functional and use it together with the perturbed energy method.

The structure of the work is as follows: To facilitate the description, firstly in Section 2, we give the fundamental definitions and theorems on function spaces that will be needed in the body of the paper and state the local existence theorem. In Section 3, the energy function $E(t)$ is defined and proved to be a nonincreasing function of time. Finally, the main result is obtained, which gives the general decay conditions:

$$g_i'(t) \leq -\xi(t) g_i^r(t), i = 1, 2. \tag{5}$$

## 2. Preliminaries

Let $L_x^p = L_x^p((0, L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left( \int_0^L x|u|^p dx \right)^{1/p}, \tag{6}$$

when $p = 2$, we get a Hilbert space, and we denote by $H = L_x^2$, it provided with the finite norm

$$\|u\|_H = \left( \int_0^L xu^2 dx \right)^{1/2}. \tag{7}$$

$V = V_x^1((0, L))$ be the Hilbert space equipped with the norm

$$\|u\|_V = \left( \|u\|_H^2 + \|u_x\|_H^2 \right)^{1/2}. \tag{8}$$

We get the following lemma by combining the Poincare inequality and (see [8]).

**Lemma 1.** *Let $V_0$ space defined as follows*

$$V_0 = \{u \in V \quad \text{such that} \quad u(L) = 0\}. \tag{9}$$

*Then, for $2 \leq p 4$, we have*

$$\int_0^L x|v|^p dx \leq C_* \|v_x\|_{H=L_x^2(0,L)}^p, \forall u \in V_0, \tag{10}$$

*where $C_*$ is a constant depending on L and p only and for $p = 2$, $C_* = C_p$ is the Poincare constant.*

*Remark 2.* It is clear that $\|u\|_{V0} = \|u_x\|_H$ defines an equivalent norm on $V_0$.

The next theorem confirms that our problem has a local solution under some condition on $p$ and the relaxation functions $g_i$, the proof can be established by following the argument of [12].

**Theorem 3.** *We take $(u_0, v_0) \in V_0^2$ and $(v_1, v_2) \in H^2$. If $p < 3$ and*

$$g_i(0) > 0, \left( 1 - \int_0^\infty g_i(s) ds \right) = l > 0, \text{for } i = 1, 2, \tag{11}$$

*then, there exists $t* > 0$ small enough such that the problem (1) has a unique local solution*

$$u \in C(0, t_*; V_0) \cap C^1(0, t_*; H). \tag{12}$$

*Remark 4.* The condition on $p$ is needed so that the embedding of $V_0$ in $L_x^2$ is Lipchitz.

We need the following assumptions to get our results.

$(\boldsymbol{G}_1)$ For $i = 1, 2$, $g_i(t)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nonincreasing $C_2$ function such that

$$\begin{cases} g_i(s) \geq 0, g_i'(s) \leq 0 \text{ and,} \\ g_i(0) \geq 0, 1 - \int_0^\infty g_i(s) ds = l_1 > 0, \end{cases} \tag{13}$$

and

$(\boldsymbol{G}_2)$

$$g_i'(t) \leq -\xi(t) g_i^\sigma(t), \quad t \geq 0, 1 \leq \sigma < \frac{3}{2}, \tag{14}$$

where $\xi(t)$ is a positive differentiable function. It satisfies for some positive constant $l$

$$\left| \frac{\xi'(t)}{\xi(t)} \right| \leq l, \xi'(t) \leq 0, \quad \int_0^\infty \xi(s) ds = +\infty, \forall t > 0. \tag{15}$$

Furthermore, for any $t_0 > 0$ and $1 < \sigma < 3/2$, there exists a positive constant $C_\sigma$ such that

$$\frac{t}{\left( 1 + \int_{t_0}^t \xi(s) ds \right)^{1/2(\sigma-1)}} \leq C_\sigma, \forall t \geq t_0. \tag{16}$$

$(\boldsymbol{G}_3)$ We take

$$\begin{aligned} f_1(u, v) &= a|u + v|^{2(r+1)}(u + v) + b|u|^r u|v|^{r+2}, \\ f_2(u, v) &= a|u + v|^{2(r+1)}(u + v) + b|v|^r v|u|^{r+2}, \end{aligned} \tag{17}$$

where $a, b > 0$ are constants and $r > -1$.

One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = 2(r + 2)F(u, v), \forall (u, v) \in \mathbb{R}^2, \tag{18}$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \quad (19)$$

$(\mathbf{G}_4)$ $\mu \geq 0, \mu > 0$ in $(L_0, L]$, where $0 \leq L_0 \leq L$.

**Lemma 5.** *For $r > -1$, there exist $\eta > 0$ such that for any $u, v \in V \cap V_0(0, L)$, we have*

$$\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)} \leq \eta\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right)^{r+2}. \quad (20)$$

*Proof.* From Minkowski inequality, we have

$$\|u+v\|_{L_x^{2(r+2)}}^2 \leq 2\left(\|u\|_{L_x^{2(r+2)}}^2 + \|v\|_{L_x^{2(r+2)}}^2\right). \quad (21)$$

We apply successively Holder's and Young's inequalities we obtain

$$\|uv\|_{L_x^{(r+2)}}^{(r+2)} \leq \|u\|_{L_x^{2(r+2)}}\|v\|_{L_x^{2(r+2)}} \leq c\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right). \quad (22)$$

We combine the two previous inequalities and the embedding $V \cap V_0(0,L), \hookrightarrow L_x^{2(r+2)}(0,L)$, we get (20).

**Lemma 6.** *There exist two positive constants $\Lambda_1$ and $\Lambda_2$ such that*

$$x|f_i(u,v)|^2dx \leq \Lambda_i\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)^{2r+3}, \forall x \in (0,L), i = 1,2. \quad (23)$$

*Proof.* It is clear that

$$\begin{aligned}|f_1(u,v)| &\leq C\left(|u+v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right)\\ &\leq C\left[|u|^{2r+3}|v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right],\end{aligned} \quad (24)$$

Applying Young's inequality with exponents $q = (2r+3)/(r+1)$, $q' = (2r+3)/(r+2)$, in the last term in the above inequality, we get

$$|f_1(u,v)| \leq C\left[|u|^{2r+3} + |v|^{2r+3}\right]. \quad (25)$$

Consequently, by using Poincaré's inequality and (20), we obtain

$$\begin{aligned}\int_0^L x|f_i(u,v)|^2dx &\leq C\left(\|u_x\|_H^{2(2r+3)} + \|v_x\|_H^{2(2r+3)}\right)\\ &\leq \Lambda_1\left(l_1\|u_x\|_H^2 + l_2\|u_x\|_H^2\right)^{(2r+3)}.\end{aligned} \quad (26)$$

Similarly, we get the inequality for $f_2$. The proof is completed.

We define the energy function as

$$\begin{aligned}E(t) = &\frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx\\ &+ \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xv_x^2(x,t)dx - \int_0^L F(u,v)dx\\ &+ \frac{1}{2}(g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ u_x)(t),\end{aligned} \quad (27)$$

where

$$(g_2 \circ u_x)(t) = \int_0^L\int_0^t xg(t-s)|u_x(x,t) - u_x(x,s)|^2dsdx. \quad (28)$$

**Lemma 7.** *Let $(u,v)$ be the solution of system (1) then for all $t \geq 0$*

$$\begin{aligned}\frac{d}{dt}[E(t)] = &-\int_0^L x\mu(x)u_t^2dx - \int_0^L x\mu(x)v_t^2dx + \frac{1}{2}\left(g' \circ u_x\right)(t)\\ &- \frac{1}{2}g_1(t)\int_0^L xu_x^2 + \frac{1}{2}\left(g' \circ v_x\right)(t) - \frac{1}{2}g_2\int_0^L xv_x^2dx.\end{aligned} \quad (29)$$

*Hence, $E(t)$ is a nonincreasing function.*

*Proof.* Multiplying the first and the second equations in (1) by $xu_t$ and $xv_t$, respectively, integrating over $(0,L)$, summing up, we obtain (30)

$$\begin{aligned}&\int_0^L xu_{tt}u_tdx - \int_0^L (xu_x)_x u_tdx + \int_0^L\int_0^t g_1(t-s)(xu_x(x,s))_x dsu_tdx\\ &+ \int_0^L xv_{tt}v_tdx - \int_0^L (xv)_x v_t + \int_o^L\int_0^t g_2(t-s)(xv_x(x,s))_x dsv_tdx\\ &= -\int_0^L x\mu(x)u_t^2dx - \int_0^L x\mu(x)v_t^2 x\mu(x)v_t^2dx\\ &+ \int_0^L\left[a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}\right]xu_tdx\\ &+ \int_0^L\left[a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}\right]xv_tdx.\end{aligned} \quad (30)$$

By integration by parts, we obtain

$$\int_0^L xu_{tt}u_tdx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_t^2dx\right], \quad (31)$$

$$\int_0^L xv_{tt}v_tdx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_t^2dx\right], \quad (32)$$

$$-\int_0^L (xu_x)_x u_tdx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_x^2dx\right], \quad (33)$$

$$-\int_0^L (xv_x)_x v_t dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_x^2 dx\right], \quad (34)$$

$$\frac{1}{2(r+2)}\int_0^L xf_1(u,v)uu_t dx + \frac{1}{2(r+t)}\int_0^L xf_2(u,v)vv_t dx = \frac{1}{2(r+2)}\frac{d}{dt}\int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]xdx, \quad (35)$$

$$\int_0^L\int_0^t g_1(t-s)(xu_x(s))_x ds u_t(t) dx = \frac{1}{2}\frac{d}{dt}\left[(g_1 \circ u_x)(t) - \int_0^t g_1(s)ds\int_0^L xu_x^2 dx\right] - \frac{1}{2}\left(g'_1 \circ u_x\right)(t) + \frac{1}{2}g_1(t)\int_0^L xu_x^2 dx, \quad (36)$$

$$\int_0^L\int_0^t g_2(t-s)(xv_x(s))_x ds v_t(t) dx = \frac{1}{2}\frac{d}{dt}\left[(g_2 \circ u_x)(t) - \int_0^t g_2(t)ds\int_0^L xv_x^2 dx\right] - \frac{1}{2}\left(g'_2 \circ u_x\right)(t) + \frac{1}{2}g_2(t)\int_0^L xv_x^2 dx, \quad (37)$$

Combining (32)–(2.22) in (31), we get (30).

## 3. Global Existence

In order to state and prove the global existence, we set the following notation

$$\begin{aligned} I(t) &:= I(u(t),v(t)) = \left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx \\ &+ \left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx + (g_1 \circ u_x)(t) \\ &+ (g_2 \circ v_x)(t) - 2(r+2)\int_0^L x\left[a|u+v|^{2(r+2)} \right. \\ &\left. + 2b|uv|^{r+2}\right]dx, \end{aligned} \quad (38)$$

$$\begin{aligned} J(t) &:= J(u(t),v(t)) = \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx \\ &+ \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx + \frac{1}{2}(g_1 \circ u_x)(t) \\ &+ \frac{1}{2}(g_2 \circ v_x)(t) - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx, \end{aligned} \quad (39)$$

we remark that

$$E(t) = J(t)\frac{1}{2}\int_0^L xu_t^2 dx + \int_0^L xv_t^2 dx. \quad (40)$$

**Lemma 8.** *Assume that ($\mathbf{G}_1$), ($\mathbf{G}_2$), and (20) hold also for any $(u_0, v_0) \in V_0^2$ and $(u_1, v_1) \in H_2$ satisfying*

$$I(0) > 0, \beta := \eta\left(\frac{2(r+2)}{(r+1)}E(0)\right)^{r+1} < 1, \quad (41)$$

*where*

$$E(0) = J(0) + \frac{1}{2}\int_0^L xu_1^2 dx + \frac{1}{2}\int_0^L xv_1^2 dx. \quad (42)$$

*Then, there exists $t_* > 0$ such that*

$$I(t) > 0, \forall t \in [0, t_*). \quad (43)$$

*Proof.* Since $I(0) > 0$, then from the continuity of $I(t)$, there exist $t_m \le t_*$ such that $I(t) \ge 0$ for all $t \in [0, t_m)$; this implies that we have a maximum time value noting $T_m$ such that

$$\{I(T_m) = 0 \quad \text{and} \quad I(t) > 0, \quad \text{for all} \quad 0 \le t < T_m\}. \quad (44)$$

From formulas of $J(t)$ and $I(t)$ together with ($\mathbf{G}_1$), we have

$$\begin{aligned} J(t) &= \frac{r+1}{2(r+2)}\left[\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 ds + \left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx\right] \\ &+ \frac{r+1}{2(r+2)}\left[(g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ v_x)(t)\right] \\ &+ \frac{1}{2(r+2)}I(t) \ge \frac{r+1}{2(r+2)}\left[\left(l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx\right) + (g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ v_x)(t)\right], \end{aligned} \quad (45)$$

hence,

$$\begin{aligned} l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx &\le \frac{2(r+2)}{r+1}J(t) \le \frac{2(r+2)}{r+1}E(t) \\ &\le \frac{2(r+2)}{r+1}E(0), \forall t \in [0, T_m), \end{aligned} \quad (46)$$

Recalling Lemma 5 and (41), we get

$$
\begin{aligned}
&2(r+2)\int_0^L F(u(T_m),v(T_m))dx \\
&\leq \eta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)^{r+2} \\
&\leq \eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right) \\
&= \beta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right) \\
&< \left(1-\int_0^t g_1(s)ds\right)\left(\int_0^L xu_x^2dx\right) \\
&\quad + \left(1-\int_0^t g_2(s)ds\right)\left(\int_0^L xv_x^2dx\right) \\
&\quad + (g_1\circ u_x)(t) + (g_2\circ v_x)(t),
\end{aligned}
\tag{47}
$$

consequently

$$
\begin{aligned}
&\left(1-\int_0^t g_1(s)ds\right)\left(\int_0^L xu_x^2dx\right) \\
&\quad + \left(1-\int_0^t g_2(s)ds\right)\left(\int_0^L xv_x^2dx\right) + (g_1\circ u_x)(t) \\
&\quad + (g_2\circ v_x)(t) - 2(r+2)\int_0^L xF(u,v)dx > 0,
\end{aligned}
\tag{48}
$$

we deduce that $I(t) > 0, \forall t \in [0, T_m)$. By repeating the procedure, $T_m$ is extended to $t_*$.

**Theorem 9.** *Suppose that* $(\mathbf{G}_1)$, $(\mathbf{G}_2)$, *and (20) hold. Then, for any* $(u_0, v_0) \in V_0^2$ *and* $(u_1, v_1) \in H^2$ *satisfying (41), the solution of system (1) is a bounded and globally in time.*

*Proof.* To achieve the proof of this theorem, it suffices to show that $\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2$ is bounded independently of $t$. As $E(t)$ is a nonincreasing function, we have

$$
E(0) \geq E(t), \tag{49}
$$

in the other hand and for the definition of $I(t)$, we have

$$
\begin{aligned}
&x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\
&\quad = I(t) - \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx \\
&\quad - \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx - (g_1\circ u_x)(t) \\
&\quad - (g_2\circ v_x)(t),
\end{aligned}
\tag{50}
$$

we introduce (49) into (50), we get

$$
\begin{aligned}
E(0) \geq E(t) &= \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx \\
&\quad + \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx \\
&\quad + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2(x,t)dx \\
&\quad + \frac{1}{2}(g_1\circ u_x)(t) + \frac{1}{2}(g_2\circ v_x)(t) + I(t),
\end{aligned}
\tag{51}
$$

by using (14), (15), and (41), (51) yields

$$
\begin{aligned}
E(0) \geq E(t) &\geq \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx \\
&\quad + \left(\frac{r+1}{2(r+2)}\right)l_1\int_0^L xu_x^2dx + \left(\frac{r+1}{2(r+2)}\right)l_2\int_0^L xv_x^2dx \\
&\geq \mu_0\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int^L xu_x^2dx + \int_0^L xv_x^2dx\right).
\end{aligned}
\tag{52}
$$

So

$$
\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 \leq \tau E(0), \tag{53}
$$

where

$$
\tau := \max\left\{2, \frac{2(r+2)}{(r+1)l_1}, \frac{2(r+2)}{(r+1)l_2}\right\}. \tag{54}
$$

The proof is completed.

## 4. Decay of Solutions

Throughout this section, we will study the asymptotic behavior of solutions' decay by constructing a suitable Lyapunov function; to do so, for $N$, $\varepsilon_1$, and $\varepsilon_2$ are positive constants, we define the following function as

$$
F(t) := E(t) + \varepsilon_1\Phi(t) + \varepsilon_2\chi(t) + \psi(t), \tag{55}
$$

where

$$
\begin{aligned}
\Phi(t) &:= \xi(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xv_t v dx \\
&\quad + \frac{\xi(t)}{2}\int_0^L x\mu(x)\left(u^2+v^2\right)dx,
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
x(t) &:= -\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&\quad - \xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx,
\end{aligned}
\tag{57}
$$

and

$$\psi(t) := \xi(t)\int_0^L x u_t h(x) u_x dx + \xi(t)\int_0^L x v_t h(x) v_x dx, \quad (58)$$

with $h \in C^1([0,L]), h(0) = h(L) = 0, (xh(x))' \le x$,

In the first step, we prove the equivalence between $F(t)$ and $E(t)$ given in the following lemma.

**Lemma 10.** *For a choice of $\varepsilon_1$ and $\varepsilon_2$ small enough, we find two positive constants $\alpha_1$ and $\alpha_2$ such that*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t). \quad (59)$$

*Proof.* By using Young inequality, follow by recalling Lemma 1 and the fact that $0 < \xi(t) \le \xi(0)$, we get

$$\left|\varepsilon_1 \xi(t)\int_0^L x u_t u dx\right| \le \frac{\varepsilon_1}{2}\xi(0)\left(\int_0^L x u_t^2 dx + C_p\int_0^L x u_x^2 dx\right), \quad (60)$$

$$\left|\varepsilon_1 \xi(t)\int_0^L x v_t v dx\right| \le \frac{\varepsilon_1}{2}\xi(0)\left(\int_0^L x v_t^2 dx + C_p\int_0^L x v_x^2 dx\right), \quad (61)$$

$$\begin{aligned}&\left|-\varepsilon_2\xi(t)\int_0^L x u_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx\right| \\ &\quad\le \frac{\varepsilon_2}{2}\xi(0)\left(\int_0^L x u_t^2 dx + C_p(1-l_1)(g_1\circ u_x)(t)\right),\end{aligned} \quad (62)$$

$$\begin{aligned}&\left|-\varepsilon_2\xi(t)\int_0^L x v_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx\right| \\ &\quad\le \frac{\varepsilon_2}{2}\xi(0)\left(\int_0^L x v_x^2 dx + C_p(1-l_2)(g_2\circ v_x)(t)\right),\end{aligned} \quad (63)$$

$$\begin{aligned}&\left|\xi(t)\int_0^L x u_t h(x) u_x dx\right| \\ &\quad\le \frac{\xi(0)}{2}\|h\|\infty\left(\int_0^L x u_x^2 dx + \int_0^L x u_t^2 dx\right),\end{aligned} \quad (64)$$

$$\begin{aligned}&\left|\xi(t)\int_0^L x v_t h(x) v_x dx\right| \\ &\quad\le \frac{\xi(0)}{2}\|h\|\infty\left(\int_0^L x v_x^2 dx + \int_0^L x v_t^2 dx\right),\end{aligned} \quad (65)$$

$$\begin{aligned}&\left|\frac{\xi(t)}{2}\int_0^L x\mu(x)\left(u^2+v^2\right)dx\right| \\ &\quad\le \frac{\xi(0)}{2}\|\mu\|_\infty C_p\left(\int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx\right),\end{aligned} \quad (66)$$

combining (60)–(66) in (55), we get

$$\begin{aligned}&|F(t) - NE(t)| \\ &\le \left(\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0) + \frac{\xi(0)}{2}\|h\|_\infty\right)\int_0^L x u_t^2 dx \\ &+ \left(\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0) + \frac{\xi(0)}{2}\|h\|_\infty\right)\int_0^L x v_t^2 dx \\ &+ \left(\frac{1+\|\mu\|_\infty}{2}\varepsilon_1 C_p\xi(0) + \frac{\xi(0)}{2}\|h\|_\infty(p+1)\right)\int_0^L x u_x^2 dx \\ &+ \left(\frac{1+\|\mu\|_\infty}{2}\varepsilon_1 C_p\xi(0) + \frac{\xi(0)}{2}\|h\|_\infty(q+1)\right)\int_0^L x v_x^2 dx \\ &+ \frac{\varepsilon_2}{2}C_p\xi(0)((1-l_1)(g_1\circ u_x)(t) + (1-l_2)(g_2\circ v_x)(t)).\end{aligned} \quad (67)$$

If we choose $\varepsilon_1$, $\varepsilon_2$ small enough, and $N$ large enough we find $\alpha_1, \alpha_2 > 0$ such that (59) holds true.

Now, we state a Lemma corresponding to the boundness of $(\mathrm{go}v_x)(t)$. It will be used in the calculus.

**Lemma 11.** *Let $w \in L^\infty((0,T);H)$ be such that $w_x \in L^\infty((0,t);H)$ and $g$ be a continuous function on $[0,T]$ and suppose that. Then, there exists a constant $C > 0$ such that*

$$\begin{aligned}(gow_x)(t) &\le C\left(\sup_{0<s<T}\|w(.,s)\|_H^2\int_0^t g^{1-\theta}(s)ds\right)^{\rho-1/\rho-1+\theta} \\ &\quad\times\left(\int_0^t g^\rho(t-s)\|w_x(.,s)\|_H^2 ds\right)^{\theta/\rho-1+\theta}, \\ &\quad\forall 0<\theta<1 \text{ and } \rho>1.\end{aligned} \quad (68)$$

$$\begin{aligned}(gow_x)(t) &\le c\left(t\|w_x(.,t)\|_H^2 + \int_0^t\|w_x(.,s)\|_H^2 ds\right)^{\rho-1/\rho} \\ &\quad\times\left(\int_0^t g^\rho(t-s)\|w_x(.,t)-w_x(.,s)\|_H^2 ds\right)^{1/2} \\ &\quad\text{for all } \rho>1.\end{aligned} \quad (69)$$

*Proof.*

(1) For any $\sigma > 1$, we have

$$\begin{aligned}(gow_x)(t) &= \int_0^t (g(t-s))^{(1-\theta)/\sigma}\|w_x(.,t)-w_x(.,s)\|^{2/\sigma} \\ &\quad\cdot (g(t-s))^{\sigma-1+\theta/\sigma}\|w_x(.,t)-w_x(.,s)\|^{2(\sigma-1)/\sigma}ds.\end{aligned} \quad (70)$$

Applying Holder's inequality with exponents $\sigma$ and $\sigma/\sigma - 1$, we get

$$(gow_x)(t) \leq \left(\int_0^t g^{1-\theta}(t-s)\|w_x(.,t)-w_x(.,s)\|^2 ds\right)^{1/\sigma} \times \left(\int_0^t g^{\sigma-1+\theta/\sigma-1}(t-s)\|w_x(.,t)-w_x(.,s)\|^2 ds\right)^{\sigma-1/\sigma}, \tag{71}$$

We set $\sigma = (\rho - 1 + \theta)/(\rho - 1)$, (71) yields

$$(gow_x)(t) \leq \left(\int_0^t g^{1-\theta}(t-s)\|w_x(.,t)-w_x(.,s)\|_H^2 ds\right)^{\rho-1/\rho-1+\theta} \times \left(\int_0^t g^{\rho}(t-s)\|w_x(.,t)-w_x(.,s)\|_H^2\right)^{\theta/\rho-1+\theta}. \tag{72}$$

It is easy to see that

$$\int_0^t g^{1-\theta}(t-s)\|w_x(.,t)-w_x(.,s)\|_H^2 ds \leq C \sup_{0<s<T}\|w_x(.,s)\|_H^2 \int_0^t g^{1-\theta}(s)ds. \tag{73}$$

We obtain (68) by combining (72) and (73).

(2) We set $\theta = 1$ in (72) and it suffices to note that

$$\int_0^t \|w_x(.,t)-w_x(.,s)\|_H^2 ds \leq 2t\|w_x(.,t)\|_H^2 + 2\int_0^t \|w_x(.,s)\|_H^2 ds, \tag{74}$$

to arrive at (69).

In the next, we present three lemmas in which we give an upper bound of each derivative's functions in $F(t)$.

**Lemma 12.** *Suppose that $r > -1$ and (39) hold. Then*

$$\begin{aligned}\Phi'(t) \leq{}& \left(1 + \frac{l}{2\delta}\right)\xi(t)\int_0^L xu_t^2 dx + \left(1 + \frac{l}{2\delta}\right)\xi(t)\int_0^L xv_t^2 dx \\ &- \xi(t)\left(\frac{l_1 - \delta C_p l}{2} - \frac{C_p\|\mu\|_\infty}{2}\right)\int_0^L xu_x^2 dx \\ &- \xi(t)\left(\frac{l_2 - \delta C_p l}{2} - \frac{C_p\|\mu\|_\infty}{2}\right)\int_0^L xv_x^2 dx - \xi(t) \\ &+ \frac{\xi(t)}{2l_1}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\ &+ \frac{\xi(t)}{2l_2}\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\ &+ \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \tag{75}$$

*For any $\delta > 0$.*

*Proof.* After derivation of (56), we recall the differential equations in (1), we get

$$\begin{aligned}\Phi'(t) ={}& \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx + (t)\int_0^L xu_{tt}u dx \\ &+ \xi'(t)\int_0^L xv_t v dx + \xi' t)\int_0^L xv_t^2 dx \\ &+ \xi(t)\int_0^L xv_{tt}v dx + \frac{\xi'(t)}{2}\int_0^L x\mu(x)\left(u^2 + v^2\right)dx \\ &+ \xi(t)\left(\int_0^L x\mu(x)(u_t u + v_t v)dx\right) \\ ={}& \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx - \xi(t)\int_0^L xu_x^2 dx \\ &+ \xi(t)\int_0^L xu_x \int_0^t g_1(t-s)u_x(s)ds dx \\ &+ \xi'(t)\int_0^L xv_t v dx + \xi(t)\int_0^L xv_t^2 dx - \xi(t)\int_0^L xv_x^2 dx \\ &+ \xi(t)\int_0^L xv_x \int_0^t g_2(t-s)v_x(s)ds dx \\ &+ \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \tag{76}$$

By Young's inequality and from, $(\boldsymbol{G}_1)$, $(\boldsymbol{G}_2)$, and Lemma 1, we arrive at

$$\begin{aligned}&\xi(t)\int_0^L xu_x(t)\left(\int_0^t g_1(t-s)u_x(s)ds\right)dx \\ &\leq \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}\int_0^L x\left(\int_0^t g_1(t-s) \cdot (|u_x(s) - u_x(t)| + |u_x(t)|)ds\right)^2 dx \\ &\leq \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}(1+\eta_1)(1+l_1)^2\int_0^L xu_x^2(t)dx \\ &\quad + \frac{\xi(t)}{2}\left(1 + \frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t)\end{aligned}$$

$$
\begin{aligned}
&= \xi(t)\left(\frac{1+(1+\eta_1)(1-l_1)^2}{2}\right)\int_0^L xu_x^2 dx \\
&\quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\
&\quad + \frac{\xi(t)}{r+2}\int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx,
\end{aligned} \tag{77}
$$

similarly, we get

$$
\begin{aligned}
&\int_0^L xv_x(t)\left(\int_0^t g_1(t-s)v_x(s)ds\right)dx \\
&\quad \le \xi(t)\left(\frac{1+(1+\eta_2)(1-l_2)^2}{2}\right)\int_0^L xv_x^2 dx \\
&\quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t).
\end{aligned} \tag{78}
$$

For any $\eta_1$ and $\eta_2 > 0$. We also have

$$
\begin{aligned}
\xi'(t)\int_0^L xu_t u dx &\le \frac{\xi(t)}{2}\left|\frac{\xi'(t)}{\xi(t)}\right|\left(C_p\delta\int_0^L xu_x^2 dx + \frac{1}{\delta}\int_0^L xu_t^2 dx\right) \\
&\le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xu_x^2 dx + \frac{l}{\delta}\int_0^L xu_t^2 dx\right), \\
&\quad \forall\delta > 0,
\end{aligned} \tag{79}
$$

and similarly, we get

$$
\xi'(t)\int_0^L xv_t v dx \le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xv_x^2 dx + \frac{l}{\delta}\int_0^L xv_t^2 dx\right). \tag{80}
$$

Also, by Lemma 1, we have

$$
\begin{aligned}
&\frac{\xi'(t)}{2}\int_0^L x\mu(x)\left(u^2+v^2\right)dx \\
&\quad \le \|\mu\|_\infty \frac{\xi(t)}{2}\left[C_p\left(\int_0^L x\left(u_x^2+v_x^2\right)dx\right]\right.
\end{aligned} \tag{81}
$$

Combining (77)–(81) in (76) leads to

$$
\begin{aligned}
\Phi'(t) &\le \left(1+\frac{l}{2\delta}\right)\xi(t)\int_0^L xu_t^2 dx + \left(1+\frac{l}{2\delta}\right)\xi(t)\int_0^L xv_t^2 dx \\
&\quad - \frac{\xi(t)}{2}\left[1-(1+\eta_1)(1-l_1)^2 - \delta C_p l - C_p\|u\|_\infty\right] \\
&\quad \cdot \int_0^L xu_t^2 dx - \frac{\xi(t)}{2}\left[1-(1+\eta_2)(1-l_2)^2 - \delta C_p l\right. \\
&\quad \left. - C_p\|u\|_\infty\right]\int_0^L xu_t^2 dx + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right) \\
&\quad \cdot \left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right) \\
&\quad \cdot \left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\
&\quad + \frac{\xi(t)}{2(r+2)}\int_0^L \left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx,
\end{aligned} \tag{82}
$$

we choose $\eta_1 = l_1/1-l_1$ and $\eta_2 = l_2/1-l_2$, hence (75) is established.

**Lemma 13.** *Suppose that $r > -1$, $(\mathbf{G}_1)$, $(\mathbf{G}_2)$, and (41) hold. Then*

$$
\begin{aligned}
\chi'(t) &\le \xi(t)\theta\left[2+c_1+c_1'+2(1-l_1)^2\right]\left(\int_0^L xu_x^2 dx\right) \\
&\quad + \xi(t)\theta\left[2+c_2+c_2'+2(1-l_2)^2\right]\left(\int_0^L xv_x^2 dx\right) \\
&\quad + \xi(t)\left[\theta - \left(\int_0^t g_1(s)ds\right) + \theta l\right]\left(\int_0^L xu_t^2 dx\right) \\
&\quad + \xi(t)\left[\theta - \left(\int_0^t g_2(s)ds\right) + \theta l\right]\left(\int_0^L xv_t^2 dx\right) \\
&\quad + \left[\frac{1}{2\theta} + 2\theta + \frac{C_p(2+l)}{4\theta}\right] \times \xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right) \\
&\quad \cdot (g_1^\sigma \circ u_x)(t) + \left[\frac{1}{2\theta} + 2\theta + \frac{C_p(2+l)}{4\theta}\right] \\
&\quad \times \xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\
&\quad - \frac{C_p}{4\theta}\xi(t)g_1(0)\left(g_1' \circ u_x\right)(t) \\
&\quad - \frac{C_p}{4\theta}\xi(t)g_2(0)\left(g_2' \circ v_x\right)(t),
\end{aligned} \tag{83}
$$

*for any $\theta > 0$.*

*Proof.* A derivation of (57) gives

$$
\begin{aligned}
\chi'(t) &= -\xi'(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))ds dx \\
&\quad - \xi(t)\int_0^L xu_{tt}\int_0^t g_1(t-s)(u(t)-u(s))ds dx \\
&\quad - \xi(t)\int_0^L xu_t\frac{d}{dt}\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\quad - \xi'(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))ds dx \\
&\quad - \xi(t)\int_0^L xv_{tt}\int_0^t g_2(t-s)(v(t)-v(s))ds dx \\
&\quad - \xi(t)\int_0^L xv_t\frac{d}{dt}\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx,
\end{aligned} \tag{84}
$$

by using Liebniz's formula, we get

$$\begin{aligned}\chi'(t) = &-\xi'(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\&-\xi(t)\int_0^L xu_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\&-\xi(t)\int_0^L xu_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx\\&-\xi(t)\left(\int_0^t g_1(s)ds\right)\int_0^L xu_t^2dx\\&-\xi'(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx\\&-\xi(t)\int_0^L xv_{tt}\int_0^t g_2(t-s)(v(t)-v(s))dsdx\\&-\xi(t)\int_0^L xv_t\int_0^t g_2'(t-s)(v(t)-v(s))dsdx\\&-\xi(t)\left(\int_0^t g_2(s)ds\right)\int_0^L xv_t^2dx.\end{aligned} \tag{85}$$

Recalling the differentials equation in (1), we get

$$\begin{aligned}\chi'(t) = &-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&+\xi(t)\int_0^L x\mu(x)u_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&+\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx\\&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\\&\cdot\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx\\&-\xi(t)\int_0^L (xf_1(u,v))\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&-\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx\\&-\xi(t)\left(\int_0^t g_1(s)ds\right)\int_0^L xu_t^2dx\\&-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\&+\xi(t)\int_0^L x\mu(x)v_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\&+\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx\\&-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)(v_x(s))ds\right)\\&\cdot\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx\\&-\xi(t)\int_0^L (xf_2(u,v))\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\&-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx\\&-\xi(t)\left(\int_0^t g_2(s)ds\right)\int_0^L xv_t^2dx.\end{aligned} \tag{86}$$

We will estimate all term in (86) by Young's inequality, Lemma 1, $(\mathbf{G}_1)$, and $(\mathbf{G}_2)$.

$$\begin{aligned}&-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&\le\xi(t)\left|\frac{\xi'(t)}{\xi(t)}\right|\left[\theta\int_0^L xu_t^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\right]\\&\le\theta l\xi(t)\int_0^L xu_t^2dx+\frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),\end{aligned} \tag{87}$$

$$\begin{aligned}&-\xi(t)\int_0^L x\mu(x)u_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&\le\xi(t)\|\mu\|_\infty\left[\theta\int_0^L xu_t^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-r}(s)ds\right)(g_1^r\circ u_x)(t)\right],\end{aligned} \tag{88}$$

and

$$\begin{aligned}&-\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx\\&\le\theta\xi(t)\int_0^L xu_x^2dx+\frac{1}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),\\&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx\\&\le 2\theta(1-l_1)^2\xi(t)\int_0^L xu_x^2dx+\left(2\theta+\frac{1}{4\theta}\right)\xi(t)\\&\cdot\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),\\&-\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx\\&\le\theta\xi(t)\int_0^L xu_t^2dx-\frac{g_1(0)}{4\theta}C_p\xi(t)\left(g_1'\circ u_x\right)(t),\end{aligned} \tag{89}$$

and

$$\begin{aligned}&-\xi(t)\int_0^L xf_1(u,v)\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\&\le\frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\\&\quad+c_1\theta\xi(t)\int_0^L xu_x^2dx+c_2\theta\xi(t)\int_0^L xv_x^2dx,\end{aligned} \tag{90}$$

where

$$\begin{cases} c_1 := \Lambda_1 \left( \dfrac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}, \\ c_2 := \Lambda_2 \left( \dfrac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}. \end{cases} \quad (91)$$

By the same technique, we obtain estimations on integrals corresponding to $v$, $g_2$, and $f_2$

$$\begin{aligned} &-\xi'(t)\int_0^L x v_t \left( \int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ &\quad \le \theta l \xi(t) \int_0^L x v_t^2 dx + \frac{C_p l}{4\theta} \xi(t) \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t), \\ &-\xi(t) \int_0^L x\mu(x) v_t \left( \int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ &\quad \le \xi(t) \|\mu\|_\infty \left[ \theta \int_0^L x v_t^2 dx + \frac{C_p}{4\theta} \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \right], \\ &\xi(t) \int_0^L x v_x \left( \int_0^t g_2(t-s)(v_x(t)-v_x(s))ds \right) dx \\ &\quad \le \theta \xi(t) \int_0^L x v_x^2 dx + \frac{1}{4\theta} \xi(t) \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t), \\ &-\xi(t) \int_0^L x \left( \int_0^t g_2(t-s) v_x(s) ds \right) \\ &\quad \cdot \left( \int_0^t g_2(t-s)(v_x(t)-v_x(s))ds \right) dx \\ &\quad \le 2\theta (1-l_2)^2 \xi(t) \int_0^L x v_x^2 dx + \left( 2\theta + \frac{1}{4\theta} \right) \xi(t) \\ &\quad \cdot \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ u_x)(t), \\ &-\xi(t) \int_0^L x v_t \left( \int_0^t g_2'(t-s)(v(t)-v(s))ds \right) dx \\ &\quad \le \theta \xi(t) \int_0^L x v_t^2 dx - \frac{g_2(0)}{4\theta} C_p \xi(t) \left( g_2' \circ v_x \right)(t), \end{aligned} \quad (92)$$

and

$$\begin{aligned} &-\frac{\xi(t)}{2(r+2)} \int_0^L x f_2(u,v) \left( \int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx \\ &\quad \le \frac{C_p}{4\theta} \xi(t) \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \\ &\quad + c_1' \theta \xi(t) \int_0^L x u_x^2 dx + c_2' \theta \xi(t) \int_0^L x v_x^2 dx, \end{aligned} \quad (93)$$

where

$$\begin{cases} c_1' := \Lambda_1' \left( \dfrac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}, \\ c_2' := \Lambda_2' \left( \dfrac{2(r+2)}{(r+1)} E(0) \right)^{2(r+1)}. \end{cases} \quad (94)$$

A combination of (87)–(93) into (86) yields (83).

**Lemma 14.** *Suppose that $r > -1$, ($\mathbf{G}_1$), ($\mathbf{G}_2$), ($\mathbf{G}_4$), and (41) hold. Then*

$$\begin{aligned} \psi(t)' &\le \xi(t) \left[ 1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty) \right] \left( \int_0^L x u_x^2 dx \right) \\ &+ \xi(t) \left[ 1 + \|h\|_\infty \theta (2 + \|\mu\|_\infty) \right] \left( \int_0^L x v_x^2 dx \right) \\ &+ \xi(t) \int_0^L \left[ 1 + \|h\|_\infty \theta \left( \frac{1}{2\theta} + \frac{1}{4\theta} \mu(x) \right) \right] x u_t^2 dx \\ &+ \xi(t) \int_0^L \left[ 1 + \|h\|_\infty \theta \left( \frac{1}{2\theta} + \frac{1}{4\theta} \mu(x) \right) \right] x v_t^2 dx \\ &+ \xi(t) \left[ \|h\|_\infty \frac{C_p}{4\theta} \right] \left( \int_0^t g_1^{2-\sigma}(s) ds \right) (g_1^\sigma \circ u_x)(t) \\ &+ \xi(t) \left[ \|h\|_\infty \frac{C_p}{4\theta} \right] \left( \int_0^t g_2^{2-\sigma}(s) ds \right) (g_2^\sigma \circ v_x)(t) \\ &+ \frac{\xi(t)}{2(r+2)} \int_0^L x \left[ a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx, \end{aligned} \quad (95)$$

*For any $\theta > 0$.*

*Proof.* We derivate (58) and integrate by part and we finish by using the differential equations in (1), we get

$$\begin{aligned} \psi'(t) &= \xi'(t) \int_0^L u_t x h(x) u_x dx + \xi'(t) \int_0^L v_t x h(x) v_x dx \\ &+ \xi(t) \int_0^L u_t x h(x) u_{tx} dx + (q+1) \xi'(t) \int_0^L v_t x h(x) v_{tx} dx \\ &+ \xi(t) \int_0^L u_{tt} x h(x) u_x dx + \xi(t) \int_0^L v_{tt} x h(x) v_{tx} dx \\ &= \xi'(t) \int_0^L u_t x h(x) u_x dx + \xi'(t) \int_0^L v_t x h(x) v_x dx \\ &- \frac{1}{2} \xi(t) \int_0^L (x h(x))' u_t^2 dx - \frac{1}{2} \xi(t) \int_0^L (x h(x))' v_t^2 dx \\ &- \frac{1}{2} \xi(t) \int_0^L (x h(x))' u_x^2 dx - \frac{1}{2} \xi(t) \int_0^L (x h(x))' v_x^2 dx \\ &- \xi(t) \int_0^L x \mu(x) h(x) u_t u_x dx - \xi(t) \int_0^L x \mu(x) h(x) v_t v_x dx \end{aligned}$$

$$-\xi(t)\int_0^L xh(x)u_x\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx$$
$$-\xi(t)\int_0^L xh(x)v_x\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx$$
$$+\frac{\xi(t)}{2(r+2)}\int_0^L (xh(x))'\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx, \tag{96}$$

Applying Young's and Poincare's inequalities, we get

$$\xi'(t)\int_0^L u_t xh(x)u_x dx \leq \xi(t)\|h\|_\infty\left(\theta\int_0^L xu_x^2dx+\frac{1}{4\theta}\int_0^L xu_t^2dx\right),$$
$$\int_0^L x\mu(x)h(x)u_tu_xdx \leq \|h\|_\infty\left(\theta\int_0^L x\mu(x)u_x^2dx+\frac{1}{4\theta}\int_0^L x\mu(x)u_t^2dx\right), \tag{97, 98}$$

finally

$$\int_0^L xh(x)u_x\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \leq \|h\|_\infty\left(\theta\int_0^L xu_x^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-r}(s)ds\right)(g_1^r\circ u_x)(t)\right), \tag{99}$$

similarly

$$\xi'(t)\int_0^L v_t xh(x)v_xdx \leq \xi(t)\|h\|_\infty\left(\theta\int_0^L xv_x^2dx+\frac{1}{4\theta}\int_0^L xv_t^2dx\right),$$
$$\int_0^L x\mu(x)h(x)v_tv_xdx \leq \|h\|_\infty\left(\theta\int_0^L x\mu(x)v_x^2dx+\frac{1}{4\theta}\int_0^L x\mu(x)v_t^2dx\right), \tag{100}$$

and

$$\int_0^L xh(x)v_x\left(\int_0^t g_1(t-s)(v(t)-v(s))ds\right)dx \leq \|h\|_\infty\left(\theta\int_0^L xv_x^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-r}(s)ds\right)(g_1^r\circ v_x)(t)\right), \tag{101}$$

Using (97)–(101) and $(xh(x))'\leq x$, we obtain (95).

Next Theorem show that solutions decreases exponentially with respect to $\xi(t)$ and $\sigma$.

**Theorem 15.** *Suppose that $r>-1$, ($\mathbf{G}_1$), and ($\mathbf{G}_2$) hold and taking $u_0, v_0\in V_0^2$, and $(u_1, v_1)\in H^2$ such that (41) hold true. Then, for each $t_0>0$, there exist positive constants $K$ and $k$ such that*

$$E(t)\leq\begin{cases} Ke^{-k\int_{t_0}^t \xi(s)ds}, & \sigma=1,\\ K\left(1+\int_{t_0}^t \xi(s)ds\right)^{-1/\sigma-1}, & 1<\sigma<\frac{3}{2}, \forall t\geq t_0.\end{cases} \tag{102}$$

*Proof.* Since $g_1$ and $g_2$ is continuous and $g_1(0)>0$, $g_2(0)>0$ then for any $t_0>0$, we have

$$\left\{\int_0^t g_i(s)ds\geq\int_{t_0}^t g_i(s)ds=g_{1,0}>0,\quad \forall t\geq t_0, i=1,2.\right. \tag{103}$$

As $\xi$ is a positive decreasing function hence $1<\xi(t)/\xi(0)$ and by recalling Lemmas 7, 12, 13, 14, and (101), we get

$$\begin{aligned}
F'(t)&=E'(t)+\varepsilon_1\Phi'(t)+\varepsilon_2\chi'+\psi'(t)\\
&\leq-\int_0^L\left[\left(N-\frac{\|h\|_\infty}{4\theta}\right)\mu(x)-\left(1+\frac{\|h\|_\infty}{4\theta}\right)\right.\\
&\quad\left.-\varepsilon_1\left(1+\frac{1}{2\delta}\right)+\varepsilon_2\left(g_{1,0}-\theta-\theta l\right)\right]\xi(t)xu_t^2dx\\
&\quad-\int_0^L\left[\left(N-\frac{\|h\|_\infty}{4\theta}\right)\mu(x)-\left(1+\frac{\|h\|_\infty}{4\theta}\right)\right.\\
&\quad\left.-\varepsilon_1\left(1+\frac{1}{2\delta}\right)+\varepsilon_2\left(g_{2,0}-\theta-\theta l\right)\right]\xi(t)xv_t^2dx\\
&\quad+(2\varepsilon_1+1)\xi(t)\int_0^L x\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx\\
&\quad+\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_1(0)\right)\left(g_1'\circ u_x\right)(t)\\
&\quad+\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_2(0)\right)\left(g_2'\circ v_x\right)(t)\\
&\quad-\left[\frac{\varepsilon_1}{2}\left(l_2-C_p\left(\delta l+\|\mu\|_\infty\right)\right)-\varepsilon_2\theta\left(1+c_2\right.\right.\\
&\quad\left.\left.+c_2'+2(1-l_2)^2\right)-\left(1+\|h\|_\infty\right)\theta\left(2+\|\mu\|_\infty\right)\right]\\
&\quad\times\xi(t)\left(\int_0^L xv_x^2dx\right)+\left[\frac{\varepsilon_1}{2l_1}\right.\\
&\quad\left.+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{2+l+C_p\|h\|_\infty+4\theta}{4\theta}\right)\right]\\
&\quad\times\xi(t)\left(\int_0^L g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\\
&\quad+\left[\frac{\varepsilon_1}{2l_2}+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{2+l+C_p\|h\|_\infty+4\theta}{4\theta}\right)\right]\\
&\quad\times\xi(t)\left(\int_0^L g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ u_x)(t).
\end{aligned} \tag{104}$$

We take $\delta < (1/2C_p l)\min\{l_1, l_2\}$, hence

$$\begin{cases} (l_1 - \delta C_p l) > \dfrac{l_1}{2}, \\ (l_2 - \delta C_p l) > \dfrac{l_2}{2}, \end{cases} \tag{105}$$

Also, we take $\theta > \max\{g_{1,0}/2(1+l), g_{2,0}/2(1+l)\}$, hence

$$\begin{cases} (g_{1,0} - (1+l)\theta) < \dfrac{1}{2} g_{1,0}, \\ (g_{2,0} - (1+l)\theta) < \dfrac{1}{2} g_{2,0}. \end{cases} \tag{106}$$

Now, we choose $\varepsilon_2$ small enough such

$$k_1 := \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_1(0)\right) > 0, \quad k_1 := \left(\frac{1}{2} - \frac{\varepsilon_2 \xi(0)}{4\theta} C_p g_2(0)\right) > 0, \tag{107}$$

As far as $\delta$, $\theta$, and $\varepsilon_2$ are fixed, we then pick $\varepsilon_1$ so small that

$$\begin{cases} k_3 := \dfrac{\varepsilon_1}{4}\left(l_1 - 2C_p\|\mu\|_\infty\right) - \varepsilon_2\theta\left(1 + c_1 + c_1' + 2(1-l_1)^2\right) - \left(1 + \|h\|_\infty \theta\left(2 + \|\mu\|_\infty\right)\right) > 0, \\ k_4 := \dfrac{\varepsilon_1}{4}\left(l_2 - 2C_p\|\mu\|_\infty\right) - \varepsilon_2\theta\left(1 + c_2 + c_2' + 2(1-l_2)^2\right) - \left(1 + \|h\|_\infty \theta\left(2 + \|\mu\|_\infty\right)\right) > 0, \end{cases}$$
$$k_5 := \left[-\varepsilon_1\left(1 + \frac{1}{2\delta}\right) + \varepsilon_2\left(g_{1,0} - (1+l)\theta\right)\right] > 0,$$
$$k_6 := \left[-\varepsilon_1\left(1 + \frac{1}{2\delta}\right) + \varepsilon_2\left(g_{2,0} - (1+l)\theta\right)\right] > 0, \tag{108}$$
$$-\left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{2 + l + C_p\|h\|_\infty + 4\theta}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)\,ds\right)\right\} > 0,$$

and

$$-\left\{\left[\frac{\varepsilon_1}{2l_3} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{2 + l + C_p\|h\|_\infty + 4\theta}{4\theta}\right)\right] \cdot \left(\int_0^t g_2^{2-\sigma}(s)\,ds\right)\right\} > 0, \tag{109}$$

Finally we choose $N > \|h\|_\infty/4\theta$ large enough such that

$$\left(N - \frac{\|h\|_\infty}{4\theta}\right)\mu(x) - \left(1 + \frac{\|h\|_\infty}{4\theta}\right) > 0. \tag{110}$$

After fixed all this choices then from (14) and (27), we obtain for some $\sigma > 0$,

$$\begin{aligned} F'(t) \le -\sigma\xi(t)\Bigg[&\int_0^L x u_t^2\,dx + \int_0^L x v_t^2\,dx \\ &- \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &+ \int_0^L x u_x^2\,dx + \int_0^L x v_x^2\,dx + (g_1^\sigma \circ u_x)(t) \\ &+ (g_2^\sigma \circ v_x)(t)\Bigg] \le -\gamma\xi(t)E(t), \forall t \ge t_0. \end{aligned} \tag{111}$$

We distingue two cases according $\sigma$

*Case 16.* $\sigma = \mathbf{1}$

We recall lemma 10, and the estimate (111) gives

$$F'(t) \le -\gamma\alpha_1\xi(t)F(t), \forall t \ge t_0. \tag{112}$$

A simple integration of the above inequality over $(t_0, t)$ leads to

$$F'(t) \le F(t_0)e^{(-\gamma\alpha_1)\int_{t_0}^t \xi(s)ds}, \forall t \ge t_0. \tag{113}$$

Hence, (102) is established.

*Case 17.* $1 < \sigma < 3/2$

By using (16), we get

$$g_i(t)^{1-\sigma} \ge (\sigma - 1)\left(\int_{t_0}^t \xi(s)\,ds\right) + g_i(t_0)^{1-\sigma}, \quad i = 1, 2. \tag{114}$$

So, for $\forall 0 < \tau < 2 - \sigma < 1$, (hence $(1 - \tau/\sigma - 1) > 1$), we have

$$\int_0^\infty g_i^{1-\tau}(s)\,ds \le \int_0^\infty \frac{1}{\left[(\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_i(t_0)^{1-\sigma}\right]^{1-\tau/\sigma-1}}\,ds. \tag{115}$$

In the other hand, by using the fact that $\int_0^\infty \xi(s)ds = +\infty$, we obtain

$$\int_0^\infty g_i^{1-\tau}(s)ds < \infty, \forall 0 < \tau < 2 - \sigma, \text{for } i = 1, 2. \tag{116}$$

So form (i) of Lemma 11 with $\theta = \tau$ and $\rho = \sigma$ and (41) yield

$$\begin{aligned}(g_i \circ w_x)(t) &\le C_i \left( E(0) \int_0^\infty g_i^{1-\tau}(s)ds \right)^{\sigma-1/\sigma-1+\tau} \\ &\cdot \left( (g_i^\sigma \circ w_x)(t) \right)^{\tau/\sigma-1+\tau} \\ &\le C_i' \left( (g_i^\sigma \circ w_x)(t) \right)^{\tau/\sigma-1+\tau},\end{aligned} \tag{117}$$

for $i = 1, 2$ and $w = u, v$, respectively, with $C_i'$ are positive constants.

Therefore, for any $\sigma_1 > 1$, we arrive at

$$\begin{aligned}E^{\sigma_1}(t) &\le C'' E^{\sigma_1-1}(0) \left( \int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \right. \\ &\left. + \int_0^L x v_x^2 dx - \int_0^L x \left[ a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \right) \\ &+ C_1'' \left( (g_1 \circ u_x)(t) \right)^{\sigma_1} + C_2'' \left( (g_2 \circ v_x)(t) \right)^{\sigma_1} \\ &\le C'' E^{\sigma_1-1}(0) \left( \int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \right. \\ &\left. + \int_0^L x v_x^2 dx - \int_0^L x \left[ a|u+v|^{2(r+2)} + 2b|uv|^{r+2} \right] dx \right) \\ &+ C_1''' \left( (g_1^\sigma \circ u_x)(t) \right)^{\tau\sigma_1/\sigma-1+\tau} \\ &+ C_2''' \left( (g_2^\sigma \circ v_x)(t) \right)^{\tau\sigma_1/\sigma-1+\tau}.\end{aligned} \tag{118}$$

By setting $\tau = 1/2$ and $\sigma 1 = 2\sigma - 1$ which give $\tau\sigma_1/\sigma-1+\tau=1$, the estimate (119) becomes,

$$\begin{aligned}E^{\sigma_1}(t) &\le \Gamma \left[ \left[ \int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx \right] \right. \\ &- \int_0^L x \left[ a|u+v|^{2(r+2)} + 2b|uv|^{r+2} dx \right. \\ &\left. + (g_1^\sigma \circ u_x)(t) + (g_2^\sigma \circ v_x)(t) \right],\end{aligned} \tag{119}$$

for some $\Gamma > 0$. Form the equivalence between $F$ and $E$ and by combining the above inequality with (111), we obtain

$$F'(t) \le -\frac{\sigma}{\Gamma} \xi(t) E^{\sigma_1}(t) \le -\frac{\sigma}{\Gamma} \alpha_1^{\sigma_1} F^{\sigma_1}(t), \quad \forall t \ge t_0. \tag{120}$$

By twice integration over $(t_0, t)$ and $(t_0, \infty)$ of (119) successively leads to

$$F(t) \le C_1^* \left( 1 + \int_{t_0}^t \xi(s)ds \right)^{-1/\sigma_1-1}, \quad \forall t \ge t_0, \tag{121}$$

$$\int_{t_0}^\infty F(t)dt \le C_1^* \int_{t_0}^\infty \frac{1}{\left( 1 + \int_{t_0}^t \xi(s)ds \right)^{1/\sigma_1-1}} dt. \tag{122}$$

Since $1/\sigma_1-1>0$ and $(1 + \int_{t_0}^t \xi(s)ds) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we get

$$\int_{t_0}^\infty F(t)dt \le \infty. \tag{123}$$

Again from (121), we have

$$tF(t) \le \frac{C_1^* t}{\left( 1 + \int_{t_0}^t \xi(s)ds \right)^{1/\sigma_1-1}} \le C_\sigma, \quad \forall t \ge t_0, \tag{124}$$

which implies that

$$\sup_{t \ge t_0} tF(t) < \infty. \tag{125}$$

Summing (123) and (125), we get

$$\int_{t_0}^\infty F(t)dt + \sup_{t \ge t_0} (tF(t)) < \infty. \tag{126}$$

By recalling (ii) of Lemma 11 with $\rho = \sigma$, Lemma 10, and by using (27) and the above bounded, we have

$$\begin{aligned}(g_i \circ w_x)(t) &\le C_{i2}^* \left( t\|w_x(x,t)\|_H^2 + \int_0^t \|w_x(x,s)\|_H^2 ds \right)^{\sigma-1/\sigma} \\ &\times \left( \int_0^t g_i^\sigma(t-s) \|w_x(x,t) - w_x(x,s)\|_H^2 ds \right)^{1/\sigma} \\ &\le C_{i2}^* \left( tF(t) + \int_{t_0}^t F(s)ds \right)^{\sigma-1/\sigma} \left( (g_i^\sigma \circ w_x)(t) \right)^{1/\sigma} \\ &\le C_{i3}^* \left( (g_i^\sigma \circ w_x)(t) \right)^{1/\sigma},\end{aligned} \tag{127}$$

Hence,

$$(g_i^\sigma \circ w_x)(t) \ge (C_{i3}^*)^{-\sigma} \left( (g_i \circ w_x)(t) \right)^\sigma, \tag{128}$$

for $i = 1,2$, $w = u,v$, respectively, and some positive constants $C_{i3}^*$.

Consequently, introduce (128) in (111) and in, (118), we find

$$F'(t) \leq -C_4\xi(t)\left[\left[\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + ((g_1 \circ u_x)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\right],\right. \tag{129}$$

and

$$E^\sigma(t) \leq C_5\xi(t)\left[\left[\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + ((g_1 \circ u_x)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\right],\right. \tag{130}$$

for all $t \geq 0$ and some positive constant $C_4$,$C_5$.

By combining the last two inequalities and along the equivalence between $F$ and $E$, we obtain

$$F'(t) \leq -C_6\xi(t)F^\sigma(t), \quad \forall t \geq t_0, \tag{131}$$

for some constant $C_6 > 0$.

A simple integration of (131) over $(t_0, t)$ gives

$$F(t) \leq C_9\left(1 + \int_{t_0}^t \xi(s)ds\right)^{-1/\sigma-1}, \quad \forall t \geq t_0. \tag{132}$$

The proof is completed by using (59).

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

## Authors' Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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