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Research Article

On Best Proximity Point Results for Some Type of Mappings

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In this paper, we give new conditions for existence and uniqueness of a best proximity point for Geraghty- and Caristi-type mappings. The presented results are most valuable generalizations of the Geraghty and Caristi fixed point theorems.

1. Introduction and Preliminaries

The Banach contraction principle (BCP) in metric spaces has been generalized and extended in various ways. As a generalization of the BCP, Geraghty [1] proved the following.

Theorem 1 [1]. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be an operator. Suppose that there exists $\beta : [0, +\infty) \longrightarrow [0, 1)$ satisfying the condition*

$$\beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \quad (1)$$

If T satisfies the following inequality

$$d(Tx, Ty) \le \beta(d(x, y))d(x, y), \text{ for all } x, y \in X, \quad (2)$$

then T has a unique fixed point.

The following are two examples of Geraghty functions β.

$$\beta(t) = \begin{cases} e^{-te^t}, & \text{if } t \in (0, \infty), \\ 0, & \text{if } t = 0, \end{cases}$$
$$a\beta(t) = \begin{cases} \dfrac{1}{1+t^2}, & \text{if } t \in (0, \infty), \\ 0, & \text{if } t = 0. \end{cases} \quad (3)$$

One of the important extensions of the BCP was given by Caristi [2].

Theorem 2 [2]. *Let Γ be a self-mapping on a complete metric space (X, d). Assume that there is a bounded below and lower semicontinuous function $\psi : X \longrightarrow \mathbb{R}$ so that*

$$d(x, \Gamma x) \le \psi(x) - \psi(\Gamma x) \quad (4)$$

for all $x \in X$. Then, Γ possesses a fixed point.

On the other hand, Kirk et al. [3] in 2003 introduced the notion of a cyclic representation.

Definition 3 [3]. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \longrightarrow A \cup B$. Then, T *is* called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

The following interesting theorem for a cyclic map was given in [3].

Theorem 4 [3]. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \longrightarrow A \cup B$ is a cyclic map such that*

$$d(Tx, Ty) \le kd(x, y) \tag{5}$$

for all $x \in A$ and $y \in B$, where $k \in [0, 1)$ is a constant. Then, T has a unique fixed point u and $u \in A \cap B$.

Let $\mathscr{M}$ and $\mathscr{N}$ be nonempty sets of a metric space (X, d). Given a map, $\Gamma : \mathscr{M} \longrightarrow \mathscr{N}$. An element $x \in \mathscr{M}$ is called a best proximity point of Γ if

$$d(x, \Gamma x) = d(\mathscr{M}, \mathscr{N}) =: \inf \{d(m, n), m, \in \mathscr{M}, n \in \mathscr{N}\}. \tag{6}$$

The set of best proximity points of Γ is denoted by P_Γ $(\mathscr{M}, \mathscr{N})$. The research of best proximity points is meaningful in optimization. The problem of existence of best proximity points in uniformly convex Banach spaces and in metric spaces as well as the convergence of sequences to such points has been focused on and successfully solved in some classic pioneering works (see [4]).

Definition 5 [5]. Let (X, d) be a metric space and A and B be subsets of X. A map $T : A \cup B \longrightarrow A \cup B$ is said to be a cyclic contractive map if it satisfies

$$T(A) \subset B, T(B) \subset A\,; \tag{7}$$

(i) $d(Tx, Ty) \le kd(x, y) + (1 - k)dist(A, B)$, for all $x \in A$ and $y \in B$.

Eldred and Veeramani [5] extended Theorem 4 to include the case $A \cap B = \varnothing$, by the following existence result of a best proximity point.

Theorem 6 [5]. *Let A and B be nonempty closed subsets of a metric space X and let $T : A \cup B \longrightarrow A \cup B$ be a cyclic contraction map. If either A or B is boundedly compact, then there exists $x \in A \cup B$ such that $d(x, Tx) = dist(A, B)$.*

A convenience attention has been recently devoted to the research on existence and uniqueness of best proximity points of self-mappings, as well as, to the investigation of associated relevant properties, for instance, stability of the iterations. The various related performed researches include the cases of cyclic ϕ-contractions [6, 7], cyclic Meir-Keeler contractions [8], weak cyclic Bianchini contractions [9], weak cyclic Kannan contractions [10], p-cyclic summing iterated contractions [11], and MF-cyclic contractions with Property UC [12]. Some contractive conditions and related properties under general contractive conditions including some proximal rational types have been also investigated [13]. In this paper, we ensure the existence of best proximity points for Geraghty and Caristi type contraction mappings.

2. A Best Proximity Point Result for Geraghty-Type Contractions

In this section, we introduce cyclic Geraghty contraction maps and give new conditions for existence and uniqueness of a best proximity point.

Definition 7. Let X be a complete metric space and $\mathscr{M}$ and $\mathscr{N}$ be subsets of X. A map $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ is a cyclic Geraghty contraction map if there exists $\beta \in \mathscr{F}$ such that

$$\Gamma(\mathscr{M}) \subset \mathscr{N}, \Gamma(\mathscr{N}) \subset \mathscr{M}\,; \tag{8}$$

(i) $d(\Gamma x, \Gamma y) \le \beta(d(x, y))d(x, y)$ for all $x, y \in \mathscr{M} \cup \mathscr{N}$

where $\mathscr{F}$ is the set of functions $\beta : [0, \infty) \longrightarrow [0, 1)$ so thatif $t_n \in [0, d(\mathscr{M}, \mathscr{N}))$ and $\beta(\Gamma_n) \longrightarrow 1$, then $t_n \longrightarrow 0$;if $t_n \in (d(\mathscr{M}, \mathscr{N}), \infty)$ and $\beta(\Gamma_n) \longrightarrow 1$, then $t_n \longrightarrow d(\mathscr{M}, \mathscr{N})$.

We give the following theorem (comparable to Theorem 3.1 of [1]).

Theorem 8. *Let $\mathscr{M}$ and $\mathscr{N}$ be closed subsets of a complete metric space X such that diam$(\mathscr{M})$, diam$(\mathscr{N}) < d(\mathscr{M}, \mathscr{N})$. Suppose $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ is a cyclic Geraghty contraction map. Then, $P_\Gamma(\mathscr{M}, \mathscr{N}) \ne 0$. Further, if $x_0 \in \mathscr{M}$ and $x_{n+1} = \Gamma$ x_n, then $\{x_{2n}\}$ converges to a best proximity point.*

Proof. Fix $x \in \mathscr{M} \cup \mathscr{N}$ and define a sequence $\{x_n\}$ in $\mathscr{M} \cup \mathscr{N}$ by $x_n = \Gamma^n x$, $n \in N_0$. First, we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(\mathscr{M}, \mathscr{N})$. We have

$$d(x_n, x_{n+1}) = d(\Gamma x_{n-1}, \Gamma x_n) \le \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n). \tag{9}$$

Hence, $\{d(x_n, x_{n+1})\}$ is monotonic decreasing and bounded below. So,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \tag{10}$$

exists. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$. It is clear that $d(\mathscr{M}, \mathscr{N}) \le \delta$. Assume that $\delta > d(\mathscr{M}, \mathscr{N})$. We have

$$\delta = \lim_{n\to\infty} d(x_n, x_{n+1}) \le \lim_{n\to\infty} \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) < \lim_{n\to\infty} d(x_{n-1}, x_n) = \delta, \tag{11}$$

so $\delta = d(\mathscr{M}, \mathscr{N})$.

Hence, $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(\mathscr{M}, \mathscr{N})$. We shall show that $d(x_{2n}, x_{2n+2}) \longrightarrow 0$ and $d(x_{2n-1}, x_{2n+1}) \longrightarrow 0$. We have

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &= d(\Gamma x_{2n-1}, \Gamma x_{2n+1}) \\ &\le \beta(d(x_{2n-1}, \Gamma x_{2n+1}))d(x_{2n-1}, \Gamma x_{2n+1}). \end{aligned} \tag{12}$$

That is, $\{d(x_{2n}, x_{2n+2})\}$ is monotonic decreasing and bounded below. Hence, $\lim_{n\to\infty} d(x_{2n}, x_{2n+2})$ exists.

Let $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = \delta$. Assume that $\delta > 0$. One writes

$$\begin{aligned} \delta &= \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) \\ &\le \lim_{n\to\infty} \beta(d(x_{2n-1}, x_{2n+1}))d(x_{2n-1}, x_{2n+1}) \\ &< \lim_{n\to\infty} d(x_{2n-2}, x_{2n}) = \delta, \end{aligned} \tag{13}$$

so $\delta = 0$. Hence, $d(x_{2n}, x_{2n+2}) \longrightarrow 0$. Similarly, we have $d(x_{2n-1}, x_{2n+1}) \longrightarrow 0$. Also, $\{x_{2n}\}$ is a Cauchy sequence. Assume that $\{x_{2n}\}$ is not Cauchy. Then,

$$\limsup_{n,m\to\infty} d(x_{2n}, x_{2m}) > 0. \tag{14}$$

By using the triangular inequality,

$$d(x_{2n}, x_{2m}) \le d(x_{2n}, x_{2n+2}) + d(x_{2n+2}, x_{2m+2}) + d(x_{2m}, x_{2m+2}). \tag{15}$$

Hence, we have

$$\begin{aligned} d(x_{2n}, x_{2m}) &\le d(x_{2n}, x_{2n+2}) + \beta(d(x_{2n+1}, x_{2m+1}))d \\ &\quad \cdot (x_{2n+1}, x_{2m+1}) + d(x_{2m}, x_{2m+2}) \\ &\le d(x_{2n}, x_{2n+2}) + \beta(d(x_{2n+1}, x_{2m+1}))\beta \\ &\quad \cdot (d(x_{2n}, x_{2m}))d(x_{2n}, x_{2m}) \\ &\quad + d(x_{2m}, x_{2m+2}), \end{aligned} \tag{16}$$

which gives us

$$\begin{aligned} d(x_{2n}, x_{2m}) &\le [(1 - \beta(d(x_{2n+1}, x_{2m+1}))\beta(d(x_{2n}, x_{2m})]^{-1} \\ &\quad \cdot [d(x_{2n}, x_{2n+2}) + d(x_{2m}, x_{2m+2})]. \end{aligned} \tag{17}$$

Since $\limsup_{n,m\to\infty} d(x_{2n}, x_{2m}) > 0$ and $\limsup_{n,m\to\infty} d(x_{2n}, x_{2n+2}) = 0$, we get

$$\limsup_{n,m\to\infty} [(1 - \beta(d(x_{2n+1}, x_{2m+1}))\beta(d(x_{2n}, x_{2m}))]^{-1} = \infty. \tag{18}$$

Observe that $\limsup_{n,m\to\infty} \beta(d(x_{2n}, x_{2m})) = 1$. Taking into account that $\beta \in \mathscr{F}$, we get $\limsup_{n,m\to\infty} d(x_{2n}, x_{2m}) = 0$ and this contradicts our assumption. Hence, $\{x_{2n}\}$ is a Cauchy sequence in $\mathscr{M}$. Because $\{x_{2n}\}$ is Cauchy, X is complete, and $\mathscr{M}$ is closed; $\lim_{n\to\infty} x_{2n} = x \in \mathscr{M}$. Now,

$$d(\mathscr{M}, \mathscr{N}) \le d(x, x_{2n-1}) \le d(x, x_{2n}) + d(x_{2n}, x_{2n-1}). \tag{19}$$

Thus, we have $d(x, x_{2n-1})$ converges to $d(\mathscr{M}, \mathscr{N})$. Since

$$d(\mathscr{M}, \mathscr{N}) \le d(x_{2n}, \Gamma x) \le \beta(d(x_{2n-1}, x))d(x_{2n-1}, x), \tag{20}$$

therefore,

$$\begin{aligned} d(\mathscr{M}, \mathscr{N}) &\le \lim_{n\to\infty} d(x_{2n}, \Gamma x) \\ &\le \lim_{n\to\infty} \beta(d(x_{2n-1}, x))d(x_{2n-1}, x) \\ &= d(\mathscr{M}, \mathscr{N}). \end{aligned} \tag{21}$$

Thus, $d(x, \Gamma x) = d(\mathscr{M}, \mathscr{N})$.

Theorem 9. *Let $\mathscr{M}$ and $\mathscr{N}$ be two nonempty closed and convex subsets of a uniformly convex Banach space X such that $\mathrm{diam}(\mathscr{M}) < d(\mathscr{M}, \mathscr{N})$. Suppose $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ is a cyclic Geraghty contraction map. Then, there exists a unique $x \in \mathscr{M}$ such that $\|x - \Gamma x\| = d(\mathscr{M}, \mathscr{N})$. Further, if $x_0 \in \mathscr{M}$ and $x_{n+1} = \Gamma x_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. By Theorem 8, $P_\Gamma(\mathscr{M}, \mathscr{N}) \neq 0$. Suppose $x, y \in P_\Gamma(\mathscr{M}, \mathscr{N})$ such that $x \neq y$. Since $\|x - \Gamma x\| = d(\mathscr{M}, \mathscr{N})$ and $\|y - \Gamma y\| = d(\mathscr{M}, \mathscr{N})$ were necessarily uniformly convexity of X, $\Gamma^2 x = x$ and $\Gamma^2 y = y$. Since $x \neq y$, we have $d(\mathscr{M}, \mathscr{N}) < \|\Gamma x - y\|$ and so $\beta(\|\Gamma x - y\|)\|\Gamma x - y\| < \|\Gamma x - y\|$. Therefore, $\|x - \Gamma y\| = \|\Gamma^2 x - \Gamma y\| \le \beta(\|\Gamma x - y\|)\|\Gamma x - y\| < \|\Gamma x - y\|$. Similarly, $\|\Gamma x - y\| < \|x - \Gamma y\|$; that is, it is a contradiction. Therefore, $x = y$.

Example 1. Let $\mathscr{M}$ and $\mathscr{N}$ be subsets of $\mathbb{R}^2$ defined by

$$\mathscr{M} = \{(x, 0) : x \ge 1\}, \mathscr{N} = \{(0, y) : y \ge 1\}. \tag{22}$$

Suppose

$$\Gamma(x, y) = (\ln(1 + y), \ln(1 + x)) \tag{23}$$

$$\beta(\Gamma) = \begin{cases} 1 + \ln(1 + t), & 0 \le t < d(\mathscr{M}, \mathscr{N}), \\ \sqrt{\dfrac{t}{d(\mathscr{M}, \mathscr{N})}}, & t > d(\mathscr{M}, \mathscr{N}). \end{cases} \tag{24}$$

Here, $d(\mathscr{M},\mathscr{N})=\sqrt{2}$. For $(x,0),(y,0)\in\mathscr{M}$, we have

$$\begin{aligned}\|\Gamma(x,0)-\Gamma(y,0)\| &= \|(0,\ln(1+x)-\ln(1+y))\| \\ &= \left|\ln\left(\frac{1+x}{1+y}\right)\right| \le 1+\ln(1+|x-y|) \\ &= \beta(\|(x,0)-(y,0)\|)\|(x,0)-(y,0)\|.\end{aligned} \tag{25}$$

For $(x,0)\in\mathscr{M}$ and $(0,y)\in\mathscr{N}$, we have

$$\begin{aligned}\Gamma(x,0)-\Gamma(0,y) &= (\ln(1+x),\ln(1+y)) \\ &= \sqrt{(\ln(1+x))^2+(\ln(1+y))^2} \\ &\le \sqrt{x^2+y^2} \le \sqrt{\frac{\|(x,0)-(0,y)\|}{d(\mathscr{M},\mathscr{N})}} \\ &\quad \cdot \|(x,0)-(0,y)\| \\ &= \beta(\|(x,0)-(0,y)\|)\|(x,0)-(0,y)\|.\end{aligned} \tag{26}$$

Then, Γ is a cyclic Geraghty contraction on $\mathscr{M}\cup\mathscr{N}$. Also, $\|(0,1)-\Gamma((1,0))\|=\sqrt{2}=d(\mathscr{M},\mathscr{N})$.

3. A Best Proximity Point Result for Caristi-Type Mappings

Recently, Du [14] established a direct proof of Caristi's fixed point theorem without using Zorn's lemma. In this section, we introduce a generalization of Caristi's fixed point theorem and provide a proof without using Zorn's lemma. We start with the following definition.

Definition 10. Let $\mathscr{M}$ and $\mathscr{N}$ be nonempty subsets of a metric space (X,d). A map $\Gamma:\mathscr{M}\longrightarrow\mathscr{N}$ is called a semicontraction if for all $u\in\mathscr{M}$ *and* $\xi\in\mathscr{N}$, we have $d(\Gamma u,\xi)\le d(u,\xi)$.

Our related result is as follows.

Theorem 11. *Let (X,d) be a complete metric space and $(\mathscr{M},\mathscr{N})$ be a pair of nonempty closed subsets of X such that $\mathscr{M}$ is boundedly compact. Also, let $\Gamma:\mathscr{M}\longrightarrow\mathscr{N}$ a semicontraction map and is a bounded below and lower semicontinuous function $\chi:\mathscr{M}\longrightarrow(-\infty,\infty]$. Assume that for all $u\in\mathscr{M}$ with $\inf_{x\in\mathscr{M}}\chi(x)<\chi(u)$, there is $v\in\mathscr{M}$ so that*

$$\begin{aligned}&d(\Gamma u,v)>d(\mathscr{M},\mathscr{N}),\\ &d(\Gamma u,v)\le\chi(u)-\chi(v).\end{aligned} \tag{27}$$

Then, there is $u_0\in\mathscr{M}$ so that $\chi(u_0)=\inf_{x\in\mathscr{M}}\chi(x)$.

Proof. Assume that $\inf_{x\in\mathscr{M}}\chi(x)<\chi(y)$ for every $y\in\mathscr{M}$. Given $\mu_0\in\mathscr{M}$. Then, $\inf_{x\in\mathscr{M}}\chi(x)<\chi(\mu_0)$. We have $\Gamma\mu_0\in\mathscr{N}$. Therefore, there is $\mu_1\in\mathscr{M}$ such that

$$d(\mu_1,\Gamma\mu_0)\le\chi(\mu_1)-\chi(\mu_0). \tag{28}$$

Let us define inductively a sequence $\{\mu_n\}\subseteq S_n$, where

$$\mathscr{S}_n:=\{w\in\mathscr{M}\mid d(\Gamma\mu_{n-1},w)\le\chi(\mu_{n-1})-\chi(w)\}, \tag{29}$$

so that

$$\chi(\mu_n)<\inf_{w\in\mathscr{S}_n}\chi(\tau)+\frac{1}{n}. \tag{30}$$

Therefore,

$$d(\Gamma\mu_{n-1},\mu_n)\le\chi(\mu_{n-1})-\chi(\mu_n). \tag{31}$$

Since $\{\mu_n\}\subseteq\mathscr{M}$ and $\mathscr{M}$ is boundedly compact, $\{\mu_n\}$ has a convergent subsequence to $x\in\mathscr{M}$. Suppose $\mu_{n_k}\longrightarrow x$, $d(\Gamma\mu_{n_k},x)\le\chi(\mu_{n_k})-\chi(x)$. Since $\Gamma x\in\mathscr{N}$, there is $z\in\mathscr{M}$ so that $d(\Gamma x,z)>d(\mathscr{M},\mathscr{N})$ and $d(\Gamma x,z)\le\chi(x)-\chi(z)$. Therefore,

$$\begin{aligned}\chi(z) &\le \chi(x)-d(\Gamma x,z)\le\chi(x)-d(\Gamma x,z)\\ &\quad +\chi\left(\mu_{n_k}\right)-\chi(x)-d\left(\Gamma\mu_{n_k},x\right)\\ &=\chi\left(\mu_{n_k}\right)-\left[d(\Gamma x,z)+d\left(\Gamma\mu_{n_k},x\right)\right]\\ &\le\chi\left(\mu_{n_k}\right)-\left[d(\Gamma x,z)+d\left(\Gamma\mu_{n_k},\Gamma x\right)\right]\\ &\le\chi\left(\mu_{n_k}\right)-d\left(\Gamma\mu_{n_k},z\right)\end{aligned} \tag{32}$$

We find that $z\in S_{n_k}$. By (30), we get

$$\chi\left(\mu_{n_k}\right)-\frac{1}{n_k}<\inf_{w\in S_{n_k}}\chi(\tau)\le\chi(z). \tag{33}$$

Thus,

$$\chi(z)<\chi(x)\le\lim_{k\to\infty}\chi\left(\mu_{n_k}\right)\le\chi(z). \tag{34}$$

It is a contradiction, so there is $u_0\in\mathscr{M}$ such that $\chi(u_0)=\inf_{x\in\mathscr{M}}\chi(x)$.

Definition 12. Let $\mathscr{M}$ and $\mathscr{N}$ be nonempty subsets of a metric space X. The mapping $\Gamma:\mathscr{M}\cup\mathscr{N}\longrightarrow\mathscr{M}\cup\mathscr{N}$ is a cyclic semicontraction if

(i) $\Gamma(\mathscr{M})\subset\mathscr{N}$ and $\Gamma(\mathscr{N})\subset\mathscr{M}$

(ii) For all $\mu\in\mathscr{M}$ and $\xi\in\mathscr{N}$, we have $d(\Gamma\mu,\xi)\le d(\mu,\xi)$

For example, let $\mathscr{M}$ and $\mathscr{N}$ be subsets of $\mathbb{R}^2$ defined by

$$\mathscr{M}=\{(x,0)\colon x\ge1\},\ \mathscr{N}=\{(0,y)\colon y\ge1\}. \tag{35}$$

Suppose $\Gamma(x,y)=(\sqrt{y},\sqrt{x})$, then Γ is cyclic semicontraction on $\mathscr{M}\cup\mathscr{N}$ and $\|(0,1)-\Gamma((1,0))\|=d(\mathscr{M},\mathscr{N})$.

Theorem 13. *Let $(\mathscr{M},\mathscr{N})$ be a pair of nonempty closed subsets of X such that $\mathscr{M}$ is boundedly compact. Suppose that $\Gamma:\mathscr{M}\cup\mathscr{N}\longrightarrow\mathscr{M}\cup\mathscr{N}$ is a cyclic semicontraction map and*

there exists a bounded below and lower semicontinuous function $\chi : \mathscr{M} \longrightarrow (-\infty, \infty]$ *so that*

$$d(\mu, \Gamma\mu) \leq \chi(\mu) - \chi(\Gamma\mu) \tag{36}$$

for all $\mu \in \mathscr{M} \cup \mathscr{N}$. *Then,* Γ *admits a best proximity point.*

Proof. The function χ is proper, so there is $\xi \in \mathscr{M}$ with $\chi(\xi) < \infty$. Put

$$H = \{\mu \in \mathscr{M} \mid d(\Gamma\xi, \mu) \leq \chi(\xi) - \chi(\mu)\}. \tag{37}$$

Let $\mu \in H$. Since $d(\mu, \Gamma\mu) \leq \chi(\mu) - \chi(\Gamma\mu)$, we have $\chi(\Gamma\mu) \leq \chi(\mu)$. Thus,

$$\begin{aligned}\chi(\Gamma^2\mu) &\leq \chi(\Gamma\mu) - d(\Gamma\mu, \Gamma^2\mu)\\ &\leq \chi(\Gamma\mu) - d(\Gamma\mu, \Gamma^2\mu) + \chi(\xi) - \chi(\mu) - d(\Gamma\xi, \mu)\\ &= \chi(\xi) - \left[d(\Gamma\mu, \Gamma^2\mu) + d(\Gamma\xi, \mu)\right]\\ &= \chi(\xi) - \left[d(\Gamma\mu, \Gamma^2\mu) + d(\Gamma\xi, \Gamma\mu)\right],\\ &\quad \text{(by definition of a cyclic semicontraction)}\\ &\leq \chi(\xi) - d(\Gamma^2\mu, \Gamma\xi).\end{aligned} \tag{38}$$

Hence,

$$d(\Gamma^2\mu, \Gamma\xi) \leq \chi(\xi) - \chi(\Gamma^2\mu), \tag{39}$$

and so $\Gamma^2\mu \in C$. Assume that $d(\mu, \Gamma\mu) > d(\mathscr{M}, \mathscr{N})$ for all $\mu \in \mathscr{M}$. Then, for each $\mu \in H$, there is $\tau \in H$ so that

$$\mu \neq \tau, d(\Gamma\mu, \tau) \leq \chi(\mu) - \chi(\tau). \tag{40}$$

By Theorem 11, there is $\mu_0 \in C$ so that $\chi(\mu_0) = \inf_{\mu\in H} \chi(\mu)$. Thus, for above $\mu_0 \in H$, we obtain

$$\begin{aligned}0 &\leq d(\mathscr{M}, \mathscr{N}) < d(\Gamma\mu, \Gamma^2\mu) \leq \chi(\mu) - \chi(\Gamma^2\mu)\\ &\leq \chi(\Gamma^2\mu) - \chi(\Gamma^2\mu) = 0,\end{aligned} \tag{41}$$

a contradiction and so Γ has a best proximity point.

From now on, $\mathscr{M}$ and $\mathscr{N}$ are nonempty subsets of a Banach space X. For $(x_0, y_0) \in \mathscr{M} \times \mathscr{N}$, the inward sets of (x_0, y_0) relative to $\mathscr{M} \times \mathscr{N}$ are as follows:

$$\begin{aligned}I_{\mathscr{N}}(x_0) &= \{ty + (1-t)x_0, y \in \mathscr{N}, t \geq 0\},\\ I_{\mathscr{M}}(y_0) &= \{\Gamma x + (1-t)y_0, x \in \mathscr{M}, t \geq 0\}.\end{aligned} \tag{42}$$

For example, let $\mathscr{M}$ and $\mathscr{N}$ be subsets of R defined by

$$\mathscr{M} = [1, 2], \mathscr{N} = [-2, -1]. \tag{43}$$

Then, for $(x, y) \in \mathscr{M} \times \mathscr{N}$, we have

$$I_{\mathscr{N}}(x) = (-\infty, x], I_{\mathscr{M}}(y) = [y, +\infty). \tag{44}$$

We now define cyclic weakly inward mappings.

Definition 14. Given $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow X$. Such Γ is said to be

(i) Cyclic inward if $(\Gamma x, \Gamma y) \in I_{\mathscr{N}}(x) \times I_{\mathscr{M}}(x)$ for all $(x, y) \in \mathscr{M} \times \mathscr{N}$

(ii) Cyclic weakly inward if $(\Gamma x, \Gamma y) \in \overline{I_{\mathscr{N}}(x)} \times \overline{I_{\mathscr{M}}(x)}$ for all $(x, y) \in \mathscr{M} \times \mathscr{N}$

(iii) Cyclic weakly inward contraction if it is cyclic weakly inward and $d(\Gamma x, \Gamma y) \leq kd(x, y)$ for all $(x, y) \in \mathscr{M} \times \mathscr{N}$, where $k \in (0, 1)$.

Theorem 15. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be closed and convex. Then* $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow X$ *is cyclic weakly inward if*

$$\lim_{h\to 0^+} \frac{d((1-h)\vartheta + h\Gamma\vartheta, \mathscr{M})}{h} = d(\mathscr{M}, \mathscr{N}), \vartheta \in \mathscr{N}, \tag{45}$$

$$\lim_{h\to 0^+} \frac{d((1-h)\vartheta + h\Gamma\vartheta, \mathscr{N})}{h} = d(\mathscr{M}, \mathscr{N}), \vartheta \in \mathscr{M}. \tag{46}$$

Proof. Suppose that ((45), (46)) hold. Fix $\vartheta \in \mathscr{M}$, given $\varepsilon > 0$. Choose $\lambda \in (0, 1)$ and $y \in \mathscr{N}$ so that

$$\|(1-\lambda)x + \lambda\Gamma\vartheta - y\| \leq d(\mathscr{M}, \mathscr{N}) + \lambda\varepsilon. \tag{47}$$

Thus,

$$\left\|\Gamma\vartheta - \left[\left(1-\lambda^{-1}\right)\vartheta + \lambda^{-1}y\right]\right\| \leq \lambda^{-1}d(\mathscr{M}, \mathscr{N}) + \varepsilon. \tag{48}$$

Consequently, $\Gamma\vartheta \in \overline{I_{\mathscr{N}}(\vartheta)}$. Similarly, for $y \in \mathscr{N}$, $\Gamma y \in \overline{I_{\mathscr{M}}(y)}$.

Conversely, assume that Γ is cyclic weakly inward; that is, $(\Gamma\vartheta, \Gamma y) \in \overline{I_{\mathscr{N}}(\vartheta)} \times \overline{I_{\mathscr{M}}(\vartheta)}$ for all $(\vartheta, y) \in \mathscr{M} \times \mathscr{N}$. Given $\varepsilon > 0$. There exists $b \in I_{\mathscr{N}}(\vartheta)$ so that

$$\|b - \Gamma\vartheta\| \leq \varepsilon. \tag{49}$$

Since $\mathscr{M}$ is convex, there is $\lambda_0 > 0$ so that

$$(1-\lambda)\vartheta + \lambda b \in \mathscr{M}, 0 < \lambda \leq \lambda_0. \tag{50}$$

Therefore,

$$\begin{aligned}&\frac{d((1-\lambda)x + \lambda\Gamma\vartheta, \mathscr{M})}{\lambda}\\ &\quad \leq \frac{\|(1-\lambda)\vartheta + \lambda\Gamma\vartheta - [(1-\lambda)\vartheta + \lambda b]\|}{\lambda} \leq \varepsilon.\end{aligned} \tag{51}$$

The condition (45) holds. Similarly, the condition (46) is satisfied.

Theorem 16. *Let $\mathscr{M}$ and $\mathscr{N}$ be closed and convex. Let $\Gamma : \mathscr{M} \cup \mathscr{N} \longrightarrow X$ be a cyclic weakly inward contraction mapping. If $\mathscr{M}$ is boundedly compact, then Γ has a unique best proximity point in $\mathscr{M}$.*

Proof. Let k $(0 < k < 1)$ denote a Lipschitz constant of Γ. Choose $\varepsilon > 0$ so that $k < 1 - \varepsilon/1 + \varepsilon$. By Theorem 15, (45) is verified. Let $\vartheta \in \mathscr{M}$ with $\|-\Gamma\vartheta\| > d(\mathscr{M}, \mathscr{N})$, then there is $\lambda \in (0, 1)$ so that

$$d((1-\lambda)\vartheta + \lambda\Gamma\vartheta, \mathscr{N}) < \lambda\varepsilon\|\vartheta - \Gamma\vartheta\|. \tag{52}$$

By the definition of a dance, there is $b \in \mathscr{N}$ so that

$$\|(1-\lambda)\vartheta + \lambda\Gamma\vartheta - b\| < \lambda\varepsilon\|\vartheta - \Gamma\vartheta\|. \tag{53}$$

Hence,

$$\begin{aligned} \lambda\varepsilon\|\vartheta - \Gamma\vartheta\| &> \|\vartheta - b - \lambda(\vartheta - \Gamma\vartheta)\| \\ &\geq \|\vartheta - b\| - \lambda\|\vartheta - \Gamma\vartheta\|, \end{aligned} \tag{54}$$

and so

$$\begin{aligned} \|b - \Gamma b\| &\leq \|b - [(1-\lambda)\vartheta + \lambda\Gamma\vartheta]\| \\ &\quad + \|(1-\lambda)\vartheta + \lambda\Gamma\vartheta - \Gamma\vartheta\|\|\Gamma\vartheta - \Gamma b\| \\ &\leq \lambda\varepsilon\|\vartheta - \Gamma\vartheta\| + (1-\lambda)\|\vartheta - \Gamma\vartheta\| + k\|\vartheta - b\| \\ &= \|\vartheta - \Gamma\vartheta\| + (\varepsilon - 1)\lambda\|\vartheta - \Gamma\vartheta\| \\ &\quad + \frac{1-\varepsilon}{1+\varepsilon}\|\vartheta - b\| - \left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)\|\vartheta - b\| \\ &< \|\vartheta - \Gamma\vartheta\| - \left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)\|\vartheta - b\|. \end{aligned} \tag{55}$$

If for all $\vartheta \in \mathscr{M}$, $\|\vartheta - \Gamma\vartheta\| > d(\mathscr{M}, \mathscr{N})$, then denote $b \in \mathscr{N}$ as above by $f\vartheta$, where $f : \mathscr{M} \longrightarrow \mathscr{N}$ is a mapping. Put

$$\chi(\vartheta) = \left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)^{-1}\|\vartheta - \Gamma\vartheta\|. \tag{56}$$

Note that $\chi : \mathscr{M} \cup \mathscr{N} \longrightarrow R^+$ is continuous and the following

$$\|\vartheta - f\vartheta\| < \chi(\vartheta) - \chi(f\vartheta) \tag{57}$$

holds. Due to Theorem 13, f admits a best proximity point, which contradicts (57).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the manuscript.

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