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Research Article

Inversion of the Attenuated X-Ray Transforms: Method of Riesz Potentials

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The attenuated X-ray transform arises from the image reconstruction in single-photon emission computed tomography. The theory of attenuated X-ray transforms is so far incomplete, and many questions remain open. This paper is devoted to the inversion of the attenuated X-ray transforms with nonnegative varying attenuation functions μ, integrable on any straight line of the plane. By constructing the symmetric attenuated X-ray transform A_μ on the plane and using the method of Riesz potentials, we obtain the inversion formula of the attenuated X-ray transforms on $L^p(\mathbb{R}^2)\,(1 \le p < 2)$ space, with nonnegative attenuation functions μ, integrable on any straight line in $\mathbb{R}^2$. These results are succinct and may be used in the type of computerized tomography with attenuation.

1. Introduction

Computerized tomography (CT) means the reconstruction of a function from its line or plane integrals. The Radon transforms are the bases of the mathematics of computerized tomography [1]. Denote the hyperplane in $\mathbb{R}^n$ with normal vector $\theta \in \mathbb{S}^{n-1}$ and with distance $|t|\,(t \in \mathbb{R})$ from the origin by

$$H(\theta, t) = \{x \in \mathbb{R}^n \colon x \cdot \theta = t\}, \tag{1}$$

where "·" is the standard inner product in $\mathbb{R}^n$. Then, the Radon transform [2–5] of a function f on $\mathbb{R}^n$ is defined by

$$Rf(\theta, t) = \int_{H(\theta,t)} f(y)\,\mathrm{d}y_H, \tag{2}$$

where $\mathrm{d}y_H$ is the Lebesgue measure on $H(\theta, t)$. When n equals 2, the hyperplane $H(\theta, t)$ is a straight line, denoted by $l(\theta, t)$, and the Radon transform becomes the X-ray transform:

$$(Xf)(\theta, t) = \int_{l(\theta,t)} f(x)\,\mathrm{d}x_l, \tag{3}$$

where $\mathrm{d}x_l$ is the Lebesgue measure on the straight line $l(\theta, t)$. Of course, there are still X-ray transforms in n-dimensional spaces for $n > 2$, see [4], Chap. 1 and [1], Chap. 2. A generalization of the Radon transform is the k-plane transform ([5], Chap. 3), which integrates a function over translates of k-dimensional subspaces of $\mathbb{R}^n$. If $k = n - 1$, then this is precisely the Radon transform. If $k = 1$, this integrates function over lines and is just the X-ray transform. In dimension 2, there is no difference between the Radon transforms and the X-ray transforms, whereas in higher dimensions, there are significant differences.

The Radon transforms are used not only in practical fields [1, 2, 5, 6] but also in theoretical fields, for example, integral geometry [3, 6]. For practical or theoretical purposes, varieties of inversion formulas of the Radon transforms are created. There are several classical methods for the inversion of the Radon transforms, such as the method of mean value operators [7, 8], the method of Riesz potentials [1, 7], the convolution-backprojection method [1, 2, 5, 6, 9, 10], and the continuous ridgelet transform method [10].

The attenuated X-ray transforms arise from the single-photon emission computed tomography (SPECT) [1]. The theory of the attenuated X-ray transforms is so far incomplete, and many questions remain open. For about twenty years, it has been an open problem whether the attenuated X-ray transform X_μ is invertible on $\mathbb{R}^2$. Until 1998, a positive answer was given by [11], in light of the theory of so-called A-analytic functions. In 2000, a breakthrough was made by Novikov [12], who found an explicit inversion formula for X_μ with general attenuation μ. And the injectivity of X_μ on $\mathbb{R}^2$ can be derived from his inversion formula. His results were known in 2000 but were formally published in 2002. In 2001, Natterer [13] gave a reformulation of the Novikov formula and proved it in a simple and convenient way for sufficiently smooth and fast decaying attenuations μ and for continuously differentiable test functions f. But, he also indicated that it is difficult to determine exactly the class of functions f for which Novikov's formula holds. In 2003, Boman and Strömberg [14] developed an inversion formula for a generalized Radon transform related to but more general than the attenuated X-ray transform. For the history of the attenuated X-ray transform, the readers could refer to [5], Sec. 5.3.

In the mathematics of single-photon emission computerized tomography, the attenuation function (nonnegative) $\mu(x)$ on $\mathbb{R}^2$ is given first, and the attenuated X-ray transform (see [2], Sec.8.8; [15]; [16], p.432; [13], p.113) is defined by

$$\left(X_\mu f\right)(\theta, s) = \int_{x\cdot\theta=s} f(x)\exp\left(-(D\mu)(x,\theta^\perp)\right)\mathrm{d}x_l, \tag{4}$$

for $(\theta, s)\in\mathbb{S}^1\times\mathbb{R}^1$, where $D\mu$ is the divergent beam transform of μ,

$$(D\mu)(x,\theta) = \int_0^{+\infty}\mu(x+t\theta)\mathrm{d}t, \text{ for } (x,\theta)\in\mathbb{R}^2\times\mathbb{S}^1, \tag{5}$$

f is the density function, $\mathrm{d}x_l$ is the Lebesgue measure on the straight line $l(\theta,s)=\{x\in\mathbb{R}^2\colon x\cdot\theta=s\}$, and $\theta^\perp=(-\theta_2,\theta_1)$ for $\theta=(\theta_1,\theta_2)\in\mathbb{S}^1$. When $\mu\equiv 0$, the attenuated X-ray transform X_μ becomes the X-ray transform X, namely, the Radon transform R on $\mathbb{R}^2$.

Obviously, $0\le\exp(-(D\mu)(x,\theta^\perp))\le 1$ for all x and θ. From the existence of the Radon transforms ([17], Theorem 4.28), we have $(X_\mu f)(\theta, s)$ which exists and is finite for almost all $(\theta,s)\in\mathbb{S}^1\times\mathbb{R}^1$ when $f\in L^p(\mathbb{R}^2)\,(1\le p<2)$. From the mapping properties of the Radon transforms ([17], Theorem 4.34), we know that

$$\left(\int_{\mathbb{S}^1}\left(\int_{\mathbb{R}}\left|\left(X_\mu f\right)(\theta,s)\right|^r\mathrm{d}s\right)^{q/r}\mathrm{d}\theta\right)^{1/q}\le c_{p,q,r}\|f\|_p, \tag{6}$$

for $1\le p<2$, $1\le q\le p/(p-1)$ and $r=p/(2-p)$, with $c_{p,q,r}$ a constant depending only on p, q, r.

This paper is devoted to the inversion of the attenuated X-ray transforms on $L^p(\mathbb{R}^2)\,(1\le p<2)$ space, with varying nonnegative attenuation functions μ, integrable on any straight line in $\mathbb{R}^2$. Generally, $(X_\mu f)(\theta, s)\ne(X_\mu f)(-\theta,-s)$ for $(\theta,s)\in\mathbb{S}^1\times\mathbb{R}^1$, that is, $X_\mu f$ has two different values on the same straight line $l(\theta,s)$. This differs from that of the X-ray transforms. The X-ray transform Xf has unique value on the same straight line $l\ (\theta,\ s)$. In other words, the X-ray transforms have the symmetry property $(Xf)(\theta, s)=(Xf)(-\theta, -s)$ for all $(\theta,s)\in\mathbb{S}^1\times\mathbb{R}^1$. Therefore, Xf can be regarded as a function on the set of straight lines in the plane, denoted by $\mathbb{P}^2$, whereas the attenuated X-ray transform $X_\mu f$ cannot be seen as a function on $\mathbb{P}^2$, but on $\mathbb{S}^1\times\mathbb{R}^1$ instead. In account of these facts, we construct the symmetric attenuated X-ray transform A_μ, and then by the method of Riesz potentials [1, 7], we obtain the inversion formula of the attenuated X-ray transform X_μ on $L^p(\mathbb{R}^2)(1\le p<2)$ space, with nonnegative attenuation function μ, integrable on any straight line in $\mathbb{R}^2$. These results are succinct and may be used in the type of computerized tomography with attenuation.

2. Preliminaries: Symmetric Attenuated X-Ray Transforms and Riesz Potentials

In this section, we construct the symmetric attenuated X-ray transform A_μ on the plane and then convert it into an operator similar to the Riesz potentials I_2^1 ([18], Sec. 25) on with the aid of the dual X-ray transform X^* ([1], Chap. 2), where I_n^α is defined by

$$\left(I_n^\alpha f\right)(x) = f * k_\alpha(x), \tag{7}$$

with $k_\alpha(y)=(\gamma_n(\alpha))^{-1}|y|^{\alpha-n}$ and $x, y\in\mathbb{R}^n$, see [18], Sec. 25. These processes are the preliminaries for the inversion of A_μ in the next section.

The symmetric attenuated X-ray transform A_μ on the plane is defined by

$$\left(A_\mu f\right)(\theta,s) = \frac{1}{2}\left[\left(X_\mu f\right)(\theta,s)+\left(X_\mu f\right)(-\theta,-s)\right], \tag{8}$$

$$= \frac{1}{2}\int_{x\cdot\theta=s} f(x)\left[\exp\left(-(D\mu)(x,\theta^\perp)\right)+\exp\left(-(D\mu)\cdot(x,-\theta^\perp)\right)\right]\mathrm{d}x_l, \tag{9}$$

for $(\theta,s)\in\mathbb{S}^1\times\mathbb{R}^1$ and nonnegative function μ on $\mathbb{R}^2$. Throughout this paper, we suppose that the attenuation function μ is nonnegative on $\mathbb{R}^2$. It is easily verified that, like the X-ray transforms, the symmetric attenuated X-ray transforms A_μ also have the symmetry property $(A_\mu f)(\theta, s)=(A_\mu f)(-\theta,-s)$ for all $(\theta,s)\in\mathbb{S}^1\times\mathbb{R}^1$, which is crucial to our inversion of the attenuated X-ray transforms. Due to this property, $A_\mu f$ can be viewed as a function on the set of straight lines in a plane.

Subsequently, based on the symmetry property, a natural idea is whether we can invert the symmetric attenuated X-ray transforms A_μ in a similar way to the X-ray transforms. In this paper, we consider the method of Riesz potentials [1, 7] to the inversion of X-ray transforms, where the dual X-ray transform X^* is used:

$$X^*g(x) = \int_{\mathbb{S}^1} g(\theta, x\cdot\theta)\mathrm{d}\theta. \tag{10}$$

We attempt to represent X^*A_μ as some convolution operator with the Riesz kernel on $\mathbb{R}^2$, namely, an operator similar to the Riesz potentials. First, some preliminaries are needed for our derivation.

Lemma 1 (see [18], Sec. 25.3). *For $f\in L^p(\mathbb{R}^2)\,(1\le p<2)$ and for almost all $x\in\mathbb{R}^2$, the Riesz potential of f,*

$$\left(I_2^1 f\right)(x) = \frac{1}{\gamma_2(1)} \int_{\mathbb{R}^2} \frac{f(x+y)}{|y|} \mathrm{d}y, \tag{11}$$

exists and is finite.

Lemma 2. *Let*

$$C\mu(x,\theta) = \frac{1}{2}\left[\exp\left(-(D\mu)(x,\theta)\right) + \exp\left(-(D\mu)(x,-\theta)\right)\right], \tag{12}$$

for $x \in \mathbb{R}^2, \theta \in \mathbb{S}^1$. *Then,*

$$C\mu(x,\theta) = C\mu(x,-\theta), \tag{13}$$

$$0 \le C\mu(x,\theta) \le 1. \tag{14}$$

Proof. The proof is simple. □

Lemma 3. *The following equalities hold:*

$$\begin{gathered} \int_{\mathbb{S}^1} \int_{\substack{l(\theta,0) \\ y\cdot\theta^\perp \ge 0}} f(y)\mathrm{d}y_l \mathrm{d}\theta = \int_{\mathbb{R}^2} \frac{f(y)}{|y|}\mathrm{d}y, \\ \int_{\mathbb{S}^1} \int_{\substack{l(\theta,0) \\ y\cdot\theta^\perp \ge 0}} f(y)\mathrm{d}y_l \mathrm{d}\theta = \int_{\mathbb{R}^2} \frac{f(y)}{|y|}\mathrm{d}y, \\ \int_{\mathbb{S}^1} \int_{l(\theta,0)} f(y)\mathrm{d}y_l \mathrm{d}\theta = 2\int_{\mathbb{R}^2} \frac{f(y)}{|y|}\mathrm{d}y, \end{gathered} \tag{15}$$

provided that either side of these equalities is finite when f is replaced by $|f|$.

Proof. Applying the formula of polar coordinate transforms for integrals, we can get the proof of this lemma.

Next, we give the main results of this section. □

Theorem 1. *For* $f \in L^p(\mathbb{R}^2)\,(1 \le p < 2)$ *and for almost all* $x \in \mathbb{R}^2$,

$$\left(X^* A_\mu\right) f(x) = \left(f_{\mu,x} * K\right)(x), \tag{16}$$

where

$$f_{\mu,x}(z) = f_\mu(z,x) = 2f(z)C\mu\left(z,(z-x)'\right), \tag{17}$$

with $y' = y/|y|$, *and*

$$K(y) = \frac{1}{|y|}. \tag{18}$$

Proof. By the definitions of A_μ and X^*, (9), and (10), we have

$$\left(X^* A_\mu\right) f(x) = \int_{\mathbb{S}^1} \int_{l(\theta,x\cdot\theta)} f(y)C\mu\left(y,\theta^\perp\right)\mathrm{d}y_l \mathrm{d}\theta, \tag{19}$$

where $C\mu\ (y,\theta)$ is defined by (12). Through the change of variables in the right-hand side, (19) gives

$$\begin{aligned} \left(X^* A_\mu\right) f(x) &= \int_{\mathbb{S}^1} \int_{l(\theta,0)} f(x+y)C\mu\left(x+y,\theta^\perp\right)\mathrm{d}y_l \mathrm{d}\theta \\ &= \int_{\mathbb{S}^1} \int_{\substack{l(\theta,0) \\ y\cdot\theta^\perp \ge 0}} f(x+y)C\mu\left(x+y,y'\right)\mathrm{d}y_l \mathrm{d}\theta \\ &\quad + \int_{\mathbb{S}^1} \int_{\substack{l(\theta,0) \\ y\cdot\theta^\perp \ge 0}} f(x+y)C\mu\left(x+y,y'\right)\mathrm{d}y_l \mathrm{d}\theta. \end{aligned} \tag{20}$$

By Lemma 1, for $f \in L^p(\mathbb{R}^2)\,(1 \le p < 2)$ and for almost all $x \in \mathbb{R}^2$, the following integral

$$\int_{\mathbb{R}^2} \frac{f(x+y)}{|y|}\mathrm{d}y, \tag{21}$$

exists and is finite. And more, $C\mu$ is bounded by (14). Thus, from Lemma 3 and (20), it follows that

$$\begin{aligned} \left(X^* A_\mu\right) f(x) &= \int_{\mathbb{R}^2} f(x+y)\frac{C\mu\left(x+y,y'\right)}{|y|}\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^2} f(x+y)\frac{C\mu\left(x+y,-y'\right)}{|y|}\mathrm{d}y, \end{aligned} \tag{22}$$

for almost all $x \in \mathbb{R}^2$. By (13), we have

$$C\mu\left(x+y,-y'\right) = C\mu\left(x+y,y'\right). \tag{23}$$

Plugging (23) into (22) gives

$$\begin{aligned} \left(X^* A_\mu\right) f(x) &= 2\int_{\mathbb{R}^2} f(x+y)\frac{C\mu\left(x+y,y'\right)}{|y|}\mathrm{d}y \\ &= 2\int_{\mathbb{R}^2} f(x-y)\frac{C\mu\left(x-y,-y'\right)}{|y|}\mathrm{d}y \\ &= \left(f_{\mu,x} * K\right)(x), \end{aligned} \tag{24}$$

where $f_{\mu,x}$ is defined by (17), and K is defined by (18). This completes the proof. □

Remark 1. In Theorem 1, the kernel function $K\ (y)$ is the Riesz kernel $|y|^{-1}$ (ignoring the constant). Thus, $X^* A_\mu$ is an operator similar to the Riesz potential I_2^1, neglecting the difference between $f_{\mu,x}$ and f.

3. Derivation of the Inversion Formula

In this section, we pursue the inversion of the operator $X^* A_\mu$ in the above section. Then, the inversion of the attenuated X-ray transforms A_μ can be consequently obtained. Indeed, if we define the operator $B_\mu = X^* A_\mu$ and denote the left inverse operator of B_μ by B_μ^{-1}, then $B_\mu^{-1} B_\mu = B_\mu^{-1} X^* A_\mu = I$, where I is the identity transform. The operator $B_\mu^{-1} X^*$ can be viewed as the left inverse operator of A_μ.

By Remark 1, we know that the operator B_μ is similar to the Riesz potential I_2^1. Then, a natural idea is whether we can invert the operator B_μ in a similar way to the Riesz potential I_2^1 ([18], Sec. 26). This is the goal of this section. First, we need to introduce the truncated hypersingular integral operator D_ϵ ($\epsilon > 0$) ([18], Sec. 26):

$$D_\varepsilon f(x) = \int_{|y|\geq\varepsilon} \frac{f(x) - f(x-y)}{|y|^3} \mathrm{d}y, \tag{25}$$

where $f(x) - f(x-y)$ is the finite difference of order 1 of function f with a step y and with center at the point x ([18], Sec. 25). Set $\varphi(x) = (B_\mu f)(x)$. Then, $D_\epsilon\varphi$ can be written as a form of convolution, stated as follows.

Theorem 2. *Let $\varphi(x) = (B_\mu f)(x)$. Then, for $f \in L^p(\mathbb{R}^2)(1 \leq p < 2)$ and for almost all $x \in \mathbb{R}^2$,*

$$(D_\varepsilon\varphi)(x) = \int_{\mathbb{R}^2} f_{\mu,x}(x - \varepsilon\xi)\widetilde{K}(\xi)\mathrm{d}\xi, \tag{26}$$

where

$$\widetilde{K}(\xi) = \frac{1}{|\xi|^2}\int_{|z|\leq|\xi|} (K(z) - K(z-e_1))\mathrm{d}z, \tag{27}$$

and K is defined by (18).

Proof. We begin the proof with the difference of φ. Applying Theorem 1 and then changing the variables of integral, we have

$$\left(B_\mu f\right)(x) - \left(B_\mu f\right)(x-y) = \left(f_{\mu,x} * K\right)(x) - \left(f_{\mu,x} * K\right)(x-y), \tag{28}$$

$$= \int_{\mathbb{R}^2} f_{\mu,x}(x-\xi)\Delta(\xi,y)\mathrm{d}\xi, \tag{29}$$

where

$$\Delta(\xi, y) = K(\xi) - K(\xi - y). \tag{30}$$

Thus, combination of (25) and (29) and then changing the order of integration give

$$(D_\varepsilon\varphi)(x) = \int_{|y|\geq\varepsilon} \frac{1}{|y|^3}\int_{\mathbb{R}^2} f_{\mu,x}(x-\xi)\Delta(\xi,y)\mathrm{d}\xi\,\mathrm{d}y, \tag{31}$$

$$= \int_{\mathbb{R}^2} f_{\mu,x}(x-\xi)\int_{|y|\geq\varepsilon}\frac{1}{|y|^3}\Delta(\xi,y)\mathrm{d}y\,\mathrm{d}\xi. \tag{32}$$

Let $e_1 = (1, 0)$ and ω_x: $e_1 \longrightarrow x/|x| = x'$ be a rotation converting e_1 into x'. It is easy to check that $\omega_{y'}^{-1}(\xi')$ and $\omega_\xi^{-1}(y')$ are symmetric with respect to the horizontal axis in $\mathbb{R}^2$. Hence,

$$|\xi - ky| = |y|\left|\frac{\omega_y^{-1}(\xi)}{|y|} - ke_1\right| = |y|\left|\frac{|\xi|}{|y|^2}\omega_\xi^{-1}(y) - ke_1\right|. \tag{33}$$

Put $(|\xi|/|y|^2)\omega_\xi^{-1}(y) = z$ in the right-hand side of (32). Then, $|\xi|/|y| = |z|$, $y = (|y|^2/|\xi|)\omega_\xi(z) = (|\xi|/|z|)\omega_\xi(z')$, and $\mathrm{d}y = |\xi|^2/|z|^4\mathrm{d}z$. Thus, from (32) and (33), it follows that

$$(D_\varepsilon\varphi)(x) = \int_{\mathbb{R}^2} f_{\mu,x}(x-\xi)\frac{1}{|\xi|^2}\int_{|z|\leq|\xi|/\varepsilon}\Delta(z)\mathrm{d}z\,\mathrm{d}\xi, \tag{34}$$

where

$$\Delta(z) = K(z) - K(z - e_1). \tag{35}$$

Through the change of variables $\xi \longrightarrow \epsilon\xi$ in the right-hand side, (34) turns into (26).

Finally, we verify the reasonability of interchanging the order of integration in the right-hand side of (31) for almost all $x \in \mathbb{R}^2$. By Fubini's theorem, the boundedness of $C\mu$ (14), and the definition of $\Delta(\xi, y)$ (30), it suffices to prove that the following integrals exist and are both finite:

$$I_1(x) = \int_{|y|\geq\varepsilon}\frac{1}{|y|^3}\int_{\mathbb{R}^2}\frac{|f(x-\xi)|}{|\xi|}\mathrm{d}\xi\,\mathrm{d}y,$$
$$I_2(x) = \int_{|y|\geq\varepsilon}\frac{1}{|y|^3}\int_{\mathbb{R}^2}\frac{|f(x-\xi)|}{|\xi-y|}\mathrm{d}\xi\,\mathrm{d}y. \tag{36}$$

Easy computation gives

$$I_1(x) = \frac{2\pi}{\varepsilon}\int_{\mathbb{R}^2}\frac{|f(x-\xi)|}{|\xi|}\mathrm{d}\xi, \tag{37}$$

which exists and is finite for $f \in L^p(\mathbb{R}^2)(1 \leq p < 2)$ for almost all $x \in \mathbb{R}^2$ and each $\epsilon > 0$ following from Lemma 1.

For $|\xi - y| \geq 1$, by Hölder's inequality, we have for all $x \in \mathbb{R}^2$,

$$\int_{|\xi-y|\geq1}\frac{|f(x-\xi)|}{|\xi-y|}\mathrm{d}\xi \lesssim \|f\|_p, \tag{38}$$

where "$\lesssim$" means less than up to a constant, which implies

$$\int_{|y|\geq\varepsilon}\frac{1}{|y|^3}\int_{|\xi-y|\geq1}\frac{|f(x-\xi)|}{|\xi-y|}\mathrm{d}\xi\,\mathrm{d}y < +\infty. \tag{39}$$

For $|\xi - y| \leq 1$, by the generalized Minkowski inequality, we have

$$\left\|\int_{|\xi-y|\leq1}\frac{|f(x-\xi)|}{|\xi-y|}\mathrm{d}\xi\right\|_{p,x} \leq 2\pi\|f\|_p, \tag{40}$$

where $\|\cdot\|_{p,x}$ denotes the L^p-norm of function with respect to variable x. Combination of (40) and the generalized Minkowski inequality yields for almost all $x \in \mathbb{R}^2$ and for each $\epsilon > 0$ that

$$\int_{|y|\geq\varepsilon}\frac{1}{|y|^3}\int_{|\xi-y|\leq1}\frac{|f(x-\xi)|}{|\xi-y|}\mathrm{d}\xi\,\mathrm{d}y < +\infty. \tag{41}$$

Hence, from (39) and (41), it follows that $I_2(x)$ exists and is finite for almost all $x \in \mathbb{R}^2$ and for each $\epsilon > 0$, which completes the verification.

Now, we consider the inversion of B_μ. For $l \in \mathbb{P}^2$, let

$$L^1(l) = \left\{\mu : \int_l |\mu(y)|\mathrm{d}y_l < +\infty\right\}. \tag{42}$$

□

Theorem 3. *The operator $Df = \lim_{\varepsilon\longrightarrow0} D_\varepsilon f$ is the left inverse to B_μ within the frames of the spaces $L^p(\mathbb{R}^2)(1 \leq p < 2)$ up to a bounded function mathcalK (x), that is,*

$$DB_\mu f(x) = \mathscr{K}(x)f(x), \tag{43}$$

for $f \in L^p(\mathbb{R}^2)(1 \leq p < 2)$ in the L^p-norm sense, where

$$\mathscr{K}(x) = 2\int_{\mathbb{R}^2}\widetilde{K}(\xi)C\mu(x,\xi')\mathrm{d}\xi, \tag{44}$$

with $\widetilde{K}$ defined by (27) and $\mu \in L^1(l)$ for all $l \in \mathbb{P}^2$.

Proof. We use the Lebesgue dominated convergence theorem to prove this theorem. By Theorem 2 and (44), we have

$$\begin{aligned}\left\|D_\varepsilon B_\mu f(x) - \mathscr{K}(x) f(x)\right\|_{p,x} &= \left\|\int_{\mathbb{R}^2} f_{\mu,x}(x-\varepsilon\xi)\widetilde{K}(\xi)\mathrm{d}\xi - 2f(x)\int_{\mathbb{R}^2}\widetilde{K}(\xi)C\mu(x,\xi')\mathrm{d}\xi\right\|_{p,x} \\ &= \left\|\int_{\mathbb{R}^2}\left[f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right]\widetilde{K}(\xi)\mathrm{d}\xi\right\|_{p,x}.\end{aligned} \tag{45}$$

Due to the generalized Minkowski inequality, we have

$$\begin{aligned}&\left\|\int_{\mathbb{R}^2}\left[f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right]\widetilde{K}(\xi)\mathrm{d}\xi\right\|_{p,x} \\ &\quad \le \int_{\mathbb{R}^2}\left\|f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right\|_{p,x}\widetilde{K}(\xi)\mathrm{d}\xi.\end{aligned} \tag{46}$$

Next, we investigate $\|f_{\mu,x}(x-\epsilon\xi) - 2f(x)\,C\mu(x,\xi')\|_{p,x}$. By the definition of $f_{\mu,x}$ (17) and (13), we have

$$\begin{aligned}&\left\|f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right\|_{p,x} \\ &= \left\|2f(x-\varepsilon\xi)C\mu(x-\varepsilon\xi,\xi') - 2f(x)C\mu(x,\xi')\right\|_{p,x} \\ &\le (\mathrm{I}) + (\mathrm{II}),\end{aligned} \tag{47}$$

where

$$(\mathrm{I}) = 2\left\|\left[f(x-\varepsilon\xi) - f(x)\right]C\mu(x-\varepsilon\xi,\xi')\right\|_{p,x}, \tag{48}$$

$$(\mathrm{II}) = 2\left\|f(x)\left[C\mu(x-\varepsilon\xi,\xi') - C\mu(x,\xi')\right]\right\|_{p,x}. \tag{49}$$

Set

$$\omega_p(f,h) = \left(\int_{\mathbb{R}^2}|f(x-h) - f(x)|^p\mathrm{d}x\right)^{1/p}. \tag{50}$$

Then, from (14) and (48), it follows that

$$(\mathrm{I}) \le 2\omega_p(f,\varepsilon\xi) \longrightarrow 0, \quad \varepsilon \longrightarrow 0, \tag{51}$$

for $f \in L^p(\mathbb{R}^2)\,(1 \le p < 2)$.

It follows from (12) that

$$\begin{aligned}\left|C\mu(x-\varepsilon\xi,\xi') - C\mu(x,\xi')\right| \le \frac{1}{2}\big|&\exp\left[-(D\mu)(x-\varepsilon\xi,\xi')\right] \\ &- \exp\left[-(D\mu)(x,\xi')\right]\big| \\ &+ \frac{1}{2}\big|\exp\left[-(D\mu)(x-\varepsilon\xi,-\xi')\right] \\ &- \exp\left[-(D\mu)(x,-\xi')\right]\big|.\end{aligned} \tag{52}$$

For all $h \in \mathbb{R}^2$ and all $\theta \in \mathbb{S}^1$,

$$|D\mu(x+\varepsilon h,\theta) - D\mu(x,\theta)| \le \int_{l(\theta^\perp,x\cdot\theta^\perp)}|\mu(y+\varepsilon h) - \mu(y)|\mathrm{d}y_l. \tag{53}$$

Because $\mu \in L^1(l)$ for all $l \in \mathbb{P}^2$,

$$\int_{l(\theta^\perp,x\cdot\theta^\perp)}|\mu(y+\varepsilon h) - \mu(y)|\mathrm{d}y_l \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{54}$$

Thus, from (53) and (54), it follows that

$$|D\mu(x+\varepsilon h,\theta) - D\mu(x,\theta)| \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{55}$$

Then, from (52) and (55),) it follows that

$$\left|C\mu(x-\varepsilon\xi,\xi') - C\mu(x,\xi')\right| \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{56}$$

By (14), (49), (56), and the Lebesgue dominated convergence theorem, we have the right-hand side of (49) which tends to 0 as ϵ tends to 0. Thus, from (49), it follows that

$$\begin{aligned}&(\mathrm{II}) \longrightarrow 0, \\ &\varepsilon \longrightarrow 0.\end{aligned} \tag{57}$$

Combination of (47), (51), and (57) yields

$$\left\|f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right\|_{p,x} \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{58}$$

Due to (14) and (17), it follows that

$$\begin{aligned}\left\|f_{\mu,x}(x-\varepsilon\xi) - 2f(x)C\mu(x,\xi')\right\|_{p,x} &\le \left\|f_{\mu,x}(x-\varepsilon\xi)\right\|_{p,x} \\ &\quad + \left\|2f(x)C\mu(x,\xi')\right\|_{p,x} \\ &\le 4\|f\|_p.\end{aligned} \tag{59}$$

The function $\widetilde{K}$ defined by (27) plays the role of the approximate kernel in the proof of Theorem 3. Thus, the integrability of $\widetilde{K}$ on $\mathbb{R}^2$ is a necessary condition, which can be obtained by Lemma 26.4 in [18]. Actually, the function $\widetilde{K}$ can be denoted by $\mathscr{K}_{l,\alpha}(|\cdot|)$ (neglecting the constant $c_{l,\alpha,n}$), defined by formula (26.30) in [18]:

$$\mathscr{K}_{l,\alpha}(|x|) = \frac{c_{l,\alpha,n}}{|x|^n}\int_{|y|\triangleleft|x|}\sum_{k=0}^{l}(-1)^l\binom{l}{k}\left|y - ke_1\right|^{\alpha-n}\mathrm{d}y, \tag{60}$$

with $l = 1$, $\alpha = 1$, $n = 2$, and $\binom{l}{k} = (l\cdot(l-1)\cdots(l-k+1))/1\cdot 2\cdots k$, $e_1 = (1,0)$. From the conclusion that $\mathscr{K}_{l,\alpha}(|\cdot|) \in L^q(\mathbb{R}^n)$, $1 - \alpha/n < 1/q \le 1$, $\alpha > 0$, $q > 0$, $l = 1,2,3,\ldots$ in Lemma 26.4 in [18], we know that $\mathscr{K}_{1,1}(|\cdot|) \in L(\mathbb{R}^2)$ (letting $l = 1$, $\alpha = 1$, and $n = 2$), namely,

$$\int_{\mathbb{R}^2} |\widetilde{K}(\xi)| d\xi < +\infty. \tag{61}$$

Thus, from (58), (59), (61) and the Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^2} \left\| f_{\mu,x}(x - \varepsilon\xi) - 2f(x) C\mu(x, \xi') \right\|_{p,x} \widetilde{K}(\xi) d\xi \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{62}$$

Finally, from (55), (56), and (62), it follows that

$$\left\| D_\varepsilon B_\mu f(x) - \mathcal{K} f(x) \right\|_{p,x} \longrightarrow 0, \quad \varepsilon \longrightarrow 0, \tag{63}$$

and the boundedness of $\mathcal{K}(x)$ follows from (14), (44), and (61). □

4. Main Results and Discussion

Based on the inversion of the operator $X^* A_\mu$ in Section 3, we have the following inversion of the attenuated X-ray transforms X_μ on $L^p(\mathbb{R}^2)(1 \le p < 2)$ space. By Theorem 3 and the definition of operator B_μ, we have the following.

Theorem 4. *For the attenuated X-ray transforms X_μ with $\mu \in L^1$ (l) for all $l \in \mathbb{P}^2$, the following formula holds:*

$$DX^* A_\mu f(x) = \mathcal{K}(x) f(x), \tag{64}$$

for $f \in L^p(\mathbb{R}^2)(1 \le p < 2)$ in the senses of L^p-norm and almost everywhere on $\mathbb{R}^2$, where $D\varphi = \lim_{\varepsilon \longrightarrow 0} D_\varepsilon \varphi$, $\mathcal{K}$ is a bounded function on $\mathbb{R}^2$ defined by (44), and A_μ is the symmetric attenuated X-ray transforms defined by (8).

The correctness of Theorem 4 ensures that the mapping $A_\mu f \longrightarrow f$ exists when $f \in L^p(\mathbb{R}^2)(1 \le p < 2)$, that is, the symmetric attenuated X-ray transforms A_μ, and thus the attenuated X-ray transforms X_μ, are invertible when $f \in L^p(\mathbb{R}^2)(1 \le p < 2)$. The uniqueness of f is then in the almost everywhere sense due to the metric in the L^p space.

Finally, we give some comparisons between some classic results and our inversion formula (64). Novikov's inversion formula [12] is little complicated, where some smoothing conditions and decaying conditions at infinity are needed for μ. Natterer's inversion formula [13] is derived but more succinct in form than Novikov's, where the class of μ is not definitely determined. The divergence operator div and the compound $He^h g$ of Hilbert transform H, $h = (1/2)(I + iH)(X\mu)$ and $g = X_\mu f$ (I, the identity operator; i, the imaginary number), are all simultaneously involved in Natterer's inversion formula, which makes the formula not simple. Our inversion formula is relatively concise, which mainly contains the operators X^* and D_ϵ, where only the integrability condition of μ is needed, namely, $\mu \in L^1$ (l) for all $l \in \mathbb{P}^2$. The operator X^* is an ordinary integral operator on $\mathbb{S}^1$, and the operator D_ϵ is a truncated hypersingular integral operator on $\mathbb{R}^2$, see [18], Sec. 26 for more details. The appearance of the bounded function $\mathcal{K}(x)$ may be a defect of our inversion formula. But, it seems inevitable for the attenuated X-ray transforms by our model. The ideal one is that $\mathcal{K}(x)$ is identically equivalent to a nonzero constant, which can be attained for $\mu(x) \equiv 0$ when the attenuated X-ray transforms become the X-ray transforms and $C\mu \equiv 1$, whereas which may never be attained for general attenuated X-ray transforms due to the nonvanishing attenuation functions μ.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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