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Research Article

Blow up of Coupled Nonlinear Klein-Gordon System with Distributed Delay, Strong Damping, and Source Terms

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This present work deals with the blow up of the coupled Klein-Gordon system with strong damping, distributed delay, and source terms, under suitable conditions.

1. Introduction

In the present paper, we consider the following system:

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \int_{\tau_2}^{\tau_1} |\mu_2(\varrho)| u_t(x, t-\varrho) d\varrho = f_1(u, v), \quad (x, t) \in \Omega \times \mathbb{R}_+, v_{tt} m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)ds + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x, t-\varrho) d\varrho = f_2(u, v), \quad (x, t) \in \Omega \times \mathbb{R}_+, \\ u(x, t) = 0, \quad v(x, t) = 0, \quad x \in \partial\Omega, \\ u_t(x, -t) = f_0(x, t), \quad v_t(x, -t) = k_0(x, t) \quad (x, t) \in \Omega \times (0, \tau_2), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where

$$\begin{cases} f_1(u, v) = a_1 |u+v|^{2(p+1)}(u+v) + b_1 |u|^p \cdot u \cdot |v|^{p+2}, \\ f_2(u, v) = a_1 |u+v|^{2(p+1)}(u+v) + b_1 |v|^p \cdot v \cdot |u|^{p+2}, \end{cases} \tag{2}$$

and $m_1, m_2, \omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, and τ_1, τ_2 are the time delay with $0 \le \tau_1 < \tau_2$, and μ_2, μ_4 are a L^∞ functions, and g, h are differentiable functions.

Viscous materials are the opposite of flexible materials that have dissipate mechanical energy and the ability to store. The mechanical properties of viscous materials are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades.

If there is only one equation and if $w_1 = 0$, that is, for absence of Δu_t, and $\mu_1 = \mu_2 = 0$. Our problem (1) has been studied by Berrimi and Messaoudi [1]. Using Galerkin's method they proved the result of local existence. They also made it clear that the local solution is global in time under suitable conditions and at the same rate of decaying (exponential or polynomial) of the kernel g. In addition, the authors themselves demonstrated that the dissipation can be deduced by the term viscous integral and that it is strong enough to stabilize the solution oscillations. Their results were also obtained under weaker conditions than those used by Cavalcanti et al. [2].

In [3], the authors considered the following problem:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma,\ u = 0, \quad (3)$$

where the authors proved the exponential decay result. This subsequent result was improved by Berrimi et al. in [1], as they showed that the viscosity elastic dissipation alone is strong enough to stabilize the problem even with the exponential rate with respect to the kernel g assumptions. In the case $\mu_1 \neq 0$, in problem (1), Kafini and Messaoudi in [4] proved a blow up result for the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds + u_t = |u|^{p-2} \cdot u, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \quad (4)$$

where g satisfies

$$\int_0^\infty g(s)ds < (2p-4)/(2p-3). \quad (5)$$

The initial data was backed by negative energy as

$$\int u_0 u_1 dx > 0. \quad (6)$$

In [5], Song and Xue considered the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t |u|^{p-2} \cdot u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \quad (7)$$

where the authors showed that there were solutions of (7) with initial energy according to suitable assumptions on g. Moreover, they showed the blow up in a finite time. Then, the same authors in [6] continued to prove that there were solutions of (7) with positive initial energy that blow up in finite time. In [7], the author studied the following problem:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2} \cdot u_t = |u|^{p-2} \cdot u, & (x,t) \in \Omega(0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \end{cases} \quad (8)$$

where they proved the exponential growth result under suitable assumptions. The authors in [8] studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t |u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \quad (9)$$

where they showed a blow up result if $p > m$ and established the global existence. In the coupled equation case, the authors in [9] studied the following system:

$$\begin{cases} u_{tt} - \Delta u + u_t |u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + v_t |v_t|^{r-2} = f_2(u,v), \end{cases} \quad (10)$$

with f_1 and f_2 nonlinear functions satisfying appropriate conditions. According to certain restrictions imposed on the initial data and parameters, they obtained numerous results on the existence of weak solutions. They obtained many results on the presence of weak solutions. In addition, by using the same techniques similar to that in [10] with negative initial, energy blows up for a finite period of time.

In [11], the authors have proved the solution of the problem:

$$\begin{cases} u_{tt} - \Delta u + \left(a|u|^k + b|v|^l\right) u_t |u_t|^{m-2} = f_1(u, v), \\ v_{tt} - \Delta v + \left(a|u|^\theta + b|v|^\vartheta\right) v_t |v_t|^{r-2} = f_2(u, v), \end{cases} \tag{11}$$

where under some restrictions on positive initial energy for certain conditions on the functions f_1 and f_2, the authors proved the blows up in finite time of solution.

The result of [11] has been extended by the authors in [12], where they studied the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds + \left(a|u|^k + b|v|^l\right) u_t |u_t|^{m-2} = f_1(u, v), \\ v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)ds + \left(a|u|^\theta + b|v|^\varrho\right) v_t |v_t|^{r-2} = f_2(u, v), \end{cases} \tag{12}$$

they proved that the solutions of a system of wave equations with degenerate damping, viscoelastic term and strong nonlinear sources acting in both equations at the same time are globally nonexisting provided that the initial data are sufficiently large in a bounded domain of Ω.

As complement to these works, we are working to prove the blow up result with distributed delay of problem (1), under appropriate assumptions, and we prove these results using the energy method. In the following, let $c, c_i > 0, i = 1, \cdots, 12$.

The present paper is organized as follows. In Section 2, we give some necessarily assumptions for the main result. In Section 3, we prove the blow up result.

2. Assumptions

We consider the following suitable assumptions.

(A1) $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are differentiable and decreasing functions such that

$$g(t) \geq 0, \quad 1 - \int_0^\infty g(s)ds = l_1 > 0, \tag{13}$$

$$h(t) \geq 0, \quad 1 - \int_0^\infty h(s)ds = l_2 > 0. \tag{14}$$

(A2) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$g'(t) \leq -\xi_1 g(t), \quad t \geq 0, \tag{15}$$

$$h'(t) \leq -\xi_2 h(t), \quad t \geq 0. \tag{16}$$

(A3) $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are a L^∞ functions so that

$$\left(\frac{2\delta - 1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1, \quad \delta > \frac{1}{2}, \tag{17}$$

$$\left(\frac{2\delta - 1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3, \quad \delta > \frac{1}{2}. \tag{18}$$

3. Blow up

In this section, we obtain the proof of the blow up result of the solution of problem (1). First, of all in [13], we introduce the new variables

$$\begin{aligned} y(x, \rho, \varrho, t) &= u_t(x, t - \varrho\rho), \\ z(x, \rho, \varrho, t) &= v_t(x, t - \varrho\rho), \end{aligned} \tag{19}$$

then, we obtain

$$\begin{aligned} &\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = u_t(x, t), \end{cases} \\ &\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = v_t(x, t). \end{cases} \end{aligned} \tag{20}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds dx. \tag{21}$$

Therefore, problem (1) get the following form:

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho = f_1(u, v), & x \in \Omega, t \geq 0, \\ v_{tt} + m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)ds + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x, 1, \varrho, t) d\varrho = f_2(u, v), & x \in \Omega, t \geq 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_p(x, \rho, \varrho, t) = 0, \\ \varrho z_t(x, \rho, \varrho, t) + z_p(x, \rho, \varrho, t) = 0, \end{cases} \tag{22}$$

with initial and boundary conditions

$$\begin{cases} u(x,t)=0, \quad v(x,t)=0, \quad x\in\partial\Omega, \\ y(x,\rho,\varrho,0)=f_0(x,\varrho\rho), \quad z(x,\rho,\varrho,0)=k_0(x,\varrho\rho), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \end{cases} \tag{23}$$

where

$$(x,\rho,\varrho,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{24}$$

Theorem 1. *Assume (14), (16), and (17) hold. Let*

$$\begin{cases} -1<p<\dfrac{4-n}{n-2}, \quad n\geq 3, \\ p\geq -1, \quad n=1,2. \end{cases} \tag{25}$$

For any initial data,

$$(u_o,u_1,v_0,v_1,f_0,k_0)\in\mathscr{H}, \tag{26}$$

where

$$\mathscr{H}=H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega) \times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2))\times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)), \tag{27}$$

then, problem (22) has a unique solution

$$u\in C([0,T];\mathscr{H}), \tag{28}$$

for some $T>0$.

Lemma 2. *There exists a function $F(u,v)$ such that*

$$\begin{aligned} F(u,v)&=\frac{1}{2(\rho+2)}\left[uf_1(u,v)+vf_2(u,v)\right], \\ &=\frac{1}{2(\rho+2)}\left[a_1|u+v|^{2(p+2)}+2b_1|uv|^{p+2}\right]\geq 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \frac{\partial F}{\partial u}&=f_1(u,v), \\ \frac{\partial F}{\partial v}&=f_2(u,v), \end{aligned} \tag{30}$$

we take $a_1=b_1=1$ for convenience.

Lemma 3. (see [12]). *There exist two positive constants c_0 and c_1 such that*

$$\begin{aligned} &\frac{c_0}{2(\rho+2)}\left(|u|^{2(p+2)}+|v|^{2(p+2)}\right) \\ &\quad\leq F(u,v)\leq\frac{c_1}{2(\rho+2)}\left(|u|^{2(p+2)}+|v|^{2(p+2)}\right). \end{aligned} \tag{31}$$

We define the energy functional (see, e.g., [14–16] and reference therein).

Lemma 4. *Assume (14), (16), (17), and (25) hold, let (u,v,y,z) be a solution of (22), then $E(t)$ is nonincreasing, that is,*

$$\begin{aligned} E(t)&=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{m_1}{2}\|u\|_2^2+\frac{m_2}{2}\|v\|_2^2 \\ &\quad+\frac{1}{2}l_1\|\nabla u\|_2^2+\frac{1}{2}l_2\|\nabla v\|_2^2+\frac{1}{2}(go\nabla u)+\frac{1}{2}(ho\nabla u) \\ &\quad+\frac{1}{2}K(y,z)-\int_\Omega F(u,v)dx, \end{aligned} \tag{32}$$

satisfies

$$\begin{aligned} E'(t)\leq -c_3\Big\{&\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_2^2+\|v\|_2^2 \\ &+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\Big\}\leq 0, \end{aligned} \tag{33}$$

where

$$\begin{aligned} K(y,z)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\big\{&|\mu_2(\varrho)|y^2(x,\rho,,\varrho,t) \\ &+|\mu_4(\varrho)|z^2(x,\rho,,\varrho,t)d\varrho d\rho dx\big\}. \end{aligned} \tag{34}$$

Proof. By multiplying $(3.4)_1$, $(3.4)_2$ by u_t, v_t and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{m_1}{2}\|u\|_2^2+\frac{m_2}{2}\|v\|_2^2+\frac{1}{2}l_1\|\nabla u\|_2^2 \\ &\quad+\frac{1}{2}l_2\|\nabla v\|_2^2+\frac{1}{2}(go\nabla u)+\frac{1}{2}(ho\nabla u)-\int_\Omega F(u,v)dx\Big\} \\ &=-\mu_1\|u_t\|_2^2-m_1\|u\|_2^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \\ &\quad-\mu_3\|v_t\|_2^2-m_2\|v\|_2^2-\int_\Omega v_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z(x,1,\varrho,t)d\varrho dx \\ &\quad+\frac{1}{2}\left(g'o\nabla u\right)-\frac{1}{2}g(t)\|\nabla u\|_2^2-\omega_1\|\nabla u_t\|_2^2+\frac{1}{2}\left(h'o\nabla u\right) \\ &\quad-\frac{1}{2}h(t)\|\nabla v\|_2^2-\omega_2\|\nabla v_t\|_2^2, \end{aligned} \tag{35}$$

and, from $(3.4)_3$, $(3.4)_4$, we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
&= -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_2(\varrho)|yy_\rho d\varrho d\rho dx, \\
&\quad+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,0,\varrho,t)d\varrho dx \\
&\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx, \\
&= \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)d\varrho\right)\|u_t\|_2^2 \\
&\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_4(\varrho)|z^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
&= -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_4(\varrho)|zz_\rho d\varrho d\rho dx, \\
&\quad+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,0,\varrho,t)d\varrho dx \\
&\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx, \\
&= \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)d\varrho\right)\|v_t\|_2^2 \\
&\quad-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx,
\end{aligned} \tag{37}$$

then, we get

$$\begin{aligned}
&\frac{d}{dt}E(t) \\
&= -\mu_1\|u_t\|_2^2 - m_1\|u\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|u_t y(x,1,\varrho,t)d\varrho dx \\
&\quad+\frac{1}{2}\left(g'o\nabla u\right) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega_1\|\nabla u_t\|_2^2 \\
&\quad+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)d\varrho\|u_t\|_2^2 - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\right. \\
&\quad-\mu_3\|v_t\|_2^2 - m_2\|v\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|v_t z(x,1,\varrho,t)d\varrho dx \\
&\quad+\frac{1}{2}\left(h'o\nabla v\right) - \frac{1}{2}h(t)\|\nabla v\|_2^2 - \omega_2\|\nabla v_t\|_2^2 \\
&\quad+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4\varrho d\varrho\right)\|v_t\|_2^2 - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx.
\end{aligned} \tag{38}$$

By (35)–(37), we get (32).

And by using Young's inequality, (14), (16), and (17) in (38), we obtain (33).

Now, we define the functional

$$\begin{aligned}
\mathbb{H}(t) = -E(t) &= -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|v_t\|_2^2 - \frac{m_1}{2}\|u\|_2^2 - \frac{m_2}{2}\|v\|_2^2 \\
&\quad-\frac{1}{2}l_1\|\nabla u\|_2^2 - \frac{1}{2}l_2\|\nabla v\|_2^2 - \frac{1}{2}(go\nabla u) - \frac{1}{2}(ho\nabla u) \\
&\quad-\frac{1}{2}K(y,z) + \frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{(p+2)}^{(p+2)}\right].
\end{aligned} \tag{39}$$

Theorem 5. *Assume (14)–(17) and (25) hold. Assume further that $E(0) < 0$, then the solution of problem (22) blow up in finite time.*

Proof. From (32), we have

$$E(t) \le E(0) \le 0. \tag{40}$$

Therefore,

$$\begin{aligned}
\mathbb{H}'(t) &= -E'(t) \\
&\ge c_3\left(\|u_t\|_2^2 + \|u\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\right. \\
&\quad\left.+\|v_t\|_2^2 + \|v\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\right),
\end{aligned} \tag{41}$$

hence,

$$\mathbb{H}'(t) \ge c_3\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \ge 0, \tag{42}$$

$$\mathbb{H}'(t) \ge c_3\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \ge 0, \tag{43}$$

$$\begin{aligned}
0 \le \mathbb{H}(0)\mathbb{H}(t) &\le \frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right], \\
&\le \frac{c_1}{2(p+2)}\left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right].
\end{aligned} \tag{44}$$

We set

$$\begin{aligned}
\mathscr{K}(t) &= \mathbb{H}^{1-\alpha} + \varepsilon\int_\Omega(uu_t + vv_t)dx + \frac{3}{2}\int_\Omega\left(\mu_1u^2 + \mu_3v^2\right)dx \\
&\quad+\frac{\varepsilon}{2}\int_\Omega\left(\omega_1(\nabla u)^2 + \omega_2(\nabla u)^2\right)dx,
\end{aligned} \tag{45}$$

where $\varepsilon > 0$ to be assigned later and

$$0 < \alpha < \frac{2p+2}{4(p+2)} < 1. \tag{46}$$

By multiplying $(3.4)_1$, $(3.4)_2$ by u, v and with a derivative of (45), we get

$$\begin{aligned}\mathscr{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right)\\ &\quad - \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \varepsilon\int_\Omega \nabla u \int_0^1 g(t-s)\nabla u(s)\,ds\,dx\\ &\quad + \varepsilon\int_\Omega \nabla v \int_0^1 h(t-s)\nabla v(s)\,ds\,dx\\ &\quad - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad + \varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right].\end{aligned} \tag{47}$$

Using Young's inequality, we get

$$\begin{aligned}&\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad\le \varepsilon\left\{\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\|u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)| y^2(x,1,\varrho,t)\,d\varrho\,dx\right\},\\ &\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad\le \varepsilon\left\{\delta_2\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,d\varrho\right)\|u\|_2^2 + \frac{1}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)| z^2(x,1,\varrho,t)\,d\varrho\,dx\right\},\end{aligned} \tag{48}$$

and we have

$$\begin{aligned}&\varepsilon\int_0^t g(t-s)\,ds\int_\Omega \nabla u\cdot\nabla u(s)\,dx\,ds\\ &\quad= \varepsilon\int_0^t g(t-s)\,ds\int_\Omega \nabla u\cdot(\nabla u(s)-\nabla u(t))\,dx\,ds\\ &\qquad+ \varepsilon\int_0^t g(s)\,ds\|\nabla u\|_2^2 \ge \frac{\varepsilon}{2}\int_0^t g(s)\,ds\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u),\\ &\varepsilon\int_0^t h(t-s)\,ds\int_\Omega \nabla v\cdot\nabla v(s)\,dx\,ds\\ &\quad= \varepsilon\int_0^t h(t-s)\,ds\int_\Omega \nabla v\cdot(\nabla v(s)-\nabla v(t))\,dx\,ds\\ &\qquad+ \varepsilon\int_0^t h(s)\,ds\|\nabla v\|_2^2 \ge \frac{\varepsilon}{2}\int_0^t h(s)\,ds\|\nabla v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla v).\end{aligned} \tag{49}$$

We obtain, from (47),

$$\begin{aligned}\mathscr{K}'(t) &\ge (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\left(\|u_t\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right)\\ &\quad - \varepsilon\left(\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2\right)\\ &\quad - \varepsilon\delta_1\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\|u\|_2^2\\ &\quad - \varepsilon\delta_2\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,d\varrho\right)\|v\|_2^2 - \frac{\varepsilon}{2}(go\nabla u)\\ &\quad - \frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)| y^2(x,1,\varrho,t)\,d\varrho\,dx - \frac{\varepsilon}{2}(ho\nabla v)\\ &\quad - \frac{\varepsilon}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)| z^2(x,1,\varrho,t)\,d\varrho\,dx\\ &\quad + \varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right].\end{aligned} \tag{50}$$

Therefore, using (43) and by setting δ_1, δ_1 so that, $1/4\delta_1 c_3 = \kappa\mathbb{H}^{-\alpha}(t)/2$ and $1/4\delta_2 c_3 = \kappa\mathbb{H}^{-\alpha}(t)/2$, substituting in (50), we get

$$\begin{aligned}\mathscr{K}'(t) &\ge [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)\\ &\quad + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right)\\ &\quad - \varepsilon\left[\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\right]\|\nabla u\|_2^2\\ &\quad - \varepsilon\left[\left(1-\frac{1}{2}\int_0^t h(s)\,ds\right)\right]\|\nabla v\|_2^2\\ &\quad - \varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\|u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u)\\ &\quad - \varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,d\varrho\right)\|v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla u)\\ &\quad + \varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right].\end{aligned} \tag{51}$$

For $0 < a < 1$, from (39),

$$\begin{aligned}&\varepsilon\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right]\\ &\quad= \varepsilon a\left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right] + \varepsilon 2(p+2)(1-a)\mathbb{H}(t)\\ &\qquad + \varepsilon(p+2)(1-a)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon(p+2)(1-a)\\ &\qquad \cdot\left(1-\int_0^t g(s)\,ds\|\nabla u\|_2^2 + \varepsilon(p+2)(1-a)\right.\\ &\qquad \cdot\left(1-\int_0^t h(s)\,ds\|\nabla v\|_2^2 - \varepsilon(p+2)(1-a)(go\nabla u)\right.\\ &\qquad - \varepsilon(p+2)(1-a)(ho\nabla v) + \varepsilon(p+2)(1-a)K(y,z),\end{aligned} \tag{52}$$

substituting in (51), we get

$$\begin{aligned}\mathscr{K}'(t) \geq & [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) \\ & + \varepsilon[(p+2)(1-\alpha)+1]\left(\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_2^2+\|v\|_2^2\right) \\ & + \varepsilon\left[(p+2)(1-\alpha)\left(1-\int_0^t g(s)ds\right)\right. \\ & \left. -\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2 \\ & + \varepsilon\left[(p+2)(1-\alpha)\left(1-\int_0^t h(s)ds\right)\right. \\ & \left. -\left(1-\frac{1}{2}\int_0^t h(s)ds\right)\right]\|\nabla v\|_2^2 \\ & - \varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\|u\|_2^2 \\ & - \varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\|v\|_2^2 \\ & + \varepsilon(p+2)(1-\alpha)K(y,z) \\ & + \varepsilon\left[(p+2)(1-\alpha)-\frac{1}{2}\right](go\nabla u+ho\nabla v) \\ & + \varepsilon a\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right] \\ & + \varepsilon 2(p+2)(1-\alpha)\mathbb{H}(t).\end{aligned} \tag{53}$$

Since (25) hold, we obtain by using (44) and (46)

$$\mathbb{H}^{\alpha}(t)\|u\|_2^2 \leq c_4\left(\|u\|_{2(p+2)}^{2\alpha(p+2)+2}+\|v\|_{2(p+2)}^{2\alpha(p+2)+2}\|u\|_2^2\right), \tag{54}$$

$$\mathbb{H}^{\alpha}(t)\|v\|_2^2 \leq c_5\left(\|v\|_{2(p+2)}^{2\alpha(p+2)+2}+\|u\|_{2(p+2)}^{2\alpha(p+2)+2}\|v\|_2^2\right), \tag{55}$$

for some positive constants c_4, c_5. By using (46) and the algebraic inequality,

$$B^{\theta} \leq (B+1) \leq \left(1+\frac{1}{b}\right)(B+b), \quad \forall B>0, \quad 0<\theta<1, \quad b>0, \tag{56}$$

we have, $\forall t>0$

$$\|u\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\mathbb{H}(0)\right) \leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\mathbb{H}(t)\right), \tag{57}$$

$$\|v\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d\left(\|v\|_{2(p+2)}^{2(p+2)}+\mathbb{H}(t)\right) \leq d\left(\|v\|_{2(p+2)}^{2(p+2)}+\mathbb{H}(t)\right), \tag{58}$$

where $d = 1 + (1/\mathbb{H}(0))$. Also, since

$$(x+y)^{\gamma} \leq C(x^{\gamma}+y^{\gamma}), \quad \forall x,y>0, \quad \gamma>0, \tag{59}$$

we conclude

$$\begin{aligned}\|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2 & \leq c_6\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_2^{2(p+2)}\right) \\ & \leq c_7\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}\right),\end{aligned} \tag{60}$$

$$\begin{aligned}\|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2 & \leq c_8\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_2^{2(p+2)}\right) \\ & \leq c_9\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right),\end{aligned} \tag{61}$$

substituting (58) and (61) in (55), we get

$$\mathbb{H}^{\alpha}(t)\|u\|_2^2 \leq c_{10}\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}\right)+c_{10}\mathbb{H}(t), \tag{62}$$

$$\mathbb{H}^{\alpha}(t)\|v\|_2^2 \leq c_{11}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)+c_{11}\mathbb{H}(t), \tag{63}$$

Combining (53) and (63), using (31), we get

$$\begin{aligned}\mathscr{K}'(t) \geq & [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) \\ & + \varepsilon[(p+2)(1-\alpha)+1]\left(\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_2^2+\|v\|_2^2\right) \\ & + \varepsilon\left\{[(p+2)(1-\alpha)-1]-\left(\int_0^t g(s)ds\right)\right. \\ & \left.\cdot\left[(p+2)(1-\alpha)-\frac{1}{2}\right]\right\}\|\nabla u\|_2^2 \\ & + \varepsilon\left\{[(p+2)(1-\alpha)-1]-\left(\int_0^t h(s)ds\right)\right. \\ & \left.\cdot\left[(p+2)(1-\alpha)-\frac{1}{2}\right]\right\}\|\nabla v\|_2^2 \\ & + \varepsilon(p+2)(1-\alpha)K(y,z) \\ & + \varepsilon\left[(p+2)(1-\alpha)-\frac{1}{2}\right](go\nabla u+ho\nabla v) \\ & + \varepsilon\left(c_0 a-\frac{\lambda_1+\lambda_2}{2c_3\kappa}\right)\left[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right] \\ & + \varepsilon\left(2(p+2)(1-\alpha)-\frac{\lambda_1+\lambda_2}{2c_3\kappa}\right)\mathbb{H}(t),\end{aligned} \tag{64}$$

where $\lambda_1 = c_{10}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho$, $\lambda_2 = c_{11}\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho$.

In this case, we take $a>0$ small enough, then

$$\alpha_1 = (p+2)(1-a)-1>0, \tag{65}$$

assuming

$$\begin{aligned}\max\left\{\int_0^{\infty} g(s)ds, \int_0^{\infty} h(s)ds\right\} & < \frac{(p+2)(1-a)-1}{((p+2)(1-a)-(1/2))} \\ & = \frac{2\alpha_1}{2\alpha_1+1},\end{aligned} \tag{66}$$

we have

$$\alpha_2 = \left\{(p+2)(1-a)-1) - \int_0^t g(s)ds\left((p+2)(1-a)-\frac{1}{2}\right)\right\} > 0,$$
$$\alpha_3 = \left\{(p+2)(1-a)-1) - \int_0^t h(s)ds\left((p+2)(1-a)-\frac{1}{2}\right)\right\} > 0, \quad (67)$$

choose κ so large that

$$\alpha_4 = ac_0 - \frac{\lambda_1+\lambda_2}{2c_3\kappa} > 0,$$
$$\alpha_5 = 2(p+2)(1-a) - \frac{\lambda_1+\lambda_2}{2c_3\kappa} > 0, \quad (68)$$

fix κ and a, we appoint ε small enough so that

$$\alpha_6 = (1-\alpha) - \varepsilon\kappa > 0. \quad (69)$$

Then, for $\beta > 0$, we estimate (64) and it becomes

$$\begin{aligned}\mathscr{K}'(t) \ge \beta\Big\{ &\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 \\ &+ \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) + K(y,z) \\ &+ \left[\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}\right]\Big\}.\end{aligned} \quad (70)$$

By (31), for $\beta_1 > 0$, we get

$$\begin{aligned}\mathscr{K}'(t) \ge \beta_1\Big\{ &\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 \\ &+ \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) + K(y,z) \\ &+ \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right]\Big\}, \\ \mathscr{K}(t) \ge \mathscr{K}&(0) > 0, t > 0.\end{aligned} \quad (71)$$

Using Holder's and Young's inequalities, we have

$$\begin{aligned}\left|\int_\Omega (uu_t + vv_t)dx\right|^{1/(1-\alpha)} \ge C\Big[&\|u\|_{2(p+2)}^{\theta/(1-\alpha)} + \|u_t\|_2^{\mu/(1-\alpha)} \\ &+ \|v\|_{2(p+2)}^{\theta/(1-\alpha)} + \|v_t\|_2^{\mu/(1-\alpha)}\Big],\end{aligned} \quad (72)$$

where $(1/\mu) + (1/\theta) = 1$ put $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \le 2(p+2). \quad (73)$$

Subsequently, for $s = 2/(1-2\alpha)$ and by using (39), we get

$$\begin{aligned}&\|u\|_{2(p+2)}^{2/(1-2\alpha)} \le d\left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\right), \\ &\|v\|_{2(p+2)}^{2/(1-2\alpha)} \le d\left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\right), \quad \forall t \ge 0.\end{aligned} \quad (74)$$

Therefore,

$$\begin{aligned}&\left|\int_\Omega (uu_t + vv_t)dx\right|^{1/(1-\alpha)} \\ &\quad \ge c_{12}\left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + \mathbb{H}(t)\right].\end{aligned} \quad (75)$$

Subsequently,

$$\begin{aligned}\mathscr{K}^{1/(1-\alpha)}(t) &= \Big(\mathbb{H}^{1-\alpha} + \varepsilon\int_\Omega (uu_t + vv_t)dx \\ &\quad + \frac{\varepsilon}{2}\int_\Omega (\mu_1 u^2 + \mu_3 v^2)dx \\ &\quad + \frac{\varepsilon}{2}\int_\Omega (\omega_1 \nabla u^2 + \omega_2 \nabla v^2)dx\Big)^{1/(1-\alpha)} \\ &\le c\Big\{\mathbb{H}(t)\left|\int_\Omega (uu_t + vv_t)dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)} \\ &\quad + \|\nabla u\|_2^{2/(1-\alpha)} + \|v\|_2^{2/(1-\alpha)} + \|\nabla v\|_2^{2/(1-\alpha)}\Big\} \\ &\le c\Big[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 \\ &\quad + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) + (ho\nabla v) \\ &\quad + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\Big].\end{aligned} \quad (76)$$

From (70) and (76), gives

$$\mathscr{K}'(t) \ge \lambda \mathscr{K}^{\frac{1}{1-\alpha}}(t), \quad (77)$$

with $\lambda > 0$, this quantity depends on β and c. By simple integration of (77), we obtain

$$\mathscr{K}^{\alpha/(1-\alpha)}(t) \ge \frac{1}{\mathscr{K}^{-\alpha/(1-\alpha)}(0) - \lambda(\alpha/(1-\alpha))t}, \quad (78)$$

Hence, $\mathscr{K}(t)$ in a situation of blow up in time, when

$$T \le T^* = \frac{1-\alpha}{\lambda\alpha\mathscr{K}^{\alpha}/^{(1-\alpha)}(0)}, \quad (79)$$

Then, this completes the proof of the theorem.

4. Conclusion

In this work, we have studied the blow up of the coupled Klein-Gordon system with strong damping, distributed delay, and source terms, under suitable conditions which are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades (see, for example, [17–19]). In the next work, we will try to apply the same technique with a new class of Boussinesq equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see, for example, [20, 21]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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