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Research Article

The Compact Finite Difference Method of Two-Dimensional Cattaneo Model

Yating Huang and Zhe Yin

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong, China

Correspondence should be addressed to Zhe Yin; zyin_sdnu@163.com

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In this paper, we propose and analyze the compact finite difference scheme of the two-dimensional Cattaneo model. The stability and convergence of the scheme are proved by the energy method, the convergence orders are 2 in time and 4 in space. We also use the variables separation method to find the true solution of the problem. On this basis, the validity and accuracy of the scheme are verified by numerical experiments.

1. Introduction

In the diffusion phenomenon described by traditional Fick law and Fourier law, the moment a point passes by, it is disturbed at a great distance, the propagation speed of the disturbance seems to be infinite. However, this property is unphysical. To solve the problem, Cattaneo modified the constitutive equation by introducing a relaxation parameter which plays the role of relaxation time, it should be noted that the relaxation parameter is small and depends on the thermodynamic properties of the material. Thus, he proposed the Cattaneo model [1].

From a mathematical point of view, Cattaneo model transforms the traditional diffusion equation into a hyperbolic equation, the speed of propagation is finite, and improves the property of infinite propagation speed. On the other hand, because of the hyperbolic nature of the Cattaneo model, it may have oscillatory solutions and negative values.

From a physical point of view, the Cattaneo model describes a physical phenomenon which is called heat waves. Although this phenomenon can only be observed under special circumstances, materials, or states [2], it is still gradually accepted by the public. It can be used to describe not only ultrasonic wave propagation in certain diluted gases but also heat pulse propagation in some pure nonmetallic crystals [3]. Straughan et al. [4] studied thermal convection with the Cattaneo-Christov model in horizontal layers of an incompressible Newtonian fluid. Haddad [5] utilized the theory of Cattaneo-Christov to investigate Brinkman's porous media. Cattaneo model is widely used in extended irreversible thermodynamics, cosmological models, and crystalline solids [6–8].

Because the fractional models have the advantage of previous memory, scholars have studied various fractional models [9–13]. With the wide application of the Cattaneo model in physics and theoretical analysis, more and more people pay attention to the generalization and solution of the Cattaneo model. Compte and Metzler [14] generalized the Cattaneo model from three perspectives which called continuous-time random walks, nonlocal transport theory, and delayed flux-force relation. Ferrillo et al. [15] compared the Cattaneo model and fractional Cattaneo model and studied the asymptotic behavior of solutions to the Cattaneo equations. Su et al. [16] studied solutions to the classical Cattaneo equation and compared with the solutions of a phase-

lagging heat transport equation. Čiegis [17] studied the numerical solutions of a class of hyperbolic heat conduction equations, proposed explicit and implicit Euler schemes for such equations, and verified that the convergence order of this scheme was $O(\tau + h^2)$. Huang and Yin [18] proposed a fourth-order compact difference scheme for the 1D Cattaneo model and verified that the convergence order of the scheme was $O(\tau^2 + h^4)$ through numerical experiments. Zhao and Sun [19] presented compact Crank-Nicolson schemes for a class fractional Cattaneo equation in an inhomogeneous medium and used the energy method to verify that the convergence order of the scheme is $O(\tau^{3-\alpha} + h^4)$. Li and Cao [20] proposed an unconditionally stable scheme with convergence order of $O(\tau^{3-\alpha} + h^2)$. Vong et al. [21] proposed a higher-order difference scheme for a class of generalized Cattaneo equations and proved the stability and convergence of the scheme by energy method.

There are many methods to solve partial differential equations, such as the finite element method [22, 23], finite volume element method [24, 25], and finite difference method [26]. Compared with other methods, the finite difference method has the advantages of a low requirement for grid nodes and simple calculation. In 1992, Lele [27] first proposed a general form of compact difference scheme. Compact difference scheme is a kind of finite difference scheme with high precision, which is widely used in solving parabolic equation and hyperbolic equation [28–33].

Previous scholars used different methods to study the integer-order Cattaneo model and the fractional Cattaneo model. The one-dimensional Cattaneo model describes the heat conduction phenomenon of the uniform thin column with side insulation, while the two-dimensional Cattaneo model needs to be studied for the heat conduction phenomenon of the thin plate with upper and lower bottom insulation. In this paper, we construct a compact difference scheme for a two-dimensional Cattaneo model. The time derivative is discretized with a central difference, and the space derivative is discretized with compact operators, so that the convergence order of the scheme reaches $O(\tau^2 + h_x^4 + h_y^4)$. We use the energy method to verify the stability and convergence of the scheme. Finally, we verify the validity and accuracy of the scheme through numerical experiments. It should be noted that the above articles are all about constructing the exact solution of the problem, and then calculating the right term, so as to conduct numerical experiments. In this paper, our example is the definite problem corresponding to the homogeneous equation. The exact solution is obtained through the separation of variables method, and then the numerical experiment is conducted.

The arrangement of the article is as follows. In the second part, we give some lemmas and symbols to derive a compact difference scheme. In the third part, we give a strict proof of the stability and convergence of the scheme by energy method. In the fourth part, we obtain the true solution of the specific problem through the separation of variable method and use numerical experiments to prove the validity of the scheme. In the fifth part, we make a final summary of the article.

2. Construction of the Compact Finite Difference Scheme

In this chapter, we consider the following two-dimensional Cattaneo model:

$$\begin{cases} \varepsilon^2 u_{tt} + u_t = D\left(u_{xx} + u_{yy}\right), & (x, y) \in \Omega, t \in (0, T], \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \\ u_t(x, y, 0) = u_1(x, y), & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y) \in \Gamma, t \in (0, T], \end{cases} \tag{1}$$

where $\Omega = (0, L_x) \times (0, L_y)$ is a rectangular region in R^2, $\Gamma = \partial\Omega$ is the boundary of the rectangular region, ε^2 is a relaxation parameter, D is the diffusion coefficient, and $u_0(x, y)$ and $u_1(x, y)$ are known functions. The above model is considered under homogeneous Dirichlet boundary conditions. The regularity assumption that problem (1) has a solution $u \in C^{6,6,4}_{x,y,t}(\bar{\Omega} \times [0, T])$ will be used in the following analysis.

In order to propose the compact finite difference scheme, we first introduce some basic symbols and preliminary lemmas. Take three positive integers M, N, and K, then let $h_x = L_x/M$, $h_y = L_y/N$, $\tau = T/K$. In this case, the spatial node can be expressed as (x_i, y_j), where $x_i = ih_x, i = 0, 1, \cdots, M$; $y_j = jh_y, j = 0, 1, \cdots, N$; $t_n = n\tau, n = 0, 1, \cdots, K$, let $\bar{\Omega}_h = \{(x_i, y_j) \mid 0 \le i \le M, 0 \le j \le N\}$, $\Omega_\tau = \{u^n \mid 0 \le n \le K\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Gamma_h = \bar{\Omega}_h \cap \Gamma$, $\omega = \{(i, j) \mid (x_i, y_j) \in \Omega_h\}$, $\partial\omega = \{(i, j) \mid (x_i, y_j) \in \Gamma_h\}$, $\bar{\omega} = \omega \cup \partial\omega$. Let $U_h = \{u \mid u = \{u_{ij} \mid (i, j) \in \bar{\omega}\}\}$ be a grid function space defined on $\bar{\Omega}_h$. For any grid function $u \in U_h$, introduce the following notations:

$$\delta_x^2 u_{ij} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2}, \delta_y^2 u_{ij} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2}. \tag{2}$$

The compact operators are defined as follows

$$A_x u_{i,j} = \begin{cases} \left(I + \dfrac{h_x^2}{12}\delta_x^2\right) u_{ij}, & 1 \le i \le M-1, 0 \le j \le N, \\ u_{i,j}, & i = 0 \text{ or } M, \end{cases}$$
$$A_y u_{i,j} = \begin{cases} \left(I + \dfrac{h_y^2}{12}\delta_y^2\right) u_{ij}, & 1 \le j \le N-1, 0 \le i \le M, \\ u_{i,j}, & j = 0 \text{ or } N. \end{cases} \tag{3}$$

Let $U_\tau = \{u \mid u = (u^0, u^1, \cdots, u^K)\}$ be grid function space defined on Ω_τ. For any grid function $u \in U_\tau$, introduce the following notations:

$$\delta_t u^n = \frac{u^{n+1} - u^{n-1}}{2\tau}, \delta_{\bar{t}} u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau}, \\ \delta_{\bar{t}} u^{n-1/2} = \frac{u^n - u^{n-1}}{\tau}, \delta_t^2 u^n = \frac{u^{n-1} - 2u^n + u^{n+1}}{\tau^2}. \tag{4}$$

The following lemmas also play an important role in constructing a scheme.

Lemma 1 [34]. *Suppose $g(x) \in C^2[c, c+h]$, it holds that*

$$g'(c) = \frac{1}{h}[g(c+h) - g(c)] - \frac{h}{2} g''(\xi_7), c < \xi_7 < c + h, \tag{5}$$

where c is a constant.

Lemma 2 [34]. *Suppose $g(x) \in C^3[t_n, t_{n+1}]$, it holds that*

$$g'' = \frac{2}{\tau}\left[\frac{g(t_{n+1}) - g(t_n)}{h} - g'\right] - \frac{\tau}{3} g'''(t_\xi), t_n < t_\xi < t_{n+1}. \tag{6}$$

Lemma 3 [34]. *Suppose $g(x) \in C^4[c-h, c+h]$, it holds that*

$$g''(c) = \frac{g(c+h) - 2g(c) + g(c-h)}{h^2} - \frac{h^2}{12} g^{(4)}(\xi_4), c - h < \xi_4 < c + h, \tag{7}$$

where c is a constant.

Lemma 4 [34]. *Suppose $g(x) \in C^6[c-h, c+h]$, it holds that*

$$\frac{1}{12}\left[g''(c-h) + 10g''(c) + g''(c+h)\right] = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] + \frac{h^4}{240} g^{(6)}(\xi_6), \tag{8}$$

where $\xi_6 \in (c-h, c+h)$, c is a constant.

Define the grid functions as $u_{ij}^n = u(x_i, y_j, t_n)$, $0 \le i \le M$, $0 \le j \le N, 0 \le n \le K$. We discrete at (x_i, y_j, t_n) and refer to the formula [34].

$$\frac{\partial^2 u(x_i, y_j, t_n)}{\partial x^2} = \frac{1}{2}\left[\frac{\partial^2 u(x_i, y_j, t_{n-1})}{\partial x^2} + \frac{\partial^2 u(x_i, y_j, t_{n+1})}{\partial x^2}\right] - \frac{\tau^2}{2} \frac{\partial^4 u(x_i, y_j, \xi_{in})}{\partial x^2 \partial t^2}, t_{n-1} < \xi_{in} < t_{n+1}. \tag{9}$$

We can obtain

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_n) + \frac{\partial u}{\partial t}(x_i, y_j, t_n) = \frac{D}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n-1}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+1}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n-1}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+1})\right] + R_{ijn}^{(1)}, (i, j) \in \omega, 1 \le n \le K - 1, \tag{10}$$

where

$$R_{ijn}^{(1)} = -\frac{D\tau^2}{2}\left[\frac{\partial^4 u(x_i, y_j, \xi_{ijn})}{\partial x^2 \partial t^2} + \frac{\partial^4 u(x_i, y_j, \eta_{ijn})}{\partial y^2 \partial t^2}\right], t_{n-1} < \xi_{ijn} < t_{n+1}, t_{n-1} < \eta_{ijn} < t_{n+1}. \tag{11}$$

Utilizing Taylor formula results in

$$\frac{\partial u(x_i, y_j, t_n)}{\partial t} = \delta_t u_{ij}^n - \frac{\tau^2}{6} \frac{\partial^3 u(x_i, y_j, \theta_{ijn})}{\partial t^3}, t_{n-1} < \theta_{ijn} < t_{n+1}. \tag{12}$$

Using the above equation and Lemma 3 result in

$$\varepsilon^2 \delta_t^2 u_{ij}^n + \delta_t u_{ij}^n = \frac{D}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n-1}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+1}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n-1}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+1})\right] + R_{ijn}^{(2)}, (i, j) \in \omega, 1 \le n \le K - 1, \tag{13}$$

where $R_{ijn}^{(2)} = R_{ijn}^{(1)} + \tau^2(1/6(\partial^3 u(x_i, y_j, \theta_{ijn})/\partial t^3) + (\varepsilon^2/12)(\partial^4 u (x_i, y_j, \rho_{ijn})/\partial t^4))$.

We apply compact operators A_x and A_y to both sides of Eq. (13) and apply Lemma 4, we get

$$\varepsilon^2 A_x A_y \delta_t^2 u_{ij}^n + A_x A_y \delta_t u_{ij}^n = \frac{D}{2}\left(A_y \delta_x^2 u_{ij}^{n-1} + A_y \delta_x^2 u_{ij}^{n+1} + A_x \delta_y^2 u_{ij}^{n-1} + A_x \delta_y^2 u_{ij}^{n+1}\right) + R_{ijn}^{(3)}, (i, j) \in \omega, 1 \le n \le K - 1, \tag{14}$$

where

$$\begin{aligned} R_{ijn}^{(3)} = A_x A_y R_{ijn}^{(2)} &+ \frac{h_x^4}{240}\left(A_y \frac{\partial^6 u}{\partial x^6}\left(\vartheta_{ijn}, y_j, t_{n-1}\right)\right. \\ &\left. + A_y \frac{\partial^6 u}{\partial x^6}\left(\mu_{ijn}, y_j, t_{n+1}\right)\right) \\ &+ \frac{h_y^4}{240}\left(A_x \frac{\partial^6 u}{\partial y^6}\left(x_i, \zeta_{ijn}, t_{n-1}\right)\right. \\ &\left. + A_x \frac{\partial^6 u}{\partial y^6}\left(x_i, \varsigma_{ijn}, t_{n+1}\right)\right). \end{aligned} \tag{15}$$

We think about the differential equation (1) at point (x_i, y_j, t_0),

$$\begin{aligned} &\varepsilon^2 \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial t^2} + \frac{\partial u\left(x_i, y_j, t_0\right)}{\partial t} \\ &= \frac{D}{2}\left(\frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial x^2} + \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial x^2}\right. \\ &\quad \left. + \frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial y^2} + \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial y^2}\right) \\ &\quad - \frac{D}{2}\left(\frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial x^2} - \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial x^2}\right. \\ &\quad \left. + \frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial y^2} - \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial y^2}\right). \end{aligned} \tag{16}$$

By applying Lemma 1 and Lemma 2, we have

$$\begin{aligned} \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial t^2} &= \frac{2}{\tau}\left[\delta_{\bar{t}} u^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right] \\ &\quad - \frac{\tau}{3} \frac{\partial^3 u\left(x_i, y_j, \varsigma_1\right)}{\partial t^3}, \varsigma_1 \in \left(t_0, t_1\right). \\ \frac{\partial u\left(x_i, y_j, t_0\right)}{\partial t} &= \delta_{\bar{t}} u^{\frac{1}{2}} - \frac{\tau}{2} \frac{\partial^2 u\left(x_i, y_j, \varsigma_2\right)}{\partial t^2}, \varsigma_2 \in \left(t_0, t_1\right). \end{aligned} \tag{17}$$

Substituting the above equation into (16) and applying $A_x A_y$ *to both sides, we have*

$$\begin{aligned} &\frac{2\varepsilon^2}{\tau} A_x A_y \left(\delta_{\bar{t}} u_{ij}^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right) + A_x A_y \delta_{\bar{t}} u_{ij}^{\frac{1}{2}} \\ &\quad = \frac{D}{2}\left(A_y \delta_x^2 u_{ij}^0 + A_y \delta_x^2 u_{ij}^1 + A_x \delta_y^2 u_{ij}^0 + A_x \delta_y^2 u_{ij}^1\right) + r_{ijn}, \end{aligned} \tag{18}$$

where

$$\left|r_{ijn}\right| = O\left(\tau + h_x^4 + h_y^4\right), (i, j) \in \omega. \tag{19}$$

Omitting small terms $R_{ijn}^{(3)}$ *and* r_{ijn}*. The numerical approximate solution is represented by the grid function* U_{ij}^n*. Replacing* u_{ij}^n *with it, the fourth-order compact difference scheme can be obtained*

$$\begin{cases} \varepsilon^2 A_x A_y \delta_t^2 U_{ij}^n + A_x A_y \delta_t U_{ij}^n = \dfrac{D}{2}\left(A_y \delta_x^2 U_{ij}^{n-1} + A_y \delta_x^2 U_{ij}^{n+1} + A_x \delta_y^2 U_{ij}^{n-1} + A_x \delta_y^2 U_{ij}^{n+1}\right), \\ (i, j) \in \omega, 1 \le n \le K - 1, \\ \dfrac{2\varepsilon^2}{\tau} A_x A_y \left(\delta_{\bar{t}} U_{ij}^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right) + A_x A_y \delta_{\bar{t}} U_{ij}^{\frac{1}{2}} = \dfrac{D}{2}\left(A_y \delta_x^2 U_{ij}^0 + A_y \delta_x^2 U_{ij}^1 + A_x \delta_y^2 U_{ij}^0 + A_x \delta_y^2 U_{ij}^1\right), \\ (i, j) \in \omega, \\ U_{ij}^0 = u_0\left(x_i, y_j\right), (i, j) \in \omega, \\ U_{ij}^n = 0, (i, j) \in \partial\omega, 0 \le n \le K. \end{cases} \tag{20}$$

From the above analysis, the following theorems can be obtained.

Theorem 5. *The truncation error of compact difference scheme (20) is*

$$|R_{ij}^n| = O\left(\tau^2 + h_x^4 + h_y^4\right), (i,j) \in \omega, 0 \le n \le K. \tag{21}$$

3. Analysis of the Compact Finite Difference Scheme

Let $u, v \in U_h = \{u \mid u \in U_h; (i,j) \in \partial\omega, u_{ij} = 0\}$, we define the following inner products and the corresponding norms

$$\begin{aligned}
&(u,v) = h_x h_y \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} u_{ij}v_{ij}, \|u\| = \sqrt{(u,u)},\\
&(u,v)_{1,x} = h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N-1} (\delta_x u_{i-1/2,j})\delta_x v_{i-1/2,j}, \|\delta_x u\| = \sqrt{(u,u)_{1,x}},\\
&(u,v)_{1,y} = h_x h_y \sum_{i=1}^{M-1}\sum_{j=1}^{N} (\delta_y u_{i,j-1/2})\delta_y v_{i,j-1/2}, \|\delta_y u\| = \sqrt{(u,u)_{1,y}},\\
&(u,v)_{x,y} = h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N} (\delta_x\delta_y u_{i-1/2,j-1/2})(\delta_x\delta_y v_{i-1/2,j-1/2}), \|\delta_x\delta_y u\| = \sqrt{(u,u)_{x,y}},\\
&(u,v)_A = (A_x A_y u, v) = (u,v) - \frac{h_x^2}{12}(u,v)_{1,x} - \frac{h_y^2}{12}(u,v)_{1,y} + \frac{h_x^2 h_y^2}{144}(u,v)_{x,y},\\
&\|\nabla_h u\|_A^2 = \left(\|\delta_x u\|^2 - \frac{h_y^2}{12}\|\delta_y\delta_x u\|^2\right) + \left(\|\delta_y u\|^2 - \frac{h_x^2}{12}\|\delta_x\delta_y u\|^2\right),\\
&\|u\|_A = \sqrt{(u,u)_A}, \|u\|_\infty = \max_{1\le i\le M-1, 1\le j\le N-1} |u_{ij}|.
\end{aligned} \tag{22}$$

Next, we introduce the Lemma that will be applied to the stability analysis of the scheme.

Lemma 6 [35]. *For any grid function $u \in U_h$, it holds that*

$$\frac{1}{3}\|u\|^2 \le \|u\|_A^2 \le \|u\|^2. \tag{23}$$

Applying the Lemma, we have the following result.

Theorem 7. *Suppose $\{v_{ij}^n \mid 0 \le i \le M, 0 \le j \le N, 0 \le n \le K\}$ is the solution of the following difference scheme*

$$\begin{aligned}
\varepsilon^2 A_x A_y \delta_t^2 v_{ij}^n + A_x A_y \delta_t v_{ij}^n = \frac{D}{2}\Big(&A_y\delta_x^2 v_{ij}^{n-1} + A_y\delta_x^2 v_{ij}^{n+1}\\
&+ A_x\delta_y^2 v_{ij}^{n-1} + A_x\delta_y^2 v_{ij}^{n+1}\Big) + g_{ij}^n, (i,j) \in \omega, 1 \le n \le K-1,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\frac{2\varepsilon^2}{\tau} A_x A_y \left(\delta_{\bar t} v_{ij}^{\frac12} - u_1(x_i, y_j)\right) + A_x A_y \delta_{\bar t} v_{ij}^{\frac12}\\
&\quad = \frac{D}{2}\left(A_y\delta_x^2 v_{ij}^0 + A_y\delta_x^2 v_{ij}^1 + A_x\delta_y^2 v_{ij}^0 + A_x\delta_y^2 v_{ij}^1\right)\\
&\quad + g_{ij}^0, (i,j) \in \omega,
\end{aligned} \tag{25}$$

$$v_{ij}^0 = u_0(x_i, y_j), (i,j) \in \omega, \tag{26}$$

$$v_{ij}^n = 0, (i,j) \in \partial\omega, 0 \le n \le K. \tag{27}$$

It holds that

$$\begin{aligned}
&\varepsilon^2\|\delta_{\bar t} v^{n+1/2}\|_A^2 + \frac{D}{2}\left(\|\nabla_h v^{n+1}\|_A^2 + \|\nabla_h v^n\|_A^2\right) \le D\|\nabla_h v^0\|_A^2\\
&\quad + \frac{3\varepsilon^4}{\tau}\|\varphi\|^2 + \frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2 + \frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2, 0 \le n \le K-1,
\end{aligned} \tag{28}$$

where $\varphi_{ij} = A_x A_y u_1(x_i, y_j)$, g_{ij}^n is a source item.

Proof. Multiplying Eq. (24) by $2\delta_t v^n$, and summing up for i from 1 to $M-1$, we have

$$\begin{aligned}
&2\varepsilon^2\left(A_x A_y \delta_t^2 v^n, \delta_t v^n\right) + 2\left(A_x A_y \delta_t v^n, \delta_t v^n\right)\\
&\quad = D\left(A_y\delta_x^2 v^{n-1} + A_y\delta_x^2 v^{n+1} + A_x\delta_y^2 v^{n-1} + A_x\delta_y^2 v^{n+1}, \delta_t v^n\right)\\
&\qquad + 2(g^n, \delta_t v^n), 1 \le n \le K-1.
\end{aligned} \tag{29}$$

Next, we deal with the two terms on the left-hand side of Eq. (29), we have

$$\begin{aligned}
&2\varepsilon^2\left(A_x A_y \delta_t^2 v^n, \delta_t v^n\right)\\
&\quad = 2\varepsilon^2\left[\left(\delta_t^2 v^n, \delta_t v^n\right) - \frac{h_x^2}{12}\left(\delta_t^2 v^n, \delta_t v^n\right)_{1,x} - \frac{h_y^2}{12}\left(\delta_t^2 v^n, \delta_t v^n\right)_{1,y}\right.\\
&\qquad \left. + \frac{h_x^2 h_y^2}{144}\left(\delta_t^2 v^n, \delta_t v^n\right)_{x,y}\right] = \frac{\varepsilon^2}{\tau}\left[\left(\|\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_{\bar t} v^{n-1/2}\|^2\right)\right.\\
&\qquad - \frac{h_x^2}{12}\left(\|\delta_x\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_x\delta_{\bar t} v^{n}\right.\\
&\qquad \left.-\frac{1}{2}\|^2\right) - \frac{h_y^2}{12}\left(\|\delta_y\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_y\delta_{\bar t} v^{n-1/2}\|^2\right)\\
&\qquad \left. + \frac{h_x^2 h_y^2}{144}\left(\|\delta_x\delta_y\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_x\delta_y\delta_{\bar t} v^{n-1/2}\|^2\right)\right]\\
&\quad = \frac{\varepsilon^2}{\tau}\left(\|\delta_{\bar t} v^{n+1/2}\|_A^2 - \|\delta_{\bar t} v^{n-1/2}\|_A^2\right), \qquad 1 \le n \le K-1.
\end{aligned} \tag{30}$$

$$2\left(A_x A_y \delta_t v^n, \delta_t v^n\right) = 2\|\delta_t v^n\|_A^2, 1 \le n \le K-1. \tag{31}$$

Then we deal with the first term on the right-hand side of

Eq. (29).

$$\begin{aligned}&\left(A_y\delta_x^2 v^{n-1}+A_y\delta_x^2 v^{n+1},\delta_t v^n\right)\\&=\left[\left(I+\frac{h_y^2}{12}\delta_y^2\right)\delta_x^2 v^{n-1}+\left(I+\frac{h_y^2}{12}\delta_y^2\right)\delta_x^2 v^{n+1},\delta_t v^n\right]\\&=\left(\delta_x^2 v^{n-1}+\delta_x^2 v^{n+1},\delta_t v^n\right)-\frac{h_y^2}{12}\left(\delta_x^2 v^{n-1}+\delta_x^2 v^{n+1},\delta_t v^n\right)_{1,y}\\&=-\frac{1}{2\tau}\left(\|\delta_x v^{n+1}\|^2-\|\delta_x v^{n-1}\|^2\right)\\&\quad+\frac{h_y^2}{24\tau}\left(\|\delta_x\delta_y v^{n+1}\|^2-\|\delta_x\delta_y v^{n-1}\|^2\right),1\le n\le K-1.\end{aligned}\tag{32}$$

Analogously

$$\begin{aligned}&\left(A_x\delta_y^2 v^{n-1}+A_x\delta_y^2 v^{n+1},\delta_t v^n\right)\\&=-\frac{1}{2\tau}\left(\|\delta_y v^{n+1}\|^2-\|\delta_y v^{n-1}\|^2\right)\\&\quad+\frac{h_x^2}{24\tau}\left(\|\delta_x\delta_y v^{n+1}\|^2-\|\delta_x\delta_y v^{n-1}\|^2\right),1\le n\le K-1.\end{aligned}\tag{33}$$

Combining (32) and (33), we have

$$\begin{aligned}&D\left(A_y\delta_x^2 v^{n-1}+A_y\delta_x^2 v^{n+1}+A_x\delta_y^2 v^{n-1}+A_x\delta_y^2 v^{n+1},\delta_t v^n\right)\\&=-\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2-\|\nabla_h v^{n-1}\|_A^2\right),1\le n\le K-1.\end{aligned}\tag{34}$$

Substituting (30), (31), (34) into (29) and applying Lemma 6, Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{\varepsilon^2}{\tau}\left(\|\delta_{\bar t} v^{n+1/2}\|_A^2-\|\delta_{\bar t} v^{n-1/2}\|_A^2\right)+2\|\delta_t v^n\|_A^2\\&=-\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2-\|\nabla_h v^{n-1}\|_A^2\right)+2(g^n,\delta_t v^n)\\&\le-\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2-\|\nabla_h v^{n-1}\|_A^2\right)+\frac{2}{3}\|\delta_t v^n\|^2+\frac{3}{2}\|g^n\|^2\\&\le-\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2-\|\nabla_h v^{n-1}\|_A^2\right)+2\|\delta_t v^n\|_A^2+\frac{3}{2}\|g^n\|^2,1\le n\le K-1.\end{aligned}\tag{35}$$

Sorting Eq. (35) and adding $D/2\|\nabla_h v^n\|_A^2$ to both sides, we have

$$\begin{aligned}&\varepsilon^2\|\delta_{\bar t} v^{n+1/2}\|_A^2+\frac{D}{2}\left(\|\nabla_h v^{n+1}\|_A^2+\|\nabla_h v^n\|_A^2\right)\le\varepsilon^2\|\delta_{\bar t} v^{n-1/2}\|_A^2\\&\quad+\frac{D}{2}\left(\|\nabla_h v^n\|_A^2+\|\nabla_h v^{n-1}\|_A^2\right)+\frac{3\tau}{2}\|g^n\|^2,1\le n\le K-1.\end{aligned}\tag{36}$$

Let $E^n=\varepsilon^2\|\delta_{\bar t} v^{n+1/2}\|_A^2+(D/2)(\|\nabla_h v^{n+1}\|_A^2+\|\nabla_h v^n\|_A^2)$, $1\le n\le K-1$.

Eq. (36) is written as

$$E^n\le E^{n-1}+\frac{3\tau}{2}\|g^n\|^2,1\le n\le K-1.\tag{37}$$

Summing over n results in

$$E^n\le E^0+\frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2,1\le n\le K-1.\tag{38}$$

Multiplying Eq. (25) by $\delta_{\bar t} v^{1/2}$, summing up for i from 1 to $M-1$, applying Lemma 4 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{2\varepsilon^2}{\tau}\left(A_xA_y\delta_{\bar t} v^{\frac12},\delta_{\bar t} v^{\frac12}\right)-\frac{2\varepsilon^2}{\tau}\left(\varphi,\delta_{\bar t} v^{\frac12}\right)+\left(A_xA_y\delta_{\bar t} v^{\frac12},\delta_{\bar t} v^{\frac12}\right)\\&\quad+\frac{D}{2\tau}\left(\|\nabla_h v^1\|_A^2-\|\nabla_h v^0\|_A^2\right)=\left(g^0,\delta_{\bar t} v^{\frac12}\right),\end{aligned}\tag{39}$$

where $\varphi_{ij}=A_xA_yu_1(x_i,y_j)$.

Sorting Eq. (39), we have

$$\begin{aligned}&\frac{2\varepsilon^2}{\tau}\|\delta_{\bar t} v^{\frac12}\|_A^2+\|\delta_{\bar t} v^{\frac12}\|_A^2+\frac{D}{2\tau}\left(\|\nabla_h v^1\|_A^2-\|\nabla_h v^0\|_A^2\right)\\&=\frac{2\varepsilon^2}{\tau}\left(\varphi,\delta_{\bar t} v^{\frac12}\right)+\left(g^0,\delta_{\bar t} v^{\frac12}\right)\le\frac{1}{3}\|\delta_{\bar t} v^{\frac12}\|^2+\frac{3\varepsilon^4}{\tau^2}\|\varphi\|^2\\&\quad+\frac{\varepsilon^2}{3\tau}\|\delta_{\bar t} v^{\frac12}\|^2+\frac{3\tau}{4\varepsilon^2}\|g^0\|^2\le\|\delta_{\bar t} v^{\frac12}\|_A^2+\frac{\varepsilon^2}{\tau}\|\delta_{\bar t} v^{\frac12}\|_A^2\\&\quad+\frac{3\varepsilon^4}{\tau^2}\|\varphi\|^2+\frac{3\tau}{4\varepsilon^2}\|g^0\|^2.\end{aligned}\tag{40}$$

Sorting Eq. (40), we have

$$\begin{aligned}&\varepsilon^2\|\delta_{\bar t} v^{\frac12}\|_A^2+\frac{D}{2}\left(\|\nabla_h v^1\|_A^2+\|\nabla_h v^0\|_A^2\right)\\&\le D\|\nabla_h v^0\|_A^2+\frac{3\varepsilon^4}{\tau}\|\varphi\|^2+\frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2.\end{aligned}\tag{41}$$

Substituting (41) into (38), we get

$$\begin{aligned}E^n&\le D\|\nabla_h v^0\|_A^2+\frac{3\varepsilon^4}{\tau}\|\varphi\|^2+\frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2\\&\quad+\frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2,1\le n\le K-1.\end{aligned}\tag{42}$$

End of the proof.

Remark 8. According to the results, it can be seen that the derivative term of initial value affects stability and convergence, so this term in the Cattaneo model will be set as zero in general. For details, please refer to the literature [15].

Theorem 9. *Suppose $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \le n \le K\}$ is the solution of Eq. (1), $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \le n \le K\}$ is the solution of the difference scheme (20). Let*

$$e_{ij}^n = u_{ij}^n - U_{ij}^n, (i,j) \in \bar{\omega}, 0 \le n \le K, \tag{43}$$

it holds that

$$\begin{aligned} &\varepsilon^2 \|\delta_{\bar{t}} e^{n+\frac{1}{2}}\|_A^2 + \frac{D}{2}\left(\|\nabla_h e^{n+1}\|_A^2 + \|\nabla_h e^n\|_A^2\right) \\ &\quad \le \left(\frac{3}{4\varepsilon^2} + \frac{3T}{2}\right) C\left(\tau^2 + h_x^4 + h_y^4\right)^2, 0 \le n \le K-1. \end{aligned} \tag{44}$$

Proof. Subtracting (14), (18), and (20), we can get the error system

$$\begin{aligned} &\varepsilon^2 A_x A_y \delta_t^2 e_{ij}^n + A_x A_y \delta_t e_{ij}^n \\ &\quad = \frac{D}{2}\left(A_y \delta_x^2 e_{ij}^{n-1} + A_y \delta_x^2 e_{ij}^{n+1} + A_x \delta_y^2 e_{ij}^{n-1} + A_x \delta_y^2 e_{ij}^{n+1}\right) \\ &\qquad + R_{ijn}^{(3)}, (i,j) \in \omega, 1 \le n \le K-1, \end{aligned} \tag{45}$$

$$\begin{aligned} &\frac{2\varepsilon^2}{\tau} A_x A_y \delta_{\bar{t}} e_{ij}^{\frac{1}{2}} + A_x A_y \delta_{\bar{t}} e_{ij}^{\frac{1}{2}} \\ &\quad = \frac{D}{2}\left(A_y \delta_x^2 e_{ij}^0 + A_y \delta_x^2 e_{ij}^1 + A_x \delta_y^2 e_{ij}^0 + A_x \delta_y^2 e_{ij}^1\right) \\ &\qquad + r_{ijn}, (i,j) \in \omega, \end{aligned} \tag{46}$$

$$e_{ij}^0 = 0, (i,j) \in \omega, \tag{47}$$

$$e_{ij}^n = 0, (i,j) \in \partial\omega, 0 \le n \le K. \tag{48}$$

Table 1: Error and convergence order of compact difference scheme.

h	τ	order	order	$errl^\infty$	$errl^2$
1/4	1/20	—	—	7.1854e-03	3.5927e-03
1/8	1/80	3.517	1.759	5.8076e-04	2.9038e-04
1/16	1/320	4.005	2.003	3.6204e-05	1.8102e-05
1/32	1/1280	3.999	2.000	2.2637e-06	1.3185e-06
1/64	1/5120	4.000	2.000	1.4151e-07	7.0753e-08

Using the Theorem 7 and noticing (47) and (48), we have

$$\begin{aligned} &\varepsilon^2 \|\delta_{\bar{t}} e^{n+1/2}\|_A^2 + \frac{D}{2}\left(\|\nabla_h e^{n+1}\|_A^2 + \|\nabla_h e^n\|_A^2\right) \\ &\quad \le D\|\nabla_h e^0\|_A^2 + \frac{3\tau^2}{4\varepsilon^2}\left(r_{ijn}\right)^2 + \frac{3\tau}{2}\sum_{k=1}^{n}\left(R_{ijn}^{(3)k}\right)^2 \\ &\quad \le \frac{3\tau^2}{4\varepsilon^2} C_1^2\left(\tau + h_x^4 + h_y^4\right)^2 + \frac{3\tau}{2} n C_2^2\left(\tau^2 + h_x^4 + h_y^4\right)^2 \\ &\quad \le \left(\frac{3}{4\varepsilon^2} + \frac{3T}{2}\right) C\left(\tau^2 + h_x^4 + h_y^4\right)^2. \end{aligned} \tag{49}$$

End of the proof.

Remark 10. Through Theorem 7 and Theorem 9, we prove the stability and convergence of the scheme. Due to the basic relationship between H^1 semi-norm and infinity norm, we can also obtain that the proposed scheme is stable and convergent in L^∞ norm.

4. Numerical Experiments

In this chapter, we verify the numerical accuracy and validity of the proposed scheme.

In the following numerical experiment, for simplicity, the domain is set to the square domain. The domain is $\Omega = (0,1) \times (0,1)$, and h_x is equal to h_y. $\varepsilon^2 = 0.1$, $D = 1$, $u_0(x,y) = \sin(\pi x)\sin(\pi y)$, $u_1(x,y) = 0$. In order to verify the numerical accuracy of the scheme, we need the following exact solution of the above problem. We can obtain the exact solution by using the separation of variables method,

$$u(x,y,t) = e^{-\frac{1}{2\varepsilon^2}t}\left(\cos\sqrt{\frac{2D\pi^2}{\varepsilon^2} - \frac{1}{4\varepsilon^2}}t + \frac{1}{2\varepsilon^2\sqrt{2D\pi^2/\varepsilon^2 - 1/4\varepsilon^2}}\sin\sqrt{\frac{2D\pi^2}{\varepsilon^2} - \frac{1}{4\varepsilon^2}}t\right)\sin(\pi x)\sin(\pi y). \tag{50}$$

Before we give the experimental results, we first give the definitions of l^2-norm of the error and l^∞-norm of the error.

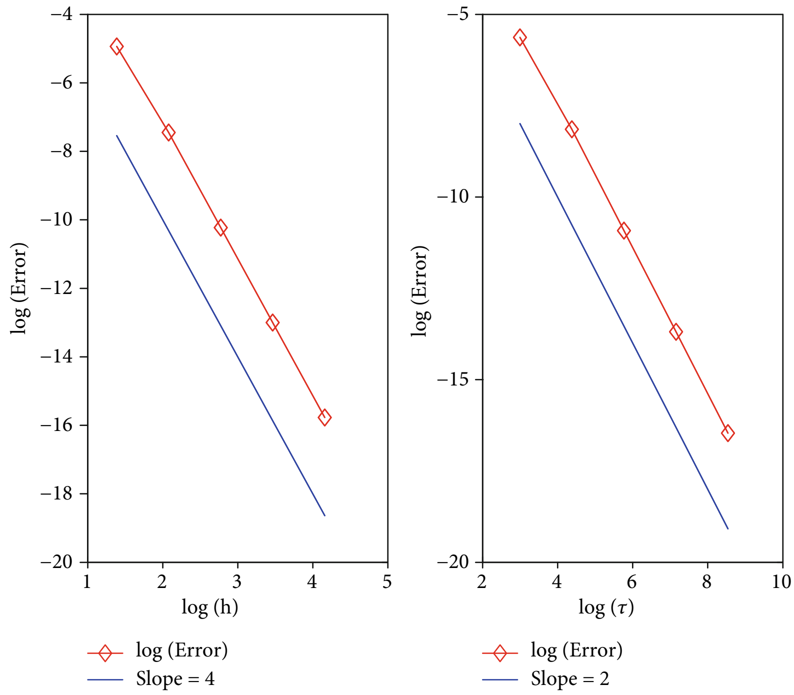

FIGURE 1: Order of spatial and temporal convergence.

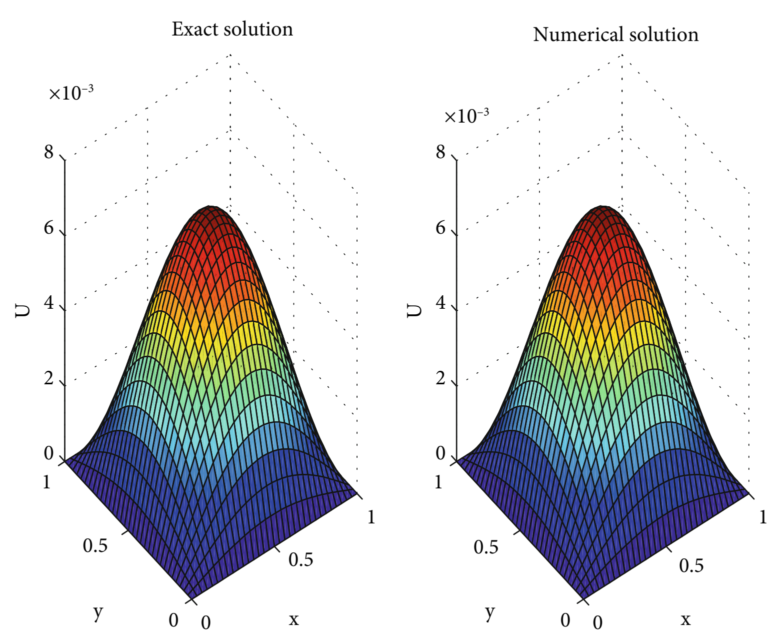

FIGURE 2: The exact solution and the numerical at fixed $D = 1$, $\varepsilon^2 = 0.1$, $T = 1, h = 1/32$ and $\tau = 1/1280$.

For all grid points, l^2-norm of the error is defined by

$$Error_{l^2} = \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{N} h_x h_y \mid u_{ij}^K - U_{ij}^K \mid}. \qquad (51)$$

For all grid points, l^∞-norm of the error is defined by

$$Error_{l^\infty} = \max_{1\le i\le M, 1\le j\le N} \left| u_{ij}^K - U_{ij}^K \right|, \qquad (52)$$

where u_{ij} is the true solution, U_{ij} is the numerical solution, and K is the last moment. Let $N = M$, order of spatial

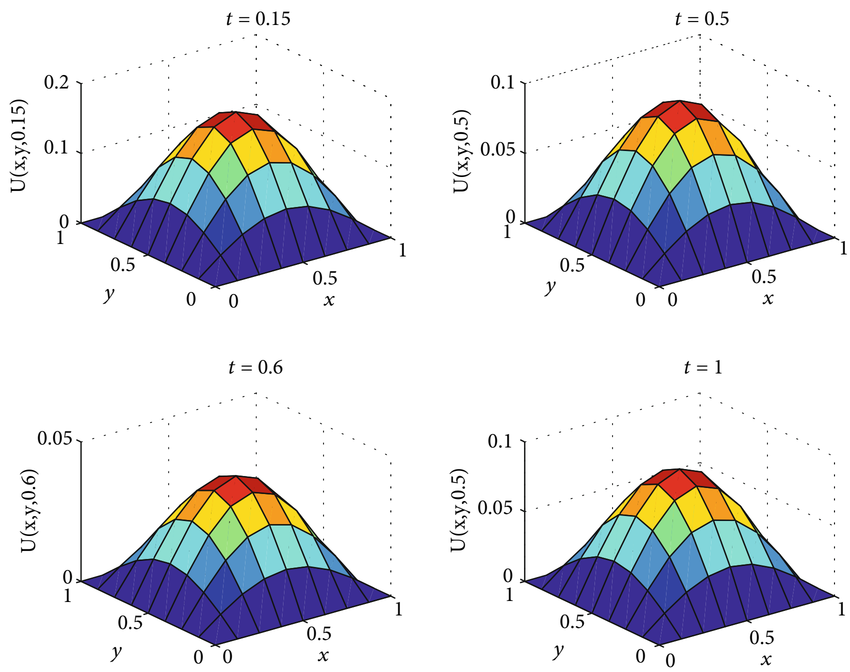

FIGURE 3: Temperature images at different times when $D = 1$, $\varepsilon^2 = 0.1$, $h = 1/16$, and $\tau = 1/80$.

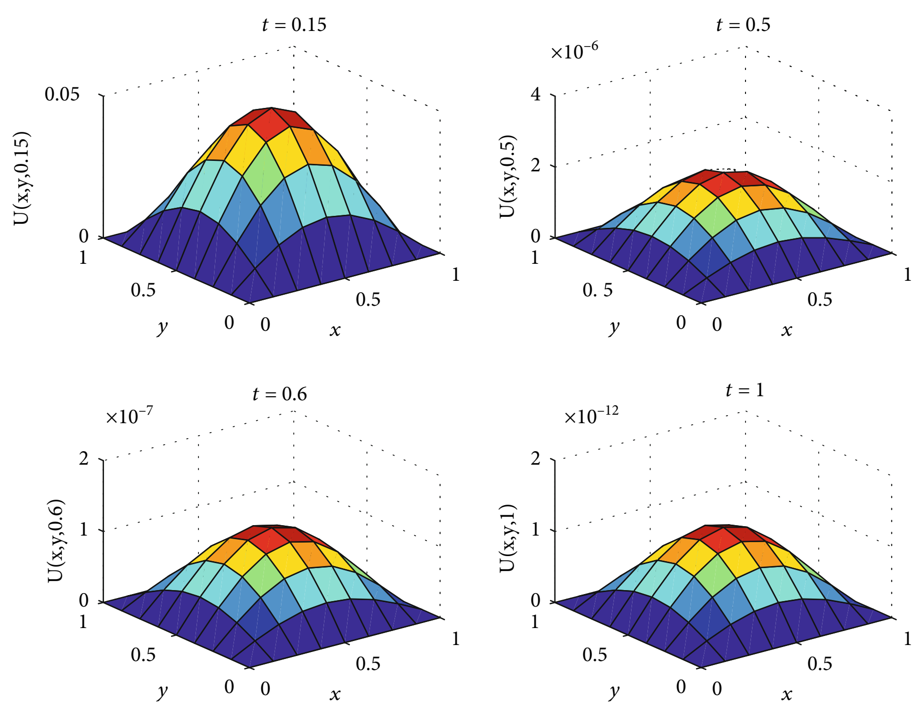

FIGURE 4: Temperature images at different times when $D = 1$, $\varepsilon^2 = 0.01$, $h = 1/16$, and $\tau = 1/80$.

convergence is denoted by P_h and order of temporal convergence is denoted by P_τ, which are defined as follows

$$P_h = \frac{\log\left(Error_1/Error_2\right)}{\log\left(h_1/h_2\right)}, P_\tau = \frac{\log\left(Error_1/Error_2\right)}{\log\left(\tau_1/\tau_2\right)} \tag{53}$$

In Table 1, let $h = h_x = h_y$, and $T = 1$, for h and τ, take the corresponding values, respectively, the compact difference scheme has the following corresponding error, order of temporal convergence and order of spatial convergence. The third column is the order of spatial convergence, and the fourth column is the order of time convergence. We can find that when the spatial step is reduced to one-

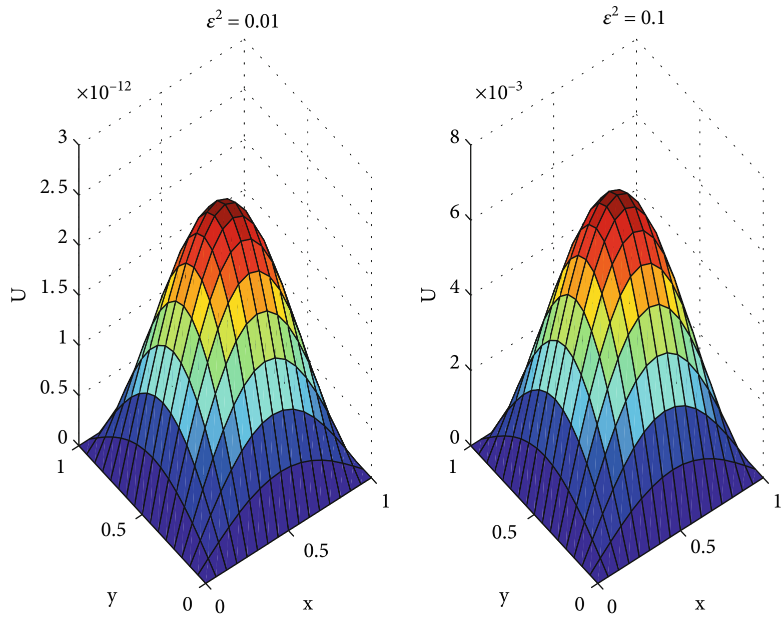

FIGURE 5: Temperature images with different relaxation parameters when $D = 1$, $h = 1/16$ and $1/320$.

half of the original and the temporal step is reduced to one-quarter of the original, the error is reduced to about one-sixteenth of the original. At the same time, the compact difference scheme keeps the fourth-order spatial rate of convergence and second-order temporal rate of convergency. So we also verify the stability and convergence of the scheme.

In order to verify the applicability of the compact difference scheme more accurately, we give the image of convergence order defined the by l^{∞}-norm. In Figure 1, we can see more clearly that the convergence order of the scheme (20) is $O(\tau^2 + h_x^4 + h_y^4)$, which also verifies that the scheme (20) proposed in this paper is stable and effective. And the relaxation parameter does not affect the convergence rate.

Figure 2 show the true solution and the numerical solution when $h = 1/32$ and $\tau = 1/1280$.

Figure 3 shows the images when t is taken at different moments. It can be seen from the images that as time increases, the temperature also decreases gradually.

Next, the value of ε^2 in the previous example is 0.01, and the temperature image of t at different moments is also obtained. See Figure 4, it has the same property as Figure 3.

Finally, we use two examples to test the influence of relaxation parameters, as shown in Figure 5. From Figure 3–5, we can see that the larger ε^2 is, the more obvious the fluctuation of temperature distribution is, and the slower the temperature change at the initial time.

In Figure 6, in order to observe the differences between the Cattaneo model and the traditional diffusion equation more conveniently, we give the temperature distribution images of the one-dimensional Cattaneo model and the heat conduction equation at different moments at the midpoint of $x = 0.5$. It can be seen from the Figure 6 that the Cattaneo model has a slower temperature transformation at the initial moment than the heat conduction equation, but at the same time, the former has negative temperature and oscillation, which is also affected by the hyperbolic property of Cattaneo model.

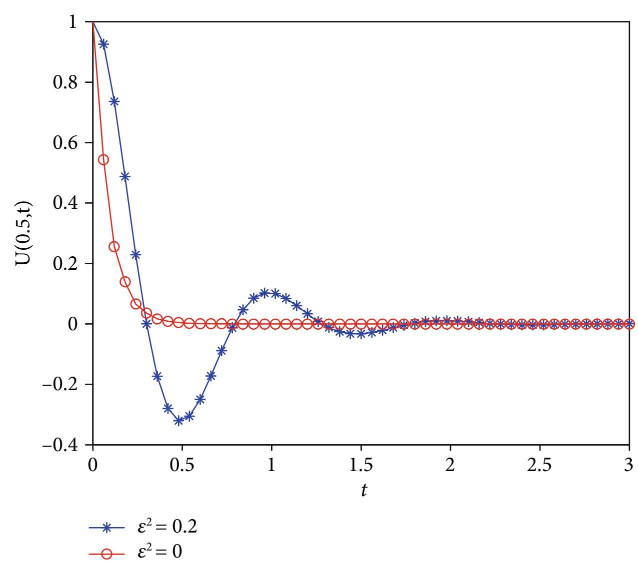

FIGURE 6: Temperature distribution images at $x = 0.5$.

5. Conclusion

In this paper, we propose a compact finite difference scheme for a two-dimensional Cattaneo model and prove the stability and convergence of the scheme by energy method. A concrete example is given, and the exact solution is obtained by the method of separating variables, it can be seen from the figures that the fitting degree of the exact solution and numerical solution is higher, it is also verified by numerical

experiments that the convergence order of the scheme is $O(\tau^2 + h_x^4 + h_y^4)$. In conclusion, the proposed scheme is simple, stable, and effective and has higher computational accuracy than some existing methods.

Data Availability

Some or all data, models, or code generated or used during the study are available from the corresponding author by request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Yating Huang carried out the main part of this article. All authors read and approved the final manuscript.

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