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Research Article

A Note on the Poly-Bernoulli Polynomials of the Second Kind

Sang Jo Yun[1] and Jin-Woo Park[2]

[1]*Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea*
[2]*Department of Mathematics Education, Daegu University, 38453, Republic of Korea*

Correspondence should be addressed to Jin-Woo Park; a0417001@knu.ac.kr

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In this paper, we define the poly-Bernoulli polynomials of the second kind by using the polyexponential function and find some interesting identities of those polynomials. In addition, we define unipoly-Bernoulli polynomials of the second kind and study some properties of those polynomials.

1. Introduction

In the book *Ars Conjectandi*, Bernoulli introduced the Bernoulli number terms of the sum of powers of consecutive integers (see [1, 2]). In [3], Luo and Srivastava defined the Apostol-Bernoulli polynomials and obtained an explicit series representation for their polynomials involving the Gaussian hypergeometric function as well as an explicit series representation involving the Hurwitz function. Frappier defined a generalized Bernoulli polynomials by using the Bessel function of the first kind and found a generalization of a well-known Fourier series representation of Bernoulli polynomials in [4]. In [5], Natalini and Bernardini defined a new class of generalized Bernoulli polynomials and showed that if a differential equation with these polynomials is of order n, then all the considered families of polynomials can be viewed as solutions of differential operators of infinite order. In [6], Kaneko defined the poly-Bernoulli polynomials and found an explicit formula and a duality theorem for those numbers. Khan et al. defined Laguerre-based Hermite-Bernoulli polynomials and derived summation formulas and related bilateral series associated with the newly introduced generating function in [7]. In [8], Jang and Kim defined type 2 degenerate Bernoulli polynomials and showed that these polynomials could be represented linear combinations of the Stirling numbers of the first and the second kinds, Bernoulli polynomials, and those numbers. Moreover, in [9], the degenerate type 2 poly-Bernoulli numbers and polynomials as degenerate versions of such numbers and polynomials were defined, and several explicit expressions and some identities for those numbers and polynomials were derived.

As is well known, *Bernoulli polynomials of order r* are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \tag{1}$$

(see [1, 5, 9, 10]).

In particular, if $r=1$, $B_n(x)=B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials. When $x=0$, $B_n^{(r)}=B_n^{(r)}(0)$ are called the *Bernoulli numbers of order r*. In [1], the relationship between the Bernoulli numbers and zeta functions was studied, and in [2, 8, 10–12], generalized Bernoulli numbers were defined, and the properties of those numbers and polynomials were investigated.

The *Bernoulli polynomials of the second kind* (or the Cauchy polynomials) are defined by the generating function to be

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x \tag{2}$$

(see [13–15]).

When $x=0$, $b_n=b_n(0)$ are called the *Bernoulli numbers of the second kind*.

For a nonnegative integer n, the *Stirling numbers of the first kind* are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^i \tag{3}$$

(see [16–18]), where $(x)_0 = x$, $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 0)$. By the direct computation of (3), we derive the following:

$$\frac{1}{n!}(\log(1+t))^n = \sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!} \tag{4}$$

(see [16–19]).

For a given nonnegative integer n, the *Stirling numbers of the second kind* are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \tag{5}$$

(see [16–18]).

By (5), we obtain

$$\left(e^t - 1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \tag{6}$$

(see [16–19]).

In [17, 19], the authors defined the generalized Stirling numbers of the first and second kinds and generalized binomial coefficients and showed that degenerated special polynomials are represented by linear combinations of those numbers.

The polyexponential function was first studied by Hardy (see [11, 20]), and Kim and Kim defined polyexponential function as an inverse to the polylogarithm function $Li_k(x) = \sum_{n=1}^{\infty}(x^n/n!)$ (see [6, 11, 20, 21]), to be

$$e_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!} \quad (k \in \mathbb{Z}) \tag{7}$$

(see [21]).

By (7), we know that $e_1(x) = e^x$.

Recently, some authors applied the polyexponential functions and the polylogarithm functions to degenerate Bernoulli polynomials, type 2 poly-Apostol-Bernoulli polynomials, type 2 degenerate poly-Euler polynomials, and poly-Genocchi polynomials and found many interesting identities about those polynomials (see [11, 12, 20–26]).

In this paper, we define poly-Bernoulli polynomials of the second kind with the polyexponential function and derive some interesting identities between the Stirling numbers of the first kind or the second kind, Bernoulli numbers, Bernoulli numbers of the second kind, and those polynomials. In addition, we define unipoly-Bernoulli polynomials of the second kind and derive some interesting identities of those polynomials.

2. The Poly-Bernoulli Polynomials of the Second Kind

By the definition of the Bernoulli polynomials of the second kind and (7), we define the *poly-Bernoulli polynomials of the second kind* by the generating function to be

$$\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} = \frac{e_k(\log(1+t))}{\log(1+t)}(1+t)^x. \tag{8}$$

In particular, if $x = 0$, $b_n^{(k)} = b_n^{(k)}(0)$ are called the *poly-Bernoulli numbers of the second kind*. By (8), we know that for each nonnegative integer n,

$$b_n^{(1)}(x) = b_n(x) \tag{9}$$

are the Bernoulli polynomials of the second kind.

Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} &= \frac{e_k(\log(1+t))}{\log(1+t)}(1+t)^x \\
&= \left(\sum_{l=0}^{\infty} b_l^{(k)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}t^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{x}{m}\binom{n}{m}m!b_{n-m}^{(k)}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{x}{n-l}\binom{n}{l}(n-l)!b_l^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{10}$$

Hence, by (10), we obtain the following theorem.

Theorem 1. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$b_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}b_l^{(k)}(x)_{n-l}, \tag{11}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$ is the k-falling factorial.

By replacing t by $e^t - 1$ in (8), we get

$$\begin{aligned}
\frac{e_k(t)}{t}e^{xt} &= \left(\frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{x^{n-m}}{(m+1)^k}\right)\frac{t^n}{n!},
\end{aligned} \tag{12}$$

and by (6), we have

$$\sum_{n=0}^{\infty} b_n^{(k)} \frac{1}{n!} \left(e^t - 1\right)^n = \sum_{n=0}^{\infty} b_n^{(k)}(x) \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)}(x) \right) \frac{t^n}{n!}. \tag{13}$$

Therefore, by (12) and (13), we obtain the following theorem.

Theorem 2. *For each nonnegative integer n, we have*

$$\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{x^{n-m}}{(m+1)^k}. \tag{14}$$

In particular, we have

$$\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)} = \frac{1}{(n+1)^k}. \tag{15}$$

From (4) and (8), we derive

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!} &= \frac{e_k(\log(1+t))}{\log(1+t)} = \frac{1}{\log(1+t)} \sum_{n=1}^{\infty} \frac{(\log(1+t))^n}{(n-1)!n^k} \\
&= \frac{1}{\log(1+t)} \sum_{n=0}^{\infty} \frac{(\log(1+t))^{n+1}}{n!(n+1)^k} \\
&= \frac{1}{\log(1+t)} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k-1}} \sum_{l=n+1}^{\infty} S_1(l,n+1) \frac{t^l}{l!} \\
&= \frac{t}{\log(1+t)} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k-1}} \sum_{l=n}^{\infty} \frac{S_1(l+1,n+1)}{l+1} \frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{1}{(l+1)^{k-1}} \frac{S_1(m+1,l+1)}{m+1} \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{b_{n-m}}{(l+1)^{k-1}} \frac{S_1(m+1,l+1)}{m+1} \right) \frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Thus, by (16), we obtain the following theorem.

Theorem 3. *For each $k \in \mathbb{Z}$ and each nonnegative integer n, we have*

$$b_n^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1,l+1)}{(l+1)^{k-1}(m+1)} b_{n-m}. \tag{17}$$

By (9) and Theorem 3, we get the following corollary.

Corollary 4. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$b_n = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1,l+1)}{m+1} b_{n-m}. \tag{18}$$

In Corollary 4, we have

$$\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1,l+1)}{(m+1)} b_{n-m} = b_n + \sum_{m=1}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1,l+1)}{(m+1)} b_{n-m}. \tag{19}$$

Therefore, we obtain the following corollary.

Corollary 5. *For each positive integer n, we have*

$$\sum_{m=1}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1,l+1)}{(m+1)} b_{n-m} = 0 \; (n \in \mathbb{N}). \tag{20}$$

In [21], the authors showed that

$$\frac{d}{dx} e_k(x) = \frac{1}{x} e_{k-1}(x) \; (k \geq 2). \tag{21}$$

From (21), we have

$$e_k(x) = \underbrace{\int_0^x \frac{1}{t} \int_0^t \frac{1}{t} \cdots \int_0^t \frac{1}{t}}_{(k-2)\text{-times}} \left(e^t - 1\right) dt dt \cdots dt \tag{22}$$

(see [9, 11, 12, 19–21, 23, 25]).

By (22), we can derive the following equations:

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(k)} \frac{x^n}{n!} &= \frac{1}{\log(1+x)} e_k(\log(1+t)) \\
&= \frac{1}{\log(1+x)} \int_0^x \frac{1}{(1+t)\log(1+t)} e_{k-1}(\log(1+t)) dt \\
&= \frac{1}{\log(1+x)} \int_0^x \frac{1}{(1+t)\log(1+t)} \\
&\quad \underbrace{\int_0^t \frac{1}{(1+t)\log(1+t)} \cdots \int_0^t \frac{t}{(1+t)\log(1+t)}}_{(k-2)\text{-times}} dt dt \cdots dt \; (k \geq 2).
\end{aligned} \tag{23}$$

It is well known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} \tag{24}$$

(see [9, 12, 23, 25]).

In particular, if we put $k=2$ in (23), then by (23) and (24), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(2)} \frac{x^n}{n!} &= \frac{1}{\log(1+x)} e_2(\log(1+t)) \\ &= \frac{1}{\log(1+x)} \int_0^x \frac{1}{(1+t)\log(1+t)} dt \\ &= \frac{1}{\log(1+x)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l!} \int_0^x t^l dt = \frac{x}{\log(1+x)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!} \\ &= \left(\sum_{m=0}^{\infty} b_m \frac{x^m}{m!}\right)\left(\sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{B_l^{(l)}}{l+1} b_{n-l}\right) \frac{x^n}{n!}.\end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 6. *For a nonnegative integer n, we have*

$$b_n^{(2)} = \sum_{l=0}^{n} \binom{n}{l} \frac{B_l^{(l)}}{l+1} b_{n-l}. \tag{26}$$

3. The Unipoly-Bernoulli Polynomials of the Second Kind

Let p be an arithmetic function which is a real or complex valued function defined on $\mathbb{N}$. In [21], Kim and Kim defined the unipoly function attached to polynomial $p(x)$ by

$$u_k(x \mid p) = \sum_{n=1}^{\infty} \frac{p(n)x^n}{n^k} \quad (k \in \mathbb{Z}). \tag{27}$$

In particular, if $p(x)=1$, then

$$u_k(x \mid 1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = Li_k(x) \tag{28}$$

is an ordinary polylogarithm function.

Note that by (27), we get

$$\frac{d}{dx} u_k(x \mid p) = \frac{1}{x} u_{k-1}(x \mid p), \tag{29}$$

for $k \geq 2$. In addition, it is well known that

$$u_k(x \mid p) = \int_0^x \frac{1}{t} \underbrace{\int_0^t \frac{1}{t} \cdots \int_0^t \frac{1}{t}}_{(k-2)\text{-times}} u_1(t \mid p) dt dt \cdots dt \tag{30}$$

(see [9, 11, 12, 19, 21, 23, 25]).

In the viewpoint of (8), we define the *unipoly-Bernoulli polynomials of the second kind* as

$$\frac{u_k(\log(1+t)|p)}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{t^n}{n!}. \tag{31}$$

From (31), we derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{u_k(\log(1+t)|p)}{\log(1+t)} (1+t)^x \\ &= \left(\sum_{n=0}^{\infty} b_{n,p}^{(k)} \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (x)_l b_{n-l,p}^{(k)}\right) \frac{t^n}{n!},\end{aligned} \tag{32}$$

and thus, by (32), we obtain the following theorem.

Theorem 7. *For each nonnegative integer n, we have*

$$b_{n,p}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} b_{n-l,p}^{(k)} (x)_l. \tag{33}$$

If we put $p(n) = 1/\Gamma(n)$, then by (31), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{1}{\log(1+t)} u_k\left(\log(1+t) \middle| \frac{1}{\Gamma}\right) (1+t)^x \\ &= \frac{1}{\log(1+t)} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k (m-1)!} (1+t)^x \\ &= \frac{e_k(\log(1+t))}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x) \frac{t^n}{n!}.\end{aligned} \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 8. *For a nonnegative integer n, if $p(n) = 1/\Gamma(n)$, then we have*

$$b_{n,p}^{(k)}(x) = b_n^{(k)}(x). \tag{35}$$

In the definition of unipoly-Bernoulli polynomials of the second kind, if $x=0$, then we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)} \frac{t^n}{n!} &= \frac{1}{\log(1+t)} \sum_{n=1}^{\infty} \frac{p(n)}{n^k} (\log(1+t))^n \\ &= \frac{1}{\log(1+t)} \sum_{n=0}^{\infty} \frac{p(n+1)}{(n+1)^k} (n+1)! \sum_{l=n+1}^{\infty} S_1(l, n+1) \frac{t^l}{l!} \\ &= \left(\frac{t}{\log(1+t)}\right) \left(\sum_{n=0}^{\infty} \frac{p(n+1)}{(n+1)^k} (n+1)! \sum_{l=0}^{\infty} S_1 \right. \\ &\quad \left. \cdot (n+1+l, n+1) \frac{t^n}{(n+1+l)!}\right) \\ &= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{p(m+1)(m+1)!}{(m+1)^k} S_1 \right. \\ &\quad \left. \cdot (n+1, m+1) \frac{t^n}{(n+1)!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{p(l+1)(l+1)!}{(l+1)^k} \frac{S_1(m+1, l+1)}{m+1} b_{n-m}\right) \frac{t^n}{n!}.\end{aligned} \tag{36}$$

Hence, by (34), we obtain the following theorem.

Theorem 9. *For each nonnegative integer n and each integer k , we have*

$$b_{n,p}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{p(l+1)(l+1)!}{(l+1)^k} \frac{S_1(m+1, l+1)}{m+1} b_{n-m}. \tag{37}$$

In particular, we have

$$b_{n,1/\Gamma}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1, n+1)}{(l+1)^{k-1}(m+1)} b_{n-m}. \tag{38}$$

By replacing t by $e^t - 1$ in (31), we have

$$\begin{aligned}\frac{u_k(t \mid p)}{t} e^{xt} &= \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{p(n) t^n}{n^k}\right) e^{xt} \\ &= \left(\sum_{n=0}^{\infty} \frac{p(n+1) t^n}{(n+1)^k}\right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{p(l+1)}{(l+1)^k} \frac{n!}{(n-l)!} x^{n-l}\right) \frac{t^n}{n!},\end{aligned} \tag{39}$$

and by (6), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{1}{n!} (e^t - 1)^n &= \sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)}(x)\right) \frac{t^n}{n!}.\end{aligned} \tag{40}$$

By (39) and (40), we obtain the following theorem.

Theorem 10. *For each nonnegative integer n and each arithmetic function $p(n)$, we have*

$$\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)}(x) = \sum_{l=0}^{n} \frac{p(l+1)}{(l+1)^k} (n)_l x^{n-l}. \tag{41}$$

In particular, we have

$$\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)} = \frac{p(n+1)}{(n+1)^k} n!. \tag{42}$$

4. Conclusion

The polyexponential function was first studied by Hardy. In [21], Kim and Kim modified that function which was again called the polyexponential functions as an inverse to the polylogarithm function. In addition, they defied the unipoly function, attached a arithmetic function p, and found some interesting identities related to Bernoulli numbers, poly-Bernoulli polynomials, and the Stirling numbers of the first kind and second kind. The polyexponential function have been used to define some special polynomials by some researcher and found many interesting identities of those polynomials (see [11, 12, 20–26]).

In this paper, we defined the poly-Bernoulli polynomials of the second kind by using the polyexponential function and found some interesting identities.

In addition, we also define the unipoly-Bernoulli polynomials of the second kind and found some identities which were related to poly-Bernoulli polynomials of the second kind, Bernoulli polynomials, and the Stirling numbers of the first and second kind.

Data Availability

No data was used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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