

Research Article

On Some Problems of Strongly Ozaki Close-to-Convex Functions

Zahra Maleki, Saeid Shams , Ali Ebadian, and Ebrahim Analouei Adegani 

Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran

Correspondence should be addressed to Saeid Shams; s.shams@urmia.ac.ir

Received 14 April 2020; Accepted 11 June 2020; Published 1 September 2020

Academic Editor: Maria Alessandra Ragusa

Copyright © 2020 Zahra Maleki et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of the current paper is to investigate some geometric properties of the class $\mathcal{F}_O(\nu, \gamma)$, called strongly Ozaki close-to-convex functions, such as strongly starlikeness and close-to-convexity. Further, we find sharp bounds on Fekete-Szegő functionals and logarithmic coefficients for functions belonging to the class $\mathcal{F}_O(\nu, \gamma)$, which incorporates some known outcomes as the specific cases.

1. Introduction and Preliminaries

Let \mathbb{U} denote the open unit disk in the complex plane \mathbb{C} . Let \mathcal{A} be the class of functions f of the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in \mathbb{U} and represent by \mathcal{S} the class of all functions of \mathcal{A} , which are univalent in \mathbb{U} . Let Ω denote the set of all analytic functions ω in \mathbb{U} that are satisfying the conditions of $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$, i.e., Ω , is considered as the family of Schwarz functions.

For two analytic functions f and F in the open unit disk \mathbb{U} , it is said that the function f is subordinate to the function F in \mathbb{U} , written $f(z) \prec F(z)$, if there exists a Schwarz function ω such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{U}$. In particular, if the function F is univalent in \mathbb{U} , the following equivalence holds:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0), \quad f(\mathbb{U}) \subset F(\mathbb{U}). \quad (2)$$

We denote by $\mathcal{S}^*(\alpha)$ the subclass of \mathcal{A} consisting of all $f \in \mathcal{A}$ for which f is a starlike of order α , with

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < 1), \quad (3)$$

and denote by $\mathcal{H}(\alpha)$ the subclass of \mathcal{A} consisting of all $f \in \mathcal{A}$ for which f is a convex of order α , with

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < 1). \quad (4)$$

Also, the subclass $\tilde{\mathcal{S}}^*(\alpha)$ of a strongly starlike function of order in \mathbb{U} is defined as

$$\tilde{\mathcal{S}}^*(\alpha) = \left\{ f \in \mathcal{A} : \left| \operatorname{Arg} \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U}, 0 \leq \alpha < 1 \right\}. \quad (5)$$

Note that $\tilde{\mathcal{S}}^*(1) = \mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{H}(0) = \mathcal{H}$ are the class of starlike functions in \mathbb{U} and the class of convex functions in \mathbb{U} , respectively.

Furthermore, $\mathcal{C}(\alpha)$ is denoted as the subclass of \mathcal{A} including functions such as close-to-convex of order α if there is a function $g \in \mathcal{S}^*$ so that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < 1), \quad (6)$$

and we denote by $\tilde{\mathcal{C}}(\alpha)$ the subclass of \mathcal{A} consisting of all of $f \in \mathcal{A}$ for which

$$\left| \text{Arg}\left(f'(z)\right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}, 0 \leq \alpha < 1). \quad (7)$$

Individually, $\mathcal{C}(0) = \mathcal{C}$ is the class of close-to-convex functions in \mathbb{U} and $\tilde{\mathcal{C}}(1) = \tilde{\mathcal{C}}$ is the subclass of close-to-convex functions in \mathbb{U} (see [1]). Here, we understand that $\text{Arg}w$ is a number in $(-\pi, \pi]$.

Recently, many authors have studied the families of analytic functions of the class \mathcal{A} and also investigated bound estimation problems, geometric property issues, and related topics for functions belonging to these families in [2–10] as well as in the references cited therein.

For example, Cho et al. [4] studied the majorization issue for a general well-known category $\mathcal{S}^*(\varphi)$ of starlike functions, which was defined by Ma and Minda [11]. Also, they investigated the majorization issue for the various subclasses $\mathcal{S}^*(\varphi)$ for different special functions φ . Moreover, estimates for the coefficients of majorized functions regarding the class $\mathcal{S}^*(\varphi)$ were given. Further, Alimohammadi et al. [2] introduced a subclass of \mathcal{A} and extended the class $\mathcal{C}(\alpha)$ ($\alpha \in (0, 1]$), defined by Nunokawa et al. in [12], consisting of all $f \in \mathcal{A}$ satisfying

$$\text{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{\alpha}{2} \quad (z \in \mathbb{U}), \quad (8)$$

and studied some geometric properties like close-to-convexity and strongly starlikeness. They determined sharp bounds of Fekete-Szegő functionals and logarithmic coefficients for this class. Kargar and Ebadian [9] considered the subclass $\mathcal{F}(\nu)$ of locally univalent functions $f \in \mathcal{A}$ in \mathbb{U} satisfying the inequality:

$$\text{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 1 - \frac{\nu}{2} \quad (z \in \mathbb{U}), \quad (9)$$

for some $-1/2 < \nu \leq 1$.

Recently, Allu et al. [3], motivated essentially by the subclass $\mathcal{F}(\nu)$, introduced the class $\mathcal{F}_O(\nu, \gamma)$ and obtained sharp bounds for three first coefficients and the corresponding inverse coefficients for the functions of this class.

Definition 1 [3]. Let $\gamma \in (0, 1]$ and $\nu \in [1/2, 1]$. Then, f is called strongly Ozaki close-to-convex if and only if

$$\mathcal{F}_O(\nu, \gamma) := \left\{ f \in \mathcal{A} : \left| \text{Arg} \left(\frac{2\nu-1}{2\nu+1} + \frac{2}{2\nu+1} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) \right| < \frac{\gamma\pi}{2}, z \in \mathbb{U} \right\}. \quad (10)$$

Note that the class $\mathcal{F}_O(\nu, 1) = \mathcal{F}_O(\nu)$ was introduced in [9] and members of this class were called Ozaki close-to-

convex functions. Also, $\mathcal{F}_O(1, 1) = \mathcal{F}(1)$ was studied by Ponnusamy et al. [13]. Furthermore, $\mathcal{F}_O(1/2, 1) = \mathcal{K}$.

It is remarkable that by means of the principle of subordination between analytic functions, the definition of the class $\mathcal{F}_O(\nu, \gamma)$ can be rewritten as follows:

$$\mathcal{F}_O(\nu, \gamma) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \frac{2\nu+1}{2} \left(\frac{1+z}{1-z} \right)^\gamma - \frac{2\nu-1}{2} := \phi(z), z \in \mathbb{U} \right\}. \quad (11)$$

The present paper was undertaken to investigate some geometric features of the class $\mathcal{F}_O(\nu, \gamma)$ such as close-to-convexity and strongly starlikeness. In addition, we found estimates for the coefficients a_n and give sharp bounds on Fekete-Szegő functionals and logarithmic coefficients for functions belonging to the class $\mathcal{F}_O(\nu, \gamma)$, which incorporates some known outcomes as the specific cases.

2. Some Geometric Properties of the Class $\mathcal{F}_O(\nu, \gamma)$

In this section, we investigate some geometric properties like strongly starlikeness and close-to-convexity for the class $\mathcal{F}_O(\nu, \gamma)$ to present the relation of this class with the well-known families of univalent functions. The key in proving is Nunokawa's lemma [14] (see also [15]), and so in order to prove our result, we require the following lemmas.

We denote by Q the class of all complex-valued functions q for which q is univalent at each $\bar{\mathbb{U}} \setminus E(q)$ and $q'(\xi) \neq 0$ for all $\xi \in \partial\mathbb{U} \setminus E(q)$ where

$$E(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \rightarrow \xi} q(z) = \infty \right\}. \quad (12)$$

Lemma 2 ([16], Lemma 2.2d (i)). *Let $q \in Q$ with $q(0) = a$ and let $p(z) = a + p_n z^n + \dots$ be analytic in \mathbb{U} with $p(z) \equiv 1$ and $n \geq 1$. If p is not subordinate to q in \mathbb{U} , then there exist $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(q)$ such that $\{p(z) : z \in \mathbb{U}, |z| < |z_0|\} \subset q(\mathbb{U})$:*

$$p(z_0) = q(\xi_0). \quad (13)$$

Lemma 3 (see [14, 15]). *Let the function p given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (14)$$

be analytic in \mathbb{U} with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ with

$$\begin{aligned} |\arg(p(z))| &< \frac{\beta\pi}{2} \quad (|z| < |z_0|), \\ |\arg(p(z_0))| &= \frac{\beta\pi}{2}, \end{aligned} \quad (15)$$

for some $\beta > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta \quad (i = \sqrt{-1}), \quad (16)$$

where

$$k \geq \frac{a + a^{-1}}{2} \geq 1, \quad \text{when } \arg(p(z_0)) = \frac{\beta\pi}{2} \quad (17)$$

$$k \leq -\frac{a + a^{-1}}{2} \leq -1, \quad \text{when } \arg(p(z_0)) = -\frac{\beta\pi}{2}, \quad (18)$$

where

$$[p(z_0)]^{1/\beta} = \pm ia, \quad a > 0. \quad (19)$$

Theorem 4. Let $\nu \in [1/2, 1]$ and $\beta_0 \leq \beta \leq 1$ where β_0 ($0 < \beta_0 < 1$) is given by $\tan(\pi/2)\beta_0 = (2/(2\nu - 1))\beta_0$. If $f \in \mathcal{A}$ satisfies the following condition:

$$\left| \text{Arg} \left(\frac{2\nu - 1}{2\nu + 1} + \frac{2}{2\nu + 1} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) \right| < \text{Arctan} \left(\frac{2\beta}{2\nu - 1} \right), \quad (20)$$

then

$$\left| \text{Arg} \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \quad (21)$$

Proof. The result is proven by contradiction. Let $f \in \mathcal{A}$ and define the function $M : \mathbb{U} \rightarrow \mathbb{C}$ by

$$M(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}). \quad (22)$$

Then, it is concluded that M is analytic in \mathbb{U} , $M(0) = 1$,

$$1 + \frac{zf''(z)}{f'(z)} = M(z) + \frac{zM'(z)}{M(z)} \quad (z \in \mathbb{U}), \quad (23)$$

and $M(z) \neq 0$ for all $z \in \mathbb{U}$. Indeed, if M has a zero $z_0 \in \mathbb{U}$ of order m , then we have

$$M(z) = (z - z_0)^m M_1(z) \quad (m \in \mathbb{N} = 1, 2, 3, \dots), \quad (24)$$

where M_1 is analytic in \mathbb{U} with $M_1(z_0) \neq 0$. Then,

$$\begin{aligned} & \frac{2\nu - 1}{2\nu + 1} + \frac{2}{2\nu + 1} \left(M(z) + \frac{zM'(z)}{M(z)} \right) \\ &= \frac{2\nu - 1}{2\nu + 1} + \frac{2}{2\nu + 1} \left(M(z) + \frac{zM_1'(z)}{M_1(z)} + \frac{mz}{z - z_0} \right). \end{aligned} \quad (25)$$

Hence, with $z \rightarrow z_0$, in the right hand of the above equality, the argument can properly take any value between $-\pi$ and π , which contradicts to (20).

Define the function $N : \bar{\mathbb{U}} \setminus \{1\} \rightarrow \mathbb{C}$ by

$$N(z) = \left(\frac{1+z}{1-z} \right)^\beta \quad (z \in \bar{\mathbb{U}} \setminus \{1\}). \quad (26)$$

Then, $N \in \mathcal{Q}$, $N(0) = 1$, and $E(N) = \{1\}$. Clearly, $|\text{Arg}(M(z))| < \beta\pi/2$ ($z \in \mathbb{U}$) if and only if $M < N$ on \mathbb{U} . Suppose $|\text{Arg}(M(z_1))| \geq \beta\pi/2$ for some $z_1 \in \mathbb{U}$. Then, M is not subordinate to N . By applying Lemma 2, there exist $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus \{1\}$ so that $\{M(z) : z \in \mathbb{U}, |z| < |z_0|\} \subset N(\mathbb{U})$ and $M(z_0) = N(\xi_0)$. Thus,

$$|\text{Arg}(M(z))| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}), \quad (27)$$

with $|z| < |z_0|$ and

$$|\text{Arg}(M(z_0))| = \frac{\beta\pi}{2}. \quad (28)$$

Therefore, Lemma 3 results in

$$\frac{zM'(z_0)}{M(z_0)} = ik\beta, \quad (29)$$

where $[M(z_0)]^{1/\beta} = \pm ia$ ($a > 0$) and κ is stated by (17) or (18).

First, let $\text{Arg}(M(z_0)) = \beta\pi/2$. Then, we write $M(z_0) = a^\beta (\cos(\beta\pi/2) + i \sin(\beta\pi/2))$, and so for $k \geq 1$, we have

$$\begin{aligned} & \text{Arg} \left(\frac{2\nu - 1}{2\nu + 1} \left(1 + \frac{2}{2\nu - 1} \left(M(z_0) + \frac{zM'(z_0)}{M(z_0)} \right) \right) \right) \\ &= \text{Arg} \left(1 + \frac{2}{2\nu - 1} a^\beta \cos \frac{\beta\pi}{2} + i \frac{2}{2\nu - 1} \left(a^\beta \sin \frac{\beta\pi}{2} + k\beta \right) \right) \\ &= \text{Arctan} \left(\frac{(2/2\nu - 1)(a^\beta \sin(\beta\pi/2) + k\beta)}{1 + (2/(2\nu - 1))a^\beta \cos(\beta\pi/2)} \right) \\ &\geq \text{Arctan} \left(\frac{(2/2\nu - 1)(a^\beta \sin(\beta\pi/2) + \beta)}{1 + (2/(2\nu - 1))a^\beta \cos(\beta\pi/2)} \right). \end{aligned} \quad (30)$$

We define the function $h : (0, a) \rightarrow \mathbb{R}$ by

$$h(s) = \frac{(2/(2\nu - 1))(s^\beta \sin(\beta\pi/2) + \beta)}{1 + (2/(2\nu - 1))s^\beta \cos(\beta\pi/2)}, \quad s \in (0, a). \quad (31)$$

Then, h is a differentiable function on $(0, a)$, and

$$h'(s) = \frac{(2\beta/(2\nu-1))s^\beta \cos(\beta\pi/2)(\tan(\beta\pi/2) - (2\beta/(2\nu-1)))}{(1 + (2/(2\nu-1))s^\beta \cos(\beta\pi/2))^2},$$

$$s \in (0, a). \quad (32)$$

Now, we define $g(\beta)$ as

$$g(\beta) = \tan\left(\frac{\beta\pi}{2}\right) - \frac{2\beta}{2\nu-1} \quad (0 < \beta_0 \leq \beta \leq 1; \beta_0 \neq 1). \quad (33)$$

Then, $g(0) = g(\beta_0) = 0$, $g'(0) = (\pi/2) - (2/(2\nu-1)) < 0$ for $\nu \in [1/2, 1]$, and

$$g''(\beta) = \frac{\pi}{2} \sec^2\left(\frac{\beta\pi}{2}\right) \tan\left(\frac{\beta\pi}{2}\right) > 0. \quad (34)$$

So, the function g has a negative value on $(0, \beta_0)$ and a positive value on $(\beta_0, 1)$.

According to the assumption and from (32), it follows that $h'(s) > 0$ for all $s \in (0, a)$ for $\beta \in (\beta_0, 1)$. Also, this shows that the function $l: (0, a) \rightarrow \mathbb{R}$ defined by

$$l(s) = \text{Arctan}(h(s)), \quad s \in (0, a), \quad (35)$$

is nondecreasing on $(0, a)$. Hence,

$$l(a) \geq \lim_{s \rightarrow 0^+} l(s) = \text{Arctan}\left(\frac{2\beta}{2\nu-1}\right). \quad (36)$$

Therefore, we get

$$\text{Arctan}\left(\frac{(2/(2\nu-1))(a^\beta \sin(\beta\pi/2) + \beta)}{1 + (2/(2\nu-1))a^\beta \cos(\beta\pi/2)}\right) \geq \text{Arctan}\left(\frac{2\beta}{2\nu-1}\right). \quad (37)$$

Now applying (30) and (37), we obtain

$$\begin{aligned} & \text{Arg}\left(\frac{2+\alpha}{2\nu-1} \left(1 + \frac{2}{2\nu-1} \left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1} \left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)\right) \\ &\geq \text{Arctan}\left(\frac{(2/(2\nu-1))(a^\beta \sin(\beta\pi/2) + \beta)}{1 + (2/(2\nu-1))a^\beta \cos(\beta\pi/2)}\right) \\ &\geq \text{Arctan}\left(\frac{2\beta}{2\nu-1}\right), \end{aligned} \quad (38)$$

which contradicts to (20).

Next, let $\text{Arg}(M(z_0)) = -(\beta\pi/2)$. Then, we write $M(z_0) = a^\beta(\cos(\beta\pi/2) - i \sin(\beta\pi/2))$. Thus, for $k \leq -1$ and utilizing (37), we get

$$\begin{aligned} & \text{Arg}\left(\frac{2\nu-1}{2\nu+1} \left(1 + \frac{2}{2\nu-1} \left(M(z_0) + \frac{z_0 M'(z_0)}{M(z_0)}\right)\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1} \left(a^\beta e^{-i\beta\pi/2} + ik\beta\right)\right) \\ &= \text{Arctan}\left(\frac{(2/(2\nu-1))(-a^\beta \sin(\beta\pi/2) + k\beta)}{1 + (2/(2\nu-1))a^\beta \cos(\beta\pi/2)}\right) \\ &\leq \text{Arctan}\left(\frac{(2/(2\nu-1))(-a^\beta \sin(\beta\pi/2) - \beta)}{1 + (2/(2\nu-1))a^\beta \cos(\beta\pi/2)}\right) \\ &= -\text{Arctan}\left(\frac{(2/(2\nu-1))(a^\beta \sin(\beta\pi/2) + \beta)}{1 + (2/(2\nu-1))a^\beta \cos(\beta\pi/2)}\right) \\ &\leq -\text{Arctan}\left(\frac{2\beta}{2\nu-1}\right), \end{aligned} \quad (39)$$

which contradicts to (20).

From the above contradictions, it results in

$$\left| \text{Arg}\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \quad (40)$$

Hence, the proof is completed.

Corollary 5. Let $\nu \in [1/2, 1]$, $\beta_0 \leq \beta \leq 1$ where $\beta_0 (0 < \beta_0 < 1)$ is given by $\tan(\pi/2)\beta_0 = (2/(2\nu-1))\beta_0$ and $\gamma = (2/\pi) \text{Arctan}(2\beta/(2\nu-1))$. If $f \in \mathcal{F}_O(\nu, \gamma)$, then $f \in \tilde{\mathcal{D}}^*(\beta)$.

Remark 6. According to the assumptions of Corollary 5, if $f \in \mathcal{F}_O(\nu, \gamma)$, then $f \in \tilde{\mathcal{D}}^*(\beta)$ implies $\text{Re}(zf'(z)/f(z)) > 0$ ($z \in \mathbb{U}$) and so it is well known that f is univalent.

Theorem 7. Let $\beta \in (0, 1]$, $\nu \in [1/2, 1]$. If $f \in \mathcal{A}$ and

$$\left| \text{Arg}\left(\frac{2\nu+1}{2\nu-1} + \frac{2}{2\nu-1} \left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \right| < \text{Arctan}\left(\frac{2\beta}{2\nu+1}\right), \quad (41)$$

then

$$\left| \text{Arg}(f'(z)) \right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \quad (42)$$

Proof. The result is proven by contradiction. To prove our result, we set the function $M: \mathbb{U} \rightarrow \mathbb{C}$ by

$$M(z) = f'(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}). \quad (43)$$

Then, M is analytic in \mathbb{U} , $p(0) = 1$, $M(z) \neq 0$ for all $z \in \mathbb{U}$, and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zM'(z)}{M(z)}. \tag{44}$$

If there is a point $z_0 \in \mathbb{U}$, then

$$|\text{Arg}(M(z))| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}), \tag{45}$$

with $|z| < |z_0|$, and

$$|\text{Arg}(M(z_0))| = \frac{\beta\pi}{2}. \tag{46}$$

Then, from Lemma 3, we have

$$\frac{zM'(z_0)}{M(z_0)} = ik\beta, \tag{47}$$

where $[M(z_0)]^{1/\beta} = \pm ia$ ($a > 0$) and k is stated by (17) or (18). For the case $\text{Arg}(M(z_0)) = \alpha\pi/2$ when

$$M(z_0)^{1/\beta} = ia \quad (a > 0) \tag{48}$$

and $k \geq 1$, we have

$$\begin{aligned} & \text{Arg}\left(\frac{2\nu-1}{2\nu+1} \left(1 + \frac{2}{2\nu-1} \left(1 + \frac{z_0M'(z_0)}{M(z_0)}\right)\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1} \left(1 + \frac{z_0M'(z_0)}{M(z_0)}\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1}(1 + ik\beta)\right) \\ &= \text{Arctan}\left(\frac{2k\beta}{2\nu+1}\right) \geq \text{Arctan}\left(\frac{2\beta}{2\nu+1}\right), \end{aligned} \tag{49}$$

which contradicts to (41).

Next, for the case $\text{Arg}(M(z_0)) = -(\alpha\pi/2)$ when

$$M(z_0) = -ia \quad (a > 0) \tag{50}$$

and $k \leq -1$, by applying the same method mentioned above, it can be concluded that

$$\begin{aligned} & \text{Arg}\left(\frac{2\nu-1}{2\nu+1} \left(1 + \frac{2}{2\nu-1} \left(1 + \frac{z_0M'(z_0)}{M(z_0)}\right)\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1} \left(1 + \frac{z_0p'(z_0)}{p(z_0)}\right)\right) \\ &= \text{Arg}\left(1 + \frac{2}{2\nu-1}(1 + ik\beta)\right) \\ &= \text{Arctan}\left(\frac{-2k\beta}{2\nu+1}\right) \leq -\text{Arctan}\left(\frac{2\beta}{2\nu+1}\right), \end{aligned} \tag{51}$$

which contradicts to (41).

As a result, from the above contradictions, we obtain

$$|\text{Arg}(f'(z))| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}), \tag{52}$$

and therefore, the proof is completed.

Corollary 8. *Let $\beta \in (0, 1]$, $\nu \in [1/2, 1]$, and $\gamma = (2/\pi)\text{Arctan}(2\beta/(2\nu+1))$. If $f \in \mathcal{F}_O(\nu, \gamma)$, then $f \in \tilde{\mathcal{C}}(\beta)$.*

Remark 9. >According to the assumptions of Corollary 8, if $f \in \mathcal{F}_O(\nu, \gamma)$, then $f \in \tilde{\mathcal{C}}(\beta)$ implies $\text{Re } f'(z) > 0$ ($z \in \mathbb{U}$) and so it is well known that f is univalent.

3. Coefficient Bounds

In this section, we find sharp bounds on Fekete-Szegő functionals and logarithmic coefficients (see [17–22]) for functions belonging to the class $\mathcal{F}_O(\nu, \gamma)$. Also, we present a general problem of coefficients in this class. To prove our main results, some requirements are needed. We remark in passing that the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by the next form:

$$\log\left(\frac{f(z)}{z}\right) = 2 \sum_{n=1}^{\infty} \gamma_n(f)z^n \quad (z \in \mathbb{U}). \tag{53}$$

These coefficients are of great significance for different estimates in the theory of univalent functions (see [19–21]).

Ma and Minda [11] defined the class consisting of several well-known classes as follows:

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z), z \in \mathbb{U} \right\}, \tag{54}$$

where in here, it is supposed that φ is a univalent function in \mathbb{U} with $\varphi(0) = 1$ such that it has the following form:

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots, z \in \mathbb{U}, \text{ with } B_1 \neq 0. \tag{55}$$

Lemma 10 ([17], Theorem 2). *Let $f \in \mathcal{K}(\varphi)$. Then, the logarithmic coefficients of f satisfy*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{56}$$

$$|\gamma_2| \leq \begin{cases} \frac{|B_1|}{12}, & \text{if } |4B_2 + B_1^2| \leq 4|B_1|, \\ \frac{|4B_2 + B_1^2|}{8}, & \text{if } |4B_2 + B_1^2| > 4|B_1|, \end{cases} \tag{57}$$

and if B_1, B_2 , and B_3 are real values,

$$|\gamma_3| \leq \frac{|B_1|}{24}H(q_1; q_2), \tag{58}$$

where $H(q_1; q_2)$ is given by [23, 24], $q_1 = (B_1 + (4B_2/B_1))/2$, and $q_2 = (B_2 + (2B_3/B_1))/2$. The bounds (56) and (57) are sharp.

Lemma 11 [23, 24]. *If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ ($z \in \mathbb{U}$), then, the following sharp estimate is given:*

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq L(q_1; q_2), \tag{59}$$

for any real numbers q_1 and q_2 where

$$L(q_1; q_2) = \begin{cases} 1, & \text{if } (q_1, q_2) \in A \cup B, \\ |q_2|, & \text{if } (q_1, q_2) \in C \cup D, \end{cases} \tag{60}$$

and the sets A, B, C, D are stated as follows:

$$\begin{aligned} A &= \left\{ (q_1, q_2) \in \mathbb{R}^2 : |q_1| \leq \frac{1}{2}, |q_2| \leq 1 \right\}, \\ B &= \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27} ((|q_1| + 1)^3 - (|q_1| + 1)) \leq q_2 \leq 1 \right\}, \\ C &= \{ (q_1, q_2) : |q_1| \leq 2, q_2 \geq 1 \}, \\ D &= \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, q_2 \geq \frac{1}{12} (q_1^2 + 8) \right\}. \end{aligned} \tag{61}$$

Lemma 12 [23, 25]. *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then,*

$$|w_2 - t w_1^2| \leq \max \{1, |t|\}, \quad \text{for all } t \in \mathbb{C}. \tag{62}$$

The result is sharp for the function $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 13 [26]. *Let φ be a convex function in \mathbb{U} with the form $\varphi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If function $f \in \mathcal{K}(\varphi)$, then*

$$|a_n| \leq \frac{\prod_{k=2}^n (k-2 + |B_1|)}{n!} \quad (n = 2, 3, \dots). \tag{63}$$

Theorem 14. *Let $f \in \mathcal{F}_O(\nu, \gamma)$. Then,*

$$\begin{aligned} |\gamma_1| &\leq \frac{(2\nu + 1)\gamma}{4}, \\ |\gamma_2| &\leq \begin{cases} \frac{(2\nu + 1)\gamma}{12}, & \text{if } \gamma \in \left(0, \frac{4}{5 + 2\nu}\right], \\ \frac{\gamma^2(2\nu + 1)(2\nu + 5)}{48}, & \text{if } \gamma \in \left[\frac{4}{5 + 2\nu}, 1\right], \end{cases} \\ |\gamma_3| &\leq \begin{cases} \frac{(2\nu + 1)\gamma}{24}, & \text{if } \gamma \in \left(0, \frac{2}{\sqrt{6\nu + 7}}\right], \\ \frac{\gamma^2(2\nu + 1)(\gamma^2(6\nu + 7) + 2)}{144}, & \text{if } \gamma \in \left[\frac{2}{\sqrt{6\nu + 7}}, 1\right]. \end{cases} \end{aligned} \tag{64}$$

All the inequalities are sharp.

Proof. Let $f \in \mathcal{F}_O(\nu, \gamma)$. From (11), it follows that

$$\begin{aligned} 1 + \frac{z f''(z)}{f'(z)} < \phi(z) &= 1 + (2\nu + 1)\gamma z + (2\nu + 1)\gamma^2 z^2 \\ &+ \frac{1}{3}(2\nu + 1)\gamma(2\gamma^2 + 1)z^3 + \dots \\ &= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots. \end{aligned} \tag{65}$$

Now, it is enough to obtain the estimate of $|\gamma_3|$. By Lemma 10 with $\varphi = \phi$ for $|\gamma_3|$, we obtain

$$\begin{aligned} |\gamma_3| &\leq \frac{|B_1|}{24} H(q_1; q_2) = \frac{(2\nu + 1)\gamma}{24} H(q_1; q_2) \\ &= \frac{(2\nu + 1)\gamma}{24} L(q_1; q_2), \end{aligned} \tag{66}$$

where $L(q_1; q_2)$ is given by Lemma 11,

$$\begin{aligned} q_1 &= \frac{B_1 + (4B_2/B_1)}{2} =: \frac{\gamma(2\nu + 5)}{2}, \\ q_2 &= \frac{B_2 + (2B_3/B_1)}{2} =: \frac{\gamma^2(6\nu + 7) + 2}{6}. \end{aligned} \tag{67}$$

First, if we consider

$$A = \left\{ (q_1, q_2) \in \mathbb{R}^2 : |q_1| \leq \frac{1}{2}, |q_2| \leq 1 \right\}, \tag{68}$$

then it is clear that $q_1 \leq (1/2)$ for $0 < \gamma \leq (1/(2\nu + 5))$ and $q_2 \leq 1$ for $0 < \gamma \leq (2/\sqrt{(6\nu + 7)})$. Therefore, we conclude $(q_1, q_2) \in A$ for $0 < \gamma \leq (1/(2\nu + 5))$.

Also, regarding

$$B = \left\{ (q_1, q_2) \in \mathbb{R}^2 : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27} ((|q_1| + 1)^3 - (|q_1| + 1)) \leq q_2 \leq 1 \right\}, \tag{69}$$

then it is clear that $(1/2) \leq q_1 \leq 2$ for $(1/(2\nu + 5)) \leq \gamma \leq (4/(2\nu + 5))$. Thus, in the above relation, $q_2 \leq 1$ is equivalent to $0 < \gamma \leq (2/\sqrt{(6\nu + 7)})$. Also,

$$\frac{4}{27} (q_1 + 1)^3 - (q_1 + 1) \leq q_2, \tag{70}$$

that is,

$$\begin{aligned} g(\nu, \gamma) &:= \frac{4(\nu + (5/2))^3}{27} \gamma^3 + \left(\frac{4(\nu + (5/2))^2}{27} \right. \\ &+ \left. \left(\nu + \frac{5}{2} \right) \left(\frac{8\nu}{27} + \frac{20}{27} \right) - \nu - \frac{7}{6} \right) \gamma^2 \\ &+ \left(-\frac{5}{9} \nu - \frac{25}{18} \right) \gamma - \frac{32}{27} \leq 0 \end{aligned} \tag{71}$$

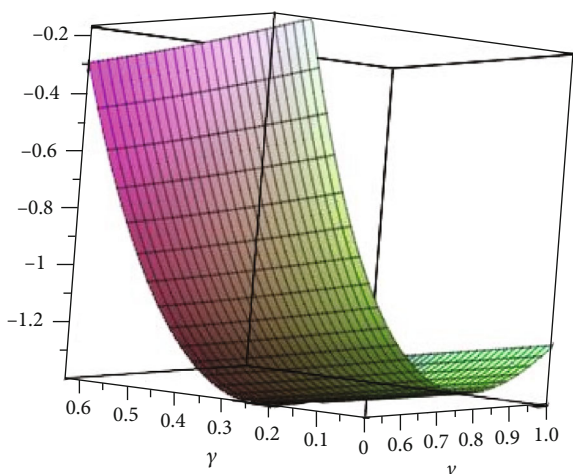


FIGURE 1: The plot of $g(v, \gamma)$ for $\gamma = 0 \cdots 2/\sqrt{(6v + 7)}$, $v = 1/2 \cdots 1$.

holds for $0 < \gamma \leq (2/\sqrt{(6v + 7)})$ (see Figure 1). Therefore, considering the above results, we conclude $(q_1, q_2) \in B$ for $\gamma \in [(1/(2v + 5)), (2/\sqrt{(6v + 7)})]$.

Next, considering

$$C = \{(q_1, q_2) \in \mathbb{R}^2 : |q_1| \leq 2, q_2 \geq 1\}, \tag{72}$$

it is clear that $q_1 \leq 2$ for $\gamma \leq (4/(2v + 5))$, and according to the above computation, the inequality $q_2 \geq 1$ holds for $\gamma \geq (2/\sqrt{(6v + 7)})$. Therefore, $(q_1, q_2) \in C$ for $\gamma \in [2/\sqrt{(6v + 7)}, 4/(2v + 5)]$.

Finally, let us consider

$$D = \left\{ (q_1, q_2) \in \mathbb{R}^2 : 2 \leq |q_1| \leq 4, q_2 \geq \frac{1}{12}(q_1^2 + 8) \right\}. \tag{73}$$

It is clear that $2 \leq q_1 \leq 4$ for $\gamma \in [4/(2v + 5), 8/(2v + 5)]$. On the other hand, the inequality $q_2 \geq ((q_1^2 + 8)/12)$ is equivalent to

$$\gamma^2 - \frac{16}{-4v^2 + 28v + 31} \geq 0, \tag{74}$$

and this last inequality holds for $\gamma \geq (4/\sqrt{(-4v^2 + 28v + 31)})$. Therefore, $(q_1, q_2) \in D$ for $\gamma \in [4/\sqrt{(-4v^2 + 28v + 31)}, 8/(2v + 5)]$. Since $(8/(2v + 5)) \geq 1$, so $(q_1, q_2) \in D$ for $\gamma \in [4/\sqrt{(-4v^2 + 28v + 31)}, 1]$.

By applying the above four conclusions from (66), it follows that

$$|\gamma_3| \leq \frac{(2v + 1)\gamma}{24} L(q_1; q_2) = \begin{cases} \frac{(2v + 1)\gamma}{24}, & \text{if } \gamma \in \left(0, \frac{2}{\sqrt{6v + 7}}\right] \text{ (see A and B),} \\ \frac{(2v + 1)\gamma}{24} q_2, & \text{if } \gamma \in \left[\frac{2}{\sqrt{6v + 7}}, 1\right] \text{ (see A and B),} \end{cases} = \begin{cases} \frac{(2v + 1)\gamma}{24}, & \text{if } \gamma \in \left(0, \frac{2}{\sqrt{6v + 7}}\right], \\ \frac{\gamma(2v + 1)(\gamma^2(6v + 7) + 2)}{144}, & \text{if } \gamma \in \left[\frac{2}{\sqrt{6v + 7}}, 1\right]. \end{cases} \tag{75}$$

Suppose that $f_{v,\gamma,n} \in \mathcal{F}_O(v, \gamma)$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \phi(z^n) \quad (z \in \mathbb{U}; n \in \mathbb{N}). \tag{76}$$

Then, the functions $f_{v,\gamma,n}$ are of the following form:

$$\begin{aligned} f_{v,\gamma,n}(z) &= \int_0^z \exp\left(\int_0^x \frac{\phi(t^n) - 1}{t} dt\right) dx \\ &= z + \frac{\gamma(2v + 1)}{n(n + 1)} z^{n+1} + \frac{\gamma^2(2v + 1)(2v + n + 1)}{2n^2(2n + 1)} z^{2n+1} \\ &\quad + \cdots \quad (z \in \mathbb{U}; n \in \mathbb{N}), \end{aligned} \tag{77}$$

and the $f_{v,\gamma,n}$ are extremal functions for diverse problems in the subclass $\mathcal{F}_O(v, \gamma)$. Hence, the extremal function for the absolute value of the coefficient γ_1 is $f = f_{v,\gamma,1}$, and those for

the absolute value of the coefficients γ_2 and γ_3 in the first relations are given as $f = f_{v,\gamma,2}$ and $f = f_{v,\gamma,3}$, respectively, whereas in the second relations, it is given as $f = f_{v,\gamma,1}$.

By taking $\gamma = 1$ and $v = (c - 1)/2$ in Theorem 14, we have the following result which is the estimates obtained by Ponnusamy et al. for $c \in [2, 3]$ in Theorem 2.7 of [27].

Corollary 15. Let $f \in \mathcal{F}_O(v)$. Then,

$$\begin{aligned} |\gamma_1| &\leq \frac{c}{4}, \\ |\gamma_2| &\leq \frac{4c + c^2}{48}, \\ |\gamma_3| &\leq \frac{2c + c^2}{48}. \end{aligned} \tag{78}$$

All the inequalities are sharp.

Theorem 16. Let $f \in \mathcal{F}_O(\nu, \gamma)$, then

$$|a_n| \leq \frac{\prod_{k=2}^n (k-2 + \gamma(2\nu+1))}{n!} \quad (n=2, 3, \dots). \quad (79)$$

Proof. Since

$$\phi(z) = \frac{2\nu+1}{2} \left(\frac{1+z}{1-z} \right)^\gamma - \frac{2\nu-1}{2} \quad (80)$$

with $B_1 = \gamma(2\nu+1)$ is convex, the result is obtained by utilization of Lemma 13.

Remark 17. Taking $\gamma = 1$ in Theorem 16, the result presented in [3], Theorem 18 is obtained.

Theorem 18. Let $f \in \mathcal{F}_O(\nu, \gamma)$. Then, we have sharp bounds for complex μ :

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma^2(2\nu+1)}{6} \left| 2(\nu+1) - \frac{3(2\nu+1)}{2} \right|, & \text{for } \left| \mu - \frac{4(\nu+1)}{3(2\nu+1)} \right| \geq \frac{2}{3\gamma(2\nu+1)}, \\ \frac{\gamma(2\nu+1)}{6}, & \text{for } \left| \mu - \frac{4(\nu+1)}{3(2\nu+1)} \right| < \frac{2}{3\gamma(2\nu+1)}. \end{cases} \quad (81)$$

Proof. Let $f \in \mathcal{F}_O(\nu, \gamma)$; then, from (11), by the concept of the subordination, there is $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \dots \quad (82)$$

So, we obtain

$$\begin{aligned} 2a_2 &= B_1 w_1, \\ 6a_3 - 4a_2^2 &= B_1 w_2 + B_2 w_1^2. \end{aligned} \quad (83)$$

Therefore,

$$|a_3 - \mu a_2^2| = \frac{1}{6} |B_1| |w_2 + \gamma w_1^2|. \quad (84)$$

The outcomes are given by applying Lemma 12 with $\gamma = [(B_2/B_1) + B_1(1 - (3\mu/2))]$ and (65). Equality occurs in the first inequality by $f = f_{\nu, \gamma, 1}$ and in the second inequality for $f = f_{\nu, \gamma, 2}$.

4. Conclusion

In the current study, we investigate some geometric features such as strongly starlikeness and close-to-convexity for the class $\mathcal{F}_O(\nu, \gamma)$. Further, sharp bounds are given on Fekete-Szegő functionals and logarithmic coefficients for functions belonging to this class.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to and approved the final manuscript.

References

- [1] A. W. Goodman, *Univalent Functions*, Mariner Publishing Company, New York, NY, USA, 1983.
- [2] D. Alimohammadi, N. E. Cho, E. Analouei Adegani, and A. Motamednezhad, "Argument and coefficient estimates for certain analytic functions," *Mathematics*, vol. 8, no. 1, p. 88, 2020.
- [3] V. Allu, D. K. Thomas, and N. Tuneski, "On Ozaki close-to-convex functions," *Bulletin of the Australian Mathematical Society*, vol. 99, no. 1, pp. 89–100, 2019.
- [4] N. E. Cho, Z. Oroujy, E. A. Adegani, and A. Ebadian, "Majorization and coefficient problems for a general class of starlike functions," *Symmetry*, vol. 12, no. 3, p. 476, 2020.
- [5] S. Gala, Q. Liu, and M. A. Ragusa, "A new regularity criterion for the nematic liquid crystal flows," *Applicable Analysis*, vol. 91, no. 9, pp. 1741–1747, 2012.
- [6] S. Gala and M. A. Ragusa, "Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices," *Applicable Analysis*, vol. 95, no. 6, pp. 1271–1279, 2016.
- [7] P. Goel and S. Sivaprasad Kumar, "Certain class of starlike functions associated with modified sigmoid function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 957–991, 2020.

- [8] S. Kanas, V. S. Masih, and A. Ebadian, "Relations of a planar domains bounded by hyperbolas with families of holomorphic functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 246, 2019.
- [9] R. Kargar and A. Ebadian, "Ozaki's conditions for general integral operator," *Sahand Communications in Mathematical Analysis*, vol. 5, no. 1, pp. 61–67, 2017.
- [10] H. M. Srivastava, D. Răducanu, and P. Zaprawa, "Certain subclass of analytic functions defined by means of differential subordination," *Filomat*, vol. 30, pp. 3743–3757, 2016.
- [11] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, pp. 157–169, Tianjin, 1992, Internat. Press, Cambridge, MA, USA.
- [12] M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda, and N. Koike, "Some results for strongly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 212, no. 1, pp. 98–106, 1997.
- [13] S. Ponnusamy, S. K. Sahoo, and H. Yanagihara, "Radius of convexity of partial sums of functions in the close-to-convex family," *Nonlinear Analysis*, vol. 95, pp. 219–228, 2014.
- [14] M. Nunokawa, "On properties of non-Carathéodory functions," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 68, pp. 152–153, 1992.
- [15] M. Nunokawa, "On the order of strongly starlikeness of strongly convex functions," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 69, pp. 234–237, 1993.
- [16] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker Inc., New York, NY, USA, 2000.
- [17] E. Analouei Adegani, N. E. Cho, and M. Jafari, "Logarithmic coefficients for univalent functions defined by subordination," *Mathematics*, vol. 7, no. 5, p. 408, 2019.
- [18] M. Fekete and G. Szegő, "Eine Bemerkung Über Ungerade Schlichte Funktionen," *Journal of the London Mathematical Society*, vol. 8, pp. 85–89, 1933.
- [19] I. M. Milin, *Univalent Functions and Orthonormal Systems*, vol. 49, American Mathematical Society Translations of Mathematical Monographs, Providence, RI, 1977.
- [20] I. M. Milin, "On a property of the logarithmic coefficients of univalent functions," in *Metric Questions in the Theory of Functions*, pp. 86–90, Naukova Dumka, Kiev, 1980.
- [21] I. M. Milin, "On a conjecture for the logarithmic coefficients of univalent functions," vol. 125, pp. 135–143, 1983.
- [22] H. M. Srivastava, N. Raza, E. S. AbuJarad, G. Srivastava, and M. H. AbuJarad, "Fekete-Szegő inequality for classes of (p, q) -starlike and (p, q) -convex functions," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A Matemáticas*, vol. 113, no. 4, pp. 3563–3584, 2019.
- [23] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Coefficient bounds for p -valent functions," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 35–46, 2007.
- [24] D. V. Prokhorov and J. Szynal, "Inverse coefficients for $(\alpha; \beta)$ -convex functions," *Annales Universitatis Mariae Curie-Skłodowska, sectio A – Mathematica*, vol. 35, pp. 125–143, 1984.
- [25] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, no. 1, pp. 8–12, 1969.
- [26] Q. H. Xu, Y. C. Gui, and H. M. Srivastava, "Coefficient estimates for certain subclasses of analytic functions of complex order," *Taiwanese Journal of Mathematics*, vol. 15, no. 5, pp. 2377–2386, 2011.
- [27] S. Ponnusamy, N. L. Sharma, and K.-J. Wirths, "Logarithmic coefficients problems in families related to starlike and convex functions," *Journal of the Australian Mathematical Society*, pp. 1–20, 2019.