Research Article

The Growth of Entire Functions Defined by Laplace–Stieltjes Transforms Related to Proximate Order and Approximation

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In this article, we discuss the growth of entire functions represented by Laplace–Stieltjes transform converges on the whole complex plane and obtain some equivalence conditions about proximate growth of Laplace–Stieltjes transforms with finite order and infinite order. In addition, we also investigate the approximation of Laplace–Stieltjes transform with the proximate order and obtain some results containing the proximate growth order, the error, A_n^* , and λ_n , which are the extension and improvement of the previous theorems given by Luo and Kong and Singhal and Srivastava.

1. Introduction

Our main aim of this paper is to investigate some problems about the growth and approximation of entire functions represented by Laplace–Stieltjes transforms which converge on the whole complex plane. Consider Laplace–Stieltjes transforms

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \quad s = \sigma + it, \quad \sigma, t \in R, \quad (1)$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y]$ ($0 < Y < +\infty$). For Laplace–Stieltjes transform (1), set

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x e^{ity} d\alpha(y) \right|, \quad (2)$$

where the sequence $\{\lambda_n\}_0^\infty$ satisfies

$$0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots, \lambda_n \longrightarrow \infty \text{ as } n \longrightarrow \infty \quad (3)$$

if

$$\limsup_{n \rightarrow +\infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \quad (4)$$

$$\limsup_{n \rightarrow +\infty} \frac{\log n}{\lambda_n} = D < +\infty, \quad (5)$$

$$\limsup_{n \rightarrow +\infty} (\lambda_{n+1} - \lambda_n) = h < +\infty,$$

similar to the method in [1], and by using the Valiron–Knopp–Bohr formula, then it yields $\sigma_u^F = +\infty$, that is, $F(s)$ is an entire function in the whole plane. We denote $\bar{L}_\beta$ to be a class of all the functions $F(s)$ of form (1) which are analytic in the half plane $\Re s < \beta$ ($-\infty < \beta < \infty$), and the sequence $\{\lambda_n\}$ satisfies (3) and (5), and denote L_∞ to be the class of all the functions $F(s)$ of form (1) which are analytic in the whole plane $\Re s < +\infty$, and the sequence $\{\lambda_n\}$ satisfies (3), (5), and (4). Thus, if $-\infty < \beta < +\infty$ and $F(s) \in L_\infty$, then $F(s) \in \bar{L}_\beta$.

By Widder [2], if $\alpha(t)$ is absolutely continuous, then $F(s)$ is the classical Laplace integral form:

$$F(s) = \int_0^\infty e^{st} g(t) dt \quad (6)$$

if $\alpha(t)$ is a step-function and a sequence $\{\lambda_n\}_0^\infty$ satisfies (3), and

$$\alpha(x) = \begin{cases} a_1 + a_2 + \cdots + a_n, & \lambda_n \leq x < \lambda_{n+1}; \\ 0, & x = 0, \end{cases} \quad (7)$$

then $F(s)$ becomes a Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \quad \sigma, t \in \mathbb{R}, \quad (8)$$

where a_n are nonzero complex numbers; if $\alpha(t)$ is an increasing continuous function which is not absolutely continuous, then integral (1) defines a class of functions $F(s)$ which cannot be expressed either in form (6) or (8) (see [2]).

In 1963, Yu [1] first proved the Valiron–Knopp–Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace–Stieltjes transform when $s = -z$.

Theorem 1. *If the sequence $\{\lambda_n\}$ satisfies (3) and (5), then Laplace–Stieltjes transform (1) satisfies*

$$\limsup_{n \rightarrow +\infty} \frac{\log B_n^*}{\lambda_n} \leq \sigma_u^F \leq \limsup_{n \rightarrow +\infty} \frac{\log B_n^*}{\lambda_n} + \limsup_{n \rightarrow +\infty} \frac{\log n}{\lambda_n}, \quad (9)$$

where σ_u^F is called the abscissa of uniformly convergence of $F(s)$, and

$$B_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x e^{-ity} d\alpha(y) \right|. \quad (10)$$

In addition, Yu [1] introduced the concept of the order of $G(z) = F(-s)$ to estimate the growth of the maximal molecule $M_u(\sigma, G)$ and the maximal term $\mu(\sigma, G)$ and gave the relation between the Borel line and the order of entire functions represented by Laplace–Stieltjes transform converge in the whole complex plane. After his wonderful works, considerable attention has been paid to the growth, and the value distribution of the functions represented by Laplace–Stieltjes transform or Dirichlet series converges in the half plane or the whole complex plane in the field of complex analysis (see [3–24]).

Set

$$\mu(\sigma, F) = \max_{n \in \mathbb{N}} \{A_n^* e^{\lambda_n \sigma}\} (\sigma < +\infty),$$

$$M_u(\sigma, F) = \sup_{0 < x < +\infty, -\infty < t < +\infty} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right|. \quad (11)$$

In view of $M_u(\sigma, F) \rightarrow +\infty$ as $\sigma \rightarrow +\infty$, in order to estimate the growth of $F(s)$ precisely, we usual use the concepts of order and type as follows.

Definition 1. If $F(s) \in L_\infty$ and

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{\sigma} = \rho, \quad 0 \leq \rho \leq +\infty, \quad (12)$$

then it is said that $F(s)$ is of order ρ in the whole plane, where $\log^+ x = \max\{\log x, 0\}$. Furthermore, if $\rho \in (0, +\infty)$, the type of $F(s)$ is defined by

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma \rho}} = T, \quad 0 \leq T \leq +\infty. \quad (13)$$

Let $F(s)$ be of order ρ in the whole plane and $0 < \rho < +\infty$, and let $\rho(\sigma) (\sigma > \sigma_0)$ be a nonnegative, continuous, and monotonous function, and it has a left-hand derivative and right-hand derivative in every $\sigma (> \sigma_0)$ such that

$$\begin{aligned} \lim_{\sigma \rightarrow +\infty} \rho(\sigma) &= \rho, \\ \lim_{\sigma \rightarrow +\infty} \sigma \rho'(\sigma) &= 0, \end{aligned} \quad (14)$$

then $\rho(\sigma)$ is called the proximate order; furthermore, if

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma \rho(\sigma)}} = T, \quad (15)$$

then T will be called the proximate type of $F(s)$ with respect to proximate order $\rho(\sigma)$.

Set $r = e^\sigma$, and let $t = \psi(r) = r^\rho (\log r) = e^{\sigma \rho(\sigma)}$ and $r = \varphi(t)$ be two reciprocally inverse functions. Then, we obtain the following theorem.

Theorem 2. *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of proximate order $\rho(\sigma)$ and order $\rho (0 < \rho < +\infty)$, then*

$$\tau \leq T \leq e^{D\rho} \tau, \quad (16)$$

where

$$\tau = \limsup_{n \rightarrow +\infty} \frac{1}{\rho e} \left[\varphi(\lambda_n) (A_n^*)^{1/\lambda_n} \right]^\rho. \quad (17)$$

Furthermore, if the sequence $\{\lambda_n\}$ satisfies

$$\liminf_{n \rightarrow +\infty} (\lambda_{n+1} - \lambda_n) = h' > 0, \quad (18)$$

then

$$T = \tau = \limsup_{n \rightarrow +\infty} \frac{1}{\rho e} \left[\varphi(\lambda_n) (A_n^*)^{1/\lambda_n} \right]^\rho. \quad (19)$$

In order to estimate the growth of Laplace–Stieltjes transform of order $\rho = +\infty$ precisely, we will use some concepts of the p -order as follows.

Definition 2. If $F(s) \in L_\infty$, $p \geq 2$ is a positive integer, and

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log_{p+1} M_u(\sigma, F)}{\sigma} = \rho_p, \quad 0 \leq \rho(p) \leq +\infty, \quad (20)$$

then we call $F(s)$ is of p -order ρ_p in the whole plane, where $\log_{p+1} x = \log_p \log x$ for $p \in \mathbb{N}_+$.

Let $F(s)$ be of p -order ρ_p in the whole plane and $0 < \rho_p < +\infty$, similar to the proximate type of $F(s)$ with respect to proximate order $\rho(\sigma)$; we can give the proximate type of $F(s)$ with respect to proximate order of p -order ρ_p as follows. Let $\rho_p(\sigma) (\sigma > \sigma_0)$ be a nonnegative, continuous, and monotonous function, and it has a left-hand derivative and right-hand derivative in every $\sigma (> \sigma_0)$ such that

$$\begin{aligned}\lim_{\sigma \rightarrow +\infty} \rho_p(\sigma) &= \rho_p, \\ \lim_{\sigma \rightarrow +\infty} \sigma \rho'_p(\sigma) &= 0,\end{aligned}\quad (21)$$

then $\rho_p(\sigma)$ is called the p -proximate order; furthermore, if

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log_p M_u(\sigma, F)}{e^{\sigma \rho_p(\sigma)}} = T_p, \quad (22)$$

then T_p will be called the proximate type of $F(s)$ with respect to p -proximate order $\rho_p(\sigma)$.

Set $r = e^\sigma$, and let $t = \psi_p(r) = r^{\rho_p(\log r)} = e^{\sigma \rho_p(\sigma)}$ and $r = \varphi_p(t)$ be two reciprocally inverse functions. Then, we obtain the following theorem.

Theorem 3. *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of p -proximate order $\rho_p(\sigma)$ and $\rho_p(0 < \rho_p < +\infty)$, $p \geq 2$, then*

$$\tau_p \leq T_p \leq e^{D_p} \tau_p, \quad (23)$$

where

$$\tau_p = \limsup_{n \rightarrow +\infty} \left[\varphi_p(\log_{p-1} \lambda_n) (A_n^*)^{1/\lambda_n} \right]^{\rho_p}. \quad (24)$$

Furthermore, if the sequence λ_n satisfies (18), then

$$T_p = \tau_p = \limsup_{n \rightarrow +\infty} \left[\varphi_p(\log_{p-1} \lambda_n) (A_n^*)^{1/\lambda_n} \right]^{\rho_p}. \quad (25)$$

The other purpose of this article is to study the approximation of the entire function represented by Laplace–Stieltjes transform converges in the whole plane. When Laplace–Stieltjes transform (1) satisfies $A_k^* \neq 0$ and $A_n^* = 0$ for $n \geq k+1$, then $F(s)$ will be called an exponential polynomial of degree k usually denoted by p_k , i.e., $p_k(s) = \int_0^{\lambda_k} \exp(sy) d\alpha(y)$. When we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i \exp(s\lambda_i)$. We also use Π_k to denote the class of all exponential polynomial of degree almost k , that is,

$$\Pi_k = \left\{ \sum_{i=1}^k b_i \exp(s\lambda_i) : (b_1, b_2, \dots, b_k) \in \mathbb{C}^k \right\}. \quad (26)$$

For $F(s) \in L_\infty$, $-\infty < \beta < +\infty$, we denote by $E_n(F, \beta)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F, \beta) = \inf_{p \in \Pi_n} \|F - p\|_\beta, \quad n = 1, 2, \dots, \quad (27)$$

where

$$\|F - p\|_\beta = \max_{-\infty < t < +\infty} |F(\beta + it) - p(\beta + it)|. \quad (28)$$

Recently, Singhal and Srivastava [15] and Xu and Liu [25] studied the approximation on Laplace–Stieltjes transforms of finite order and obtained the following theorems.

Theorem 4 (see [15]). *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of order $\rho(0 < \rho < \infty)$ and of type T , then for any real number $-\infty < \beta < +\infty$, we have*

$$\begin{aligned}\rho &= \limsup_{n \rightarrow +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta) \exp(-\beta \lambda_n)} \\ &= \limsup_{n \rightarrow +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta)}, \\ T &= \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\rho e} (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\rho/\lambda_n} \\ &= \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\rho \exp(\rho\beta + 1)} (E_{n-1}(F, \beta))^{\rho/\lambda_n}.\end{aligned}\quad (29)$$

Theorem 5 (see [25], Theorem 2.5). *If the Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of the lower order $\lambda(0 \leq \lambda \neq \rho < \infty)$ and if $\lambda_n \sim \lambda_{n+1}$, then for any real number $-\infty < \beta < +\infty$, we have*

$$\tau_\lambda \geq \liminf_{n \rightarrow +\infty} \left(\frac{\lambda_n}{e\lambda} \right) (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\lambda/\lambda_n}, \quad (0 \leq \tau_\lambda \leq \infty). \quad (30)$$

Furthermore, there exists a positive integer n_0 such that

$$\xi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n} \quad (31)$$

forms a nondecreasing function of n for $n > n_0$, and then we have

$$\tau_\lambda = \liminf_{n \rightarrow +\infty} \left(\frac{\lambda_n}{e\lambda} \right) (E_{n-1}(F, \beta) \exp(-\beta \lambda_n))^{\lambda/\lambda_n} \quad (0 \leq \tau_\lambda \leq \infty), \quad (32)$$

i.e.,

$$\exp(\beta\lambda)e\lambda\tau_\lambda = \liminf_{n \rightarrow +\infty} \lambda_n (E_{n-1}(F, \beta))^{\lambda/\lambda_n}. \quad (33)$$

Corresponding to Theorems 2 and 3, we obtain the following results concerning the approximation.

Theorem 6. *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of proximate order $\rho(\sigma)$ and order $\rho(0 < \rho < +\infty)$, then for any real number $-\infty < \beta < +\infty$, we have*

$$\tau_a \leq T \leq e^{D_p} \tau_a, \quad (34)$$

where

$$\tau_a = \limsup_{n \rightarrow +\infty} \frac{1}{\rho \exp(\rho\beta + 1)} [\varphi(\lambda_n) (E_{n-1}(F, \beta))^{1/\lambda_n}]^\rho. \quad (35)$$

Furthermore, if the sequence λ_n satisfies (18), then

$$T = \tau_a = \limsup_{n \rightarrow +\infty} \frac{1}{\rho \exp(\rho\beta + 1)} [\varphi(\lambda_n) (E_{n-1}(F, \beta))^{1/\lambda_n}]^\rho. \quad (36)$$

Theorem 7. *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of p -proximate order $\rho_p(\sigma)$ and $\rho_p(0 < \rho_p < +\infty)$, $p \geq 2$, then for any real number $-\infty < \beta < +\infty$, we have*

$$\tau_{pa} \leq T_p \leq e^{D\rho_p} \tau_{pa}, \quad (37)$$

where

$$\tau_{pa} = \limsup_{n \rightarrow +\infty} \frac{1}{\exp(\beta\rho_p)} [\varphi_p(\log_{p-1}\lambda_n)(E_{n-1}(F, \beta))^{1/\lambda_n}]^{\rho_p}. \quad (38)$$

Furthermore, if the sequence λ_n satisfies (18), then

$$T_p = \tau_{pa} = \limsup_{n \rightarrow +\infty} \frac{1}{\exp(\beta\rho_p)} [\varphi_p(\log_{p-1}\lambda_n)(E_{n-1}(F, \beta))^{1/\lambda_n}]^{\rho_p}. \quad (39)$$

2. Some Lemmas

Lemma 1. For the functions ψ, φ , and $r = e^\sigma$, we have

$$\lim_{\sigma \rightarrow +\infty} \frac{r\psi'(r)}{\psi(r)} = \rho, \quad (40)$$

$$\lim_{t \rightarrow +\infty} \frac{t\varphi'(t)}{\varphi(t)} = \frac{1}{\rho},$$

$$\lim_{\sigma \rightarrow +\infty} \frac{\psi(kr)}{\psi(r)} = k^\rho, \quad (41)$$

$$\lim_{t \rightarrow +\infty} \frac{\varphi(kt)}{\varphi(t)} = k^{1/\rho}, \quad k > 0.$$

Proof. Since $(t\varphi'(t)/\varphi(t)) = (\psi(r)/r\psi'(r))$ and

$$\lim_{\sigma \rightarrow +\infty} \left(\frac{r\psi'(r)}{\psi(r)} \right) = \lim_{\sigma \rightarrow +\infty} \frac{e^\sigma e^{\sigma\rho(\sigma)} [\rho(\sigma) + \sigma\rho'(\sigma)] e^{-\sigma}}{e^{\sigma\rho(\sigma)}} = \rho, \quad (42)$$

then (40) follows. Moreover,

$$\lim_{\sigma \rightarrow +\infty} \frac{\psi(kr)}{\psi(r)} = \lim_{\sigma \rightarrow +\infty} \frac{k^{\rho(\sigma)} \psi(r)}{\psi(r)} = k^\rho. \quad (43)$$

Next, we prove the last equality. Since $\varphi(t) = e^\sigma$ and $t = e^{\sigma\rho(\sigma)}$, thus, it follows $\sigma \rightarrow +\infty$ as $t \rightarrow +\infty$ and $kt = ke^{\sigma\rho(\sigma)}$, and $\varphi(kt) = k^{(1/\rho(\sigma))} e^\sigma$. Since $\lim_{\sigma \rightarrow +\infty} \rho(\sigma) = \rho$, then it yields

$$\lim_{t \rightarrow +\infty} \frac{\varphi(kt)}{\varphi(t)} = \lim_{\sigma \rightarrow +\infty} \frac{k^{1/\rho(\sigma)} e^\sigma}{e^\sigma} = k^{1/\rho}. \quad (44)$$

Therefore, this completes the proof of Lemma 1. $\square$

Remark 1. Obviously, the functions ψ_p and φ_p have the same conclusions as ψ and φ in Lemma 1.

Lemma 2. Let $a, b > 0$; then, the function

$$h(r) = a\psi(r) - b \log r \quad (45)$$

can obtain the minimum

$$b \left\{ \frac{1}{\rho(1+o(1))} - \log \varphi \left(\frac{b}{a\rho(1+o(1))} \right) \right\} \quad \text{in } r = \varphi \left(\frac{b}{a\rho(1+o(1))} \right). \quad (46)$$

Proof. Let

$$h'(r) = a\psi'(r) - \frac{b}{r} = \frac{1}{r} (ar\psi'(r) - b) = 0. \quad (47)$$

Then, it means $ar\psi'(r) - b = 0$. In view of the first equality in (40), we deduce $b = a\rho(1+o(1))\psi(r)$, that is,

$$r = \varphi \left(\frac{b}{a\rho(1+o(1))} \right), \quad \text{as } b \rightarrow +\infty. \quad (48)$$

We can see that, as the value of r increases to the given value above, the value of $h(r)$ changes from a negative value to a positive value. Thus, $h(r)$ can obtain minimum

$$b \left\{ \frac{1}{\rho(1+o(1))} - \log \varphi \left(\frac{b}{a\rho(1+o(1))} \right) \right\}, \quad \text{in } r = \varphi \left(\frac{b}{a\rho(1+o(1))} \right). \quad (49)$$

Therefore, this completes the proof of Lemma 2. $\square$

Lemma 3. Let $a > 0$; then, the function

$$g(t) = at - t \log \varphi(t) \quad (50)$$

can obtain the maximum

$$\frac{1}{\rho} (1+o(1)) \psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right), \quad \text{in } t = \psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right) (a \rightarrow +\infty). \quad (51)$$

Proof. Let

$$g'(t) = a - \log \varphi(t) - \frac{t\varphi'(t)}{\varphi(t)} = 0. \quad (52)$$

Then, it follows from (40) that $a - \log \varphi(t) - 1/\rho(1+o(1)) = 0$. In view of the first equality in (40), we deduce $\varphi(t) = (e^a/e^{(1+o(1))/\rho})$, that is,

$$t = \psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right), \quad \text{as } a \rightarrow +\infty. \quad (53)$$

It is easy to see that $g(t)$ obtain the maximum

$$\psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right) \left[a - a + \frac{1}{\rho} (1+o(1)) \right] = \frac{1}{\rho} (1+o(1)) \psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right), \quad (a \rightarrow +\infty), \quad \text{in } t = \psi \left(\frac{e^a}{e^{(1+o(1))/\rho}} \right). \quad (54)$$

Therefore, this completes the proof of Lemma 3. $\square$

Lemma 4. If Laplace–Stieltjes transform $F(s) \in L_\infty$, for any $\sigma (0 < \sigma < +\infty)$ and $\varepsilon (> 0)$, we have

$$\frac{1}{2} \mu(\sigma, F) \leq M_u(\sigma, F) \leq Ce^{K\sigma} \mu(\sigma + D + \varepsilon, F), \quad (55)$$

where C and K are constants.

Proof. We will adapt the method as in Yu [1] and Kong and Hong [7]. Set

$$I(x; \sigma + it) = \int_0^x \exp\{(\sigma + it)y\} d\alpha(y). \quad (56)$$

For $x > \lambda_n$, it follows

$$\begin{aligned} \int_{\lambda_n}^x \exp\{it y\} d\alpha(y) &= \int_{\lambda_n}^x \exp\{-\sigma y\} d_y I(y; \sigma + it) \\ &= I(y; \sigma + it) \exp\{-\sigma y\} \Big|_{\lambda_n}^x \\ &\quad + \sigma \int_{\lambda_n}^x \exp\{-\sigma y\} I(y; \sigma + it) dy. \end{aligned} \quad (57)$$

Then, for any $\sigma (> 0)$ and any $x \in (\lambda_n, \lambda_{n+1}]$, it yields

$$\begin{aligned} \left| \int_{\lambda_n}^x \exp\{it y\} d\alpha(y) \right| &\leq 2M_u(\sigma, F) |\exp\{-\sigma \lambda_n\} - \exp\{-x\sigma\}| \\ &\leq 2M_u(\sigma, F) \exp\{-\lambda_n \sigma\}, \end{aligned} \quad (58)$$

which implies

$$\frac{1}{2} \mu(\sigma, F) \leq M_u(\sigma, F). \quad (59)$$

On the contrary, for any $x > 0$, it follows that there exists a positive integer $n \in \mathbb{N}_+$ such that $\lambda_n < x \leq \lambda_{n+1}$. Thus, it follows

$$\begin{aligned} \int_0^x \exp\{(\sigma + it)y\} d\alpha(y) &= \sum_{k=1}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} \exp\{(\sigma + it)y\} d\alpha(y) \\ &\quad + \int_{\lambda_n}^x \exp\{(\sigma + it)y\} d\alpha(y). \end{aligned} \quad (60)$$

Set $I_k(x; it) = \int_{\lambda_n}^x \exp\{it y\} d\alpha(y)$, ($\lambda_k < x \leq \lambda_{k+1}$); then, for any real number t and $\sigma > 0$, it follows $|I_k(x; it)| \leq A_k^* \leq \mu(\sigma, F) e^{-\lambda_k \sigma}$. Thus, for any $\varepsilon \in (0, 1)$ and $\sigma > 0$, it yields $|I_k(x; it)| \leq \mu(\sigma + D + \varepsilon, F) e^{-\lambda_k(\sigma + D + \varepsilon)}$ and

$$\begin{aligned} \int_0^x \exp\{(\sigma + it)y\} d\alpha(y) &= \sum_{k=1}^{n-1} \left[\exp\{\lambda_{k+1} \sigma\} I_k(\lambda_{k+1}; it) \right. \\ &\quad \left. - \sigma \int_{\lambda_k}^{\lambda_{k+1}} \exp\{\sigma y\} I_k(y; it) dy \right] \\ &\quad + \exp\{x\sigma\} I_n(x; it) - \sigma \int_{\lambda_n}^x \exp\{\sigma y\} I_n(y; it) dy. \end{aligned} \quad (61)$$

In view of (5), there exists a constant K such that $\lambda_n + 1 \leq \lambda_n + K$ for $n \geq 1$. So, we can deduce

$$\begin{aligned} &\left| \int_0^x \exp\{(\sigma + it)y\} d\alpha(y) \right| \\ &\leq \sum_{k=1}^{n-1} \mu(\sigma + D + \varepsilon, F) \exp\{-\lambda_k(\sigma + D + \varepsilon)\} (\exp\{\lambda_{k+1} \sigma\} + |\exp\{\lambda_{k+1} \sigma\} - \exp\{\lambda_k \sigma\}|) \\ &\quad + \mu(\sigma + D + \varepsilon, F) \exp\{-\lambda_n(\sigma + D + \varepsilon)\} (\exp\{x\sigma\} + |\exp\{x\sigma\} - \exp\{\lambda_n \sigma\}|) \\ &\leq 2\mu(\sigma + D + \varepsilon, F) e^{K\sigma} \sum_{k=1}^n \exp\{-\lambda_k(D + \varepsilon)\} \\ &= 2\mu(\sigma + D + \varepsilon, F) e^{K\sigma} \sum_{k=1}^n \exp\left\{-\lambda_k \left(D + \frac{1}{2}\varepsilon\right) \frac{D + \varepsilon}{D + (1/2)\varepsilon}\right\}. \end{aligned} \quad (62)$$

In view of (5), for the above ε and sufficiently large n , we have $\log n < \lambda_n(D + (1/2)\varepsilon)$,

$$\exp\left\{-\lambda_k \left(D + \frac{1}{2}\varepsilon\right) \frac{D + \varepsilon}{D + (1/2)\varepsilon}\right\} < \frac{1}{n}^{(D + \varepsilon/D + (1/2)\varepsilon)}, \quad (63)$$

and $\sum_{k=1}^{+\infty} (1/n)^{(D + \varepsilon/D + (1/2)\varepsilon)}$ is convergent. Therefore, this completes the proof of Lemma 4. $\square$

3. Proofs of Theorems 2 and 3

3.1. The Proof of Theorem 2. We firstly prove $\tau \leq T$. Assume that $T < +\infty$. From

$$T = \limsup_{\sigma \rightarrow +\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma\rho(\sigma)}}, \quad (64)$$

then for any positive number ε and sufficiently large σ , it yields

$$\log^+ M_u(\sigma, F) < e^{\sigma\rho(\sigma)} (T + \varepsilon) = \psi(r)(T + \varepsilon). \quad (65)$$

By Lemmas 2 and 4, it follows

$$\log A_n^* < \psi(r)(T + \varepsilon) - \lambda_n \log r + \log 2$$

$$\leq \lambda_n \left\{ \frac{1}{\rho(1 + o(1))} - \log \varphi \left(\frac{\lambda_n}{(T + \varepsilon)\rho(1 + o(1))} \right) \right\},$$

$$n \rightarrow \infty.$$

$$(66)$$

Hence, in view of Lemma 1, we can deduce

$$\begin{aligned}
(A_n^*)^{1/\lambda_n} &\leq \frac{\exp(1/(\rho(1+o(1))))}{\varphi(\lambda_n/((T+\varepsilon)\rho(1+o(1))))} \\
&= \frac{\exp(1/(\rho(1+o(1))))[\rho(T+\varepsilon)(1+o(1))]^{1/\rho}}{\varphi(\lambda_n)}, \quad n \rightarrow \infty,
\end{aligned} \quad (67)$$

that is,

$$\left[\varphi(\lambda_n)(A_n^*)^{1/\lambda_n} \right]^\rho \leq \exp(1+o(1))[\rho(T+\varepsilon)(1+o(1))], \quad n \rightarrow \infty. \quad (68)$$

Since ε is arbitrary, we conclude from the above inequality that

$$\tau = \limsup_{n \rightarrow \infty} \frac{1}{\rho e} \left[\varphi(\lambda_n)(A_n^*)^{1/\lambda_n} \right]^\rho \leq T, \quad (69)$$

and $\tau < +\infty$ as $T < +\infty$.

Next, we prove $T \leq e^{D\rho}\tau$. Assume that $0 \leq \tau < +\infty$; then, for any $\varepsilon > 0$, there exists a constant K_1 such that

$$A_n^* < K_1 \left\{ \frac{[\rho e(\tau + \varepsilon)]^{1/\rho}}{\varphi(\lambda_n)} \right\}^{\lambda_n}, \quad n \geq 1. \quad (70)$$

Thus, from Lemma 3, Lemma 4, and (70), it yields

$$\begin{aligned}
\log M_u(\sigma, F) &\leq \log C + \log K_1 + K\sigma \\
&+ \max_n \left\{ \log \left[\frac{e^{\sigma+D+\varepsilon}[\rho e(\tau + \varepsilon)]^{1/\rho}}{\varphi(\lambda_n)} \right]^{\lambda_n} \right\} \log M_u(\sigma, F) \\
&\leq \log C + \log K_1 + K\sigma + \max_n \left\{ \log \left[\frac{e^{\sigma+D+\varepsilon}[\rho e(\tau + \varepsilon)]^{1/\rho}}{\varphi(\lambda_n)} \right]^{\lambda_n} \right\} \\
&\leq \log C + \log K_1 + K\sigma + \frac{1}{\rho} (1+o(1)) \\
&\quad \psi \left(e^{D+\varepsilon-\rho^{-1}(1+o(1))} [\rho e(\tau + \varepsilon)]^{1/\rho} e^\sigma \right) \\
&= \log C + \log K_1 + K\sigma + \frac{1}{\rho} (1+o(1)) e^{\rho(D+\varepsilon)+o(1)} \rho(\tau + \varepsilon) \psi(e^\sigma) \\
&= \log C + \log K_1 + K\sigma \\
&\quad + [e^{\rho(D+\varepsilon)+o(1)} (\tau + \varepsilon) + o(1)] \psi(e^\sigma), \quad (\sigma \rightarrow +\infty),
\end{aligned} \quad (71)$$

which implies

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log M_u(\sigma, F)}{\psi(e^\sigma)} \leq e^{\rho(D+\varepsilon)} (\tau + \varepsilon). \quad (72)$$

Since ε is arbitrary, let $\varepsilon \rightarrow 0$ in (72); it follows that $T \leq e^{D\rho}\tau$ and $T < +\infty$ as $\tau < +\infty$.

Therefore, (16) is proved.

Furthermore, if the sequence $\{\lambda_n\}$ satisfies (18), then for any $\varepsilon \in (0, h')$, there exists a positive integer n_0 such that $\lambda_{n+1} - \lambda_n > h' - \varepsilon$ for $n \geq n_0$. Thus, for any positive integer k , it follows

$$\lambda_{k+n_0} - \lambda_{n_0} = \lambda_{k+n_0} - \lambda_{k-1+n_0} + \dots + \lambda_{1+n_0} - \lambda_{n_0} > k(h' - \varepsilon). \quad (73)$$

Hence,

$$\frac{k+n_0}{\lambda_{k+n_0}} < \frac{\lambda_{k+n_0} - \lambda_{n_0}}{\lambda_{k+n_0}} \frac{1}{h' - \varepsilon} + \frac{n_0}{\lambda_{k+n_0}}. \quad (74)$$

Since ε is arbitrary, then in view of (74), we conclude

$$\begin{aligned}
D &= \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} \leq \limsup_{n \rightarrow \infty} \frac{\log n}{n} \limsup_{n \rightarrow \infty} \frac{n}{\lambda_n} \\
&\leq \limsup_{n \rightarrow \infty} \frac{\log n}{n} \limsup_{k \rightarrow \infty} \frac{k+n_0}{\lambda_{k+n_0}} \leq \frac{1}{h'} \limsup_{n \rightarrow \infty} \frac{\log n}{n},
\end{aligned} \quad (75)$$

that is, $D = 0$. Thus, in view of (16), it follows $T = \tau = \limsup_{n \rightarrow \infty} (1/\rho e) [\varphi(\lambda_n)(A_n^*)^{1/\lambda_n}]^\rho$.

Therefore, this completes the proof of Theorem 2.

3.2. The Proof of Theorem 3. We firstly prove $\tau_p \leq T_p$. Assume that $T_p < +\infty$. From

$$T_p = \limsup_{\sigma \rightarrow +\infty} \frac{\log_p M_u(\sigma, F)}{e^{\sigma\rho(\sigma)}}, \quad (76)$$

then for any positive number ε and sufficiently large σ , it yields

$$\log_p M_u(\sigma, F) < e^{\sigma\rho(\sigma)} (T_p + \varepsilon) = \psi_p(r)(T_p + \varepsilon), \quad (77)$$

where $r = e^\sigma$. By Lemmas 2 and 4, for sufficiently large σ , it follows

$$\begin{aligned}
\log A_n^* &\leq \log M_u(\sigma, F) - \lambda_n \sigma + \log 2 < \exp_{p-1} \{ \psi_p(r)(T_p + \varepsilon) \} \\
&\quad - \lambda_n \log r + \log 2,
\end{aligned} \quad (78)$$

where $\exp_0(x) = x$, $\exp_p(x) = \exp(\exp_{p-1}(x))$, ($p \geq 1$). In view of Lemma 2, it is easy to see that $\exp_{p-1} \{ \psi_p(r)(T_p + \varepsilon) \} - \lambda_n \log r + \log 2$ can attain the minimum when r satisfies the following equation:

$$r \psi_p'(r) \prod_{j=1}^{p-1} \exp_j \{ (T_p + \varepsilon) \psi_p(r) \} = \frac{\lambda_n}{T_p + \varepsilon}. \quad (79)$$

By Lemma 1, we have $r \psi_p'(r) = \psi_p(r) \rho_p(1+o(1))$ as $\sigma \rightarrow +\infty$. Thus, it follows from (79) that

$$\prod_{j=1}^{p-1} \exp_j \{ (T_p + \varepsilon) \psi_p(r) \} = \frac{\lambda_n}{\rho_p(1+o(1))}, \quad n \rightarrow +\infty. \quad (80)$$

Hence, the function $\exp_{p-1} \{ \psi_p(r)(T_p + \varepsilon) \} - \lambda_n \log r + \log 2$ can obtain the minimum

$$-(1+o(1)) \lambda_n \log \varphi_p \left(\frac{1}{T_p + \varepsilon} \log_{p-1} \frac{\lambda_n}{\rho_p(1+o(1))} \right), \quad n \rightarrow +\infty, \quad (81)$$

as r satisfies equation (80). From $p \geq 2$ and Lemma 1, it yields

$$\begin{aligned}
\varphi_p\left(\frac{1}{T_p + \varepsilon} \log_{p-1} \frac{\lambda_n}{\rho_p(1 + o(1))}\right) &= \varphi_p\left(\frac{1 + o(1)}{T_p + \varepsilon} \log_{p-1} \lambda_n\right) \\
&= \left(\frac{1 + o(1)}{T_p + \varepsilon}\right)^{1/\rho_p} \varphi_p(\log_{p-1} \lambda_n), \\
n &\longrightarrow +\infty.
\end{aligned} \tag{82}$$

So, we can deduce from (78), (81), and (82) that

$$\begin{aligned}
\log A_n^* &\leq -(1 + o(1)) \lambda_n \log \varphi_p\left(\frac{1}{T_p + \varepsilon} \log_{p-1} \frac{\lambda_n}{\rho_p(1 + o(1))}\right) \\
&\leq -(1 + o(1)) \lambda_n \log \left(\frac{1 + o(1)}{T_p + \varepsilon}\right)^{1/\rho_p} \varphi_p(\log_{p-1} \lambda_n), \\
n &\longrightarrow +\infty,
\end{aligned} \tag{83}$$

that is,

$$\left[\varphi_p(\log_{p-1} \lambda_n) (A_n^*)^{1/\lambda_n}\right]^{\rho_p} \leq (1 + o(1))(T_p + \varepsilon), \quad n \longrightarrow \infty. \tag{84}$$

Since ε is arbitrary, we conclude from the above inequality that

$$\tau_p = \limsup_{n \longrightarrow +\infty} \left[\varphi_p(\log_{p-1} \lambda_n) (A_n^*)^{1/\lambda_n}\right]^{\rho_p} \leq T_p, \tag{85}$$

and $\tau_p < +\infty$ as $T_p < +\infty$.

Next, we prove $T_p \leq e^{D\rho_p} \tau_p$. Assume that $0 \leq \tau_p < +\infty$; then, for any $\varepsilon > 0$, there exists a constant K_2 such that

$$A_n^* < K_2 \left\{ \frac{(\tau_p + \varepsilon)^{1/\rho_p}}{\varphi_p(\log_{p-1} \lambda_n)} \right\}^{\lambda_n}, \quad n \geq 1. \tag{86}$$

Thus, from Lemma 3, Lemma 4, and (86), it yields

$$\begin{aligned}
\log M_u(\sigma, F) &\leq \log C + \log K_2 + K\sigma \\
&\quad + \max_n \left\{ \log \left[\frac{e^{\sigma+D+\varepsilon} [(\tau_p + \varepsilon)]^{1/\rho_p}}{\varphi_p(\log_{p-1} \lambda_n)} \right]^{\lambda_n} \right\}.
\end{aligned} \tag{87}$$

By using the same argument as in Lemma 3, we can conclude that the function

$$\log \left[\frac{e^{\sigma+D+\varepsilon} [(\tau_p + \varepsilon)]^{1/\rho_p}}{\varphi_p(\log_{p-1} \lambda_n)} \right]^{\lambda_n} \tag{88}$$

can attain the maximum

$$\frac{1 + o(1)}{\rho_p \prod_{j=0}^{p-2} \log_j \lambda_n \exp_{p-1} \left(\psi_p \left((1 + o(1)) \exp(\sigma + D + \varepsilon) (\tau_p + \varepsilon)^{1/\rho_p} \right) \right)}, \quad n \longrightarrow +\infty, \tag{89}$$

as λ_n satisfies

$$\log \left(\exp(\sigma + D + \varepsilon) (\tau_p + \varepsilon)^{1/\rho_p} \right) - \log \varphi_p(\log_{p-1} \lambda_n) - \frac{1 + o(1)}{\rho_p \prod_{j=0}^{p-2} \log_j \lambda_n} = 0, \quad n \longrightarrow +\infty. \tag{90}$$

Thus, by Lemma 1, we can deduce from (87) and (89) that

$$\limsup_{\sigma \longrightarrow +\infty} \frac{\log_p M_u(\sigma, F)}{\psi_p(e^\sigma)} \leq e^{\rho_p(D+\varepsilon)} (\tau_p + \varepsilon). \tag{91}$$

Since ε is arbitrary, let $\varepsilon \longrightarrow 0$ in (89); it follows that $T_p \leq e^{D\rho_p} \tau_p$ and $T_p < +\infty$ as $\tau_p < +\infty$. Hence, we prove (23).

Furthermore, if the sequence $\{\lambda_n\}$ satisfies (18), then we see $D = 0$ from the proof of Theorem 2. Thus, it is easy to prove that $T_p = \tau_p$ from (23).

Therefore, this completes the proof of Theorem 3.

4. Proofs of Theorems 6 and 7

4.1. The Proof of Theorem 6. Firstly of all, we prove $\tau_a \leq T$. Assume that $T < +\infty$. By using the same argument as in the proof of Theorem 2, for any positive number ε and sufficiently large σ , we have (64). Since $F(s) \in L_\infty$, then we have $F(s) \in \bar{L}_\beta$ for any $\beta (-\infty < \beta < +\infty)$. Hence, from the definitions of $E_n(F, \beta)$ and p_n , it follows

$$\begin{aligned}
E_n(F, \beta) &\leq \|F - p_n\|_\beta \leq |F(\beta + it) - p_n(\beta + it)| \\
&\leq \left| \int_0^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) - \int_0^{\lambda_n} \exp\{(\beta + it)y\} d\alpha(y) \right| \\
&= \left| \int_{\lambda_n}^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) \right|, \quad \beta < \sigma < +\infty.
\end{aligned} \tag{92}$$

Let

$$\begin{aligned}
I(x; it) &= \int_0^x \exp\{it y\} d\alpha(y), \\
J_{j+k}(b; it) &= \int_{\lambda_{j+k}}^b \exp\{it y\} d\alpha(y), \\
(\lambda_{j+k} < b \leq \lambda_{j+k+1}).
\end{aligned} \tag{93}$$

Then, $|J_{j+k}(b; it)| \leq A_{j+k}^*$. Since

$$\begin{aligned}
&\left| \int_{\lambda_k}^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) \right| \\
&= \lim_{b \rightarrow +\infty} \left| \int_{\lambda_k}^b \exp\{(\beta + it)y\} d\alpha(y) \right|, \quad -\infty < \beta < +\infty,
\end{aligned} \tag{94}$$

then we can deduce

$$\begin{aligned}
&\left| \int_{\lambda_k}^b \exp\{(\beta + it)y\} d\alpha(y) \right| \\
&= \left| \sum_{j=k}^{n+k-1} \int_{\lambda_j}^{\lambda_{j+1}} \exp\{\beta y\} d_y J_j(y; it) + \int_{\lambda_{n+k}}^b \exp\{\beta y\} d_y J_{n+k}(y; it) \right| \\
&= \left| \left[\sum_{j=k}^{n+k-1} e^{\lambda_{j+1}\beta} J_j(\lambda_{j+1}; it) - \beta \int_{\lambda_j}^{\lambda_{j+1}} e^{\beta y} J_j(y; it) dy \right] \right. \\
&\quad \left. + e^{\beta b} J_{n+k}(b; it) - \beta \int_{\lambda_{n+k}}^b e^{\beta y} J_{n+k}(y; it) dy \right| \\
&\leq \sum_{j=k}^{n+k-1} [A_j^* e^{\lambda_{j+1}\beta} + A_j^* (e^{\lambda_{j+1}\beta} - e^{\lambda_j\beta})].
\end{aligned} \tag{95}$$

Hence, from the above inequalities, we conclude

$$\left| \int_{\lambda_k}^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) \right| \leq 2 \sum_{n=k}^{+\infty} A_n^* \exp\{\beta \lambda_{n+1}\}, \quad \text{as } n \rightarrow +\infty. \tag{96}$$

Thus, for any σ ($\beta < \sigma < +\infty$), from (92) and (96) and by combining $A_n^* \leq 2M_u(\sigma, F)e^{-\sigma\lambda_n}$, it yields

$$E_n(F, \beta) \leq 2 \sum_{k=n+1}^{+\infty} A_{k-1}^* \exp\{\beta \lambda_k\} \leq 4M_u(\sigma, F) \sum_{k=n+1}^{+\infty} \exp\{(\beta - \sigma)\lambda_k\}. \tag{97}$$

Choosing H ($0 < H < h$) such that $(\lambda_{n+1} - \lambda_n) \geq H$ for $n \geq 0$, thus, for $\sigma \geq \beta + 1$, we can deduce from (97) that

$$\begin{aligned}
E_n(F, \beta) &\leq 4M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \sum_{k=n+1}^{+\infty} \exp\{(\lambda_k - \lambda_{n+1})(\beta - \sigma)\} \\
&\leq 4M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \exp\{H(n+1)\} \sum_{k=n+1}^{+\infty} (\exp\{-Hk\}) \\
&= 4M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} (1 - \exp\{-H\})^{-1},
\end{aligned} \tag{98}$$

that is,

$$E_{n-1}(F, \beta) \leq K_3 M_u(\sigma, F) \exp\{\lambda_n(\beta - \sigma)\}, \tag{99}$$

where K_3 is a constant.

Thus, it follows from (65) and (99) that

$$\log[E_{n-1}(F, \beta) \exp(-\beta \lambda_n)] < \psi(r)(T + \varepsilon) - \lambda_n \log r + \log K_3. \tag{100}$$

In view of (100) and using the same argument as in the proof of Theorem 2, we conclude from the above inequality that

$$\tau_a = \limsup_{n \rightarrow +\infty} \frac{1}{\rho \exp(\beta \rho + 1)} [\varphi(\lambda_n) (E_{n-1}(F, \beta))^{1/\lambda_n}]^\rho \leq T, \tag{101}$$

and $\tau_a < +\infty$ as $T < +\infty$.

Next, we prove $T \leq e^{D\rho} \tau_a$. Assume that $0 \leq \tau_a < +\infty$; then, for any $\varepsilon > 0$, there exists a constant K_4 such that

$$E_{n-1}(F, \beta) \exp(-\beta \lambda_n) < K_4 \left\{ \frac{[\rho e(\tau_a + \varepsilon)]^{1/\rho}}{\varphi(\lambda_n)} \right\}^{\lambda_n}, \quad n \geq 1. \tag{102}$$

And since

$$\begin{aligned}
A_n^* \exp\{\beta \lambda_n\} &= \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x \exp\{it y\} d\alpha(y) \right| \exp\{\beta \lambda_n\} \\
&\leq \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x \exp\{(\beta + it)y\} d\alpha(y) \right| \\
&\leq \sup_{-\infty < t < +\infty} \left| \int_{\lambda_n}^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) \right|,
\end{aligned} \tag{103}$$

thus, it means

$$A_n^* \exp\{\beta \lambda_n\} \leq |F(\beta + it) - p(\beta + it)| \leq \|F - p\|_\beta, \tag{104}$$

for any $p \in \Pi_{n-1}$. In addition, there exists $p_1 \in \Pi_{n-1}$ such that

$$\|F - p_1\| \leq 2E_{n-1}(F, \beta). \tag{105}$$

Thus, for any $\beta < +\infty$ and $F(s) \in L_\infty$, from (105) and (104), it yields

$$A_n^* \exp\{\beta \lambda_n\} \leq 2E_{n-1}(F, \beta). \tag{106}$$

Hence, in view of (102) and (106) and by using the same argument as in the proof of Theorem 2, we conclude

$$\limsup_{\sigma \rightarrow +\infty} \frac{\log M_u(\sigma, F)}{\psi(e^\sigma)} \leq e^{\rho(D+\varepsilon)} (\tau_a + \varepsilon), \tag{107}$$

which implies $T \leq e^{D\rho} \tau_a$ and $T < +\infty$ as $\tau_a < +\infty$. Hence, (34) holds.

Furthermore, if the sequence λ_n satisfies (18), from the proof of Theorem 2, it follows $D = 0$, which means $T = \tau_a$.

Therefore, this completes the proof of Theorem 6.

4.2. The Proof of Theorem 7. By using the same argument as in the proof of Theorem 3 and combining (99), (102), and (106), we can prove the conclusions of Theorem 7 easily.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

H. Y. Xu was responsible for conceptualization and writing and preparing the original draft; H. Y. Xu and W. J. Tang contributed to writing the review and editing; and W. J. Tang, H. Y. Xu, and J. Chen were responsible for funding acquisition.

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