Research Article

On the Uniqueness of Meromorphic Functions on Annuli in terms of Deficiencies

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The purpose of this article is to study the uniqueness of meromorphic functions on annuli. Under a certain condition about deficiencies, we prove some new uniqueness theorems of meromorphic functions on the annulus \( A = \{ z : (1/R_0) < |z| < R_0 \} \), where \( 1 < R_0 \leq +\infty \).

1. Introduction and Main Results

In this article, we assume that the readers are familiar with the classical notations and definitions of Nevanlinna theory (refer to [1, 2]). The main purpose of this article is to study the uniqueness of meromorphic functions on annuli. For the necessary concepts and notations of the Nevanlinna theory of meromorphic functions on annuli, such as \( T_0(r, f) \), \( m_0(r, f) \), and \( N_0(r, f) \), refer to the excellent summarizations [3–11].

Let \( \Lambda \) be a value in the extended complex plane \( \mathbb{C} \), and let \( f \) and \( g \) be two meromorphic functions on the annulus \( \Lambda \) defined by\( \Lambda = \{ z : (1/R_0) < |z| < R_0 \} \), where \( 1 < R_0 \leq +\infty \). Then, we say that \( f \) and \( g \) share a IM (ignoring multiplicities) when \( f - a \) and \( g - a \) have the same zeros, and furthermore, we say that \( f \) and \( g \) share a CM (counting multiplicities) when \( f - a \) and \( g - a \) have the same zeros with the same multiplicities.

As mentioned in [3, 4], the reduced counting function \( \overline{N}_0(r, 1/(f - a)) \) is defined by

\[
\frac{1}{r} \int_{1/r}^{1} \overline{N}_1(t, 1/(f - a)) \, dt + \frac{1}{r} \int_{1/r}^{1} \overline{N}_2(t, 1/(f - a)) \, dt, \tag{1}
\]

where \( \overline{N}_1(t, 1/(f - a)) \) and \( \overline{N}_2(t, 1/(f - a)) \) are the functions in counting only one of the zeros of \( f - a \) \( \{ z : 0 < |z| < 1 \} \) and \( \{ z : 1 < |z| < t \} \), respectively. Similarly, we denote by \( \overline{N}_0(r, a) \) \( (N_0(r, a)) \) the reduced counting function of common zeros (different zeros) of \( f - a \) and \( g - a \) on \( \Lambda \), where \( \overline{N}_0^D(r, a) = \overline{N}_0(r, 1/(f - a)) + \overline{N}_0(r, 1/(g - a)) - 2\overline{N}_0^D(r, a) \).

For a nonconstant meromorphic function \( f \) on the annulus \( \Lambda \), it is named as a transcendental meromorphic function on \( \Lambda \) provided that

\[
\limsup_{r \to +\infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty, \tag{2}
\]

or

\[
\limsup_{r \to +\infty} \frac{T_0(r, f)}{-\log(R_0 - r)} = 0, \quad 1 < r < R_0 < +\infty,
\]

respectively. In fact, the transcendental meromorphic functions are also known as admissible meromorphic functions. If \( f \) is a transcendental meromorphic function on \( \Lambda \), then we have \( S(r, f) \equiv o(T_0(r, f)) \) for all \( 1 < r < R_0 \) except for a set \( \Delta \), such that \( \int_{\Delta} r^{1+} \, dr < +\infty \) or a set \( \Delta' \), such that \( \int_{\Delta'} r^{1+} \, dr < +\infty \), respectively.

There existed many famous results about the uniqueness theory of meromorphic functions sharing values. In 1926, Nevanlinna [12] proved the celebrated five-value theorem.
Theorem 1. Let $f$ and $g$ be two nonconstant meromorphic functions in $\mathbb{C}$, and let $a_i (i = 1, 2, 3, 4, 5)$ be five distinct values in $\mathbb{C}$. If $f$ and $g$ share the values $a_i$ IM for $i = 1, 2, 3, 4, 5$ in $\mathbb{C}$, then $f \equiv g$. 

Since that time, a series of results emerged in large numbers, which discussed and generalized the five-value theorem (Theorem 1). For the main results about the generalizations of Theorem 1 in simply connected regions, we can refer to [7, 13–16]. For instance, Zheng [15, 16] and generalizations of $fT$ theorem 1 in simply connected regions, complex numbers in $\mathbb{C}$. Let $f$ and $g$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_i (i = 1, 2, 3, 4, 5)$ be five distinct values in $\mathbb{C}$. If $f$ and $g$ share the values $a_i$ IM for $i = 1, 2, 3, 4, 5$ on $\mathbb{A}$, then $f \equiv g$.

From the very point of sharing small functions, we studied above theorems in [19] and provided the following uniqueness theorem of meromorphic functions sharing four small functions on annuli.

Theorem 3. Let $f$ and $g$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_i (i = 1, 2, 3, 4)$ be four distinct small functions with respect to $f$ and $g$ on $\mathbb{A}$. If $f$ and $g$ share $a_i (i = 1, 2, 3, 4)$ IM and

$$\sum_{i=1}^{4} \tilde{N}_0(r, a_i) \neq S(r, f),$$

then $f \equiv g$, where $\tilde{N}_0(r, a_i)$ is the reduced counting function which counts the multiple common zeros of $f - a_i$ and $g - a_i$ on $\mathbb{A}$.

In this article, we mainly investigate whether Theorem 2 holds if $f$ and $g$ dissatisfy the condition of sharing values. From the very point of deficiencies, we deal with this question and propose the following uniqueness theorem without conditions of sharing values. This theorem generalizes Theorem 2.

Theorem 4. Let $f$ and $g$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and let $a_i (i = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\mathbb{C}$. Then, we have $f \equiv g$ provided that

$$\sum_{i=1}^{5} \delta_0^D(a_i) > \frac{14}{3},$$

where the deficiencies $\delta_0^D(a_i)$ are defined as

$$1 - \limsup_{r \to \infty} \frac{\tilde{N}_0^D(r, a_i)}{T_0(r, f) + T_0(r, g)},$$

when $R_0 = +\infty$, or

$$1 - \lim_{r \to R_0} \frac{\tilde{N}_0^D(r, a_i)}{T_0(r, f) + T_0(r, g)},$$

when $R_0 < +\infty$, respectively.

In special, if $f$ and $g$ share $a_i (i = 1, 2, 3, 4, 5)$ "IM," then it is obvious that $\sum_{i=1}^{5} \delta_0^D(a_i) = 5$, which satisfies the condition

$$\sum_{i=1}^{5} \delta_0^D(a_i) > \frac{14}{3},$$

of Theorem 4. In view of the discussion above, we deduce a corollary as follows. This corollary partly improves Theorem 2 in the sense that IM is replaced with "IM."

Corollary 1. Let $f$ and $g$ be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and let $a_i (i = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\mathbb{C}$. If $f$ and $g$ share $a_i (i = 1, 2, 3, 4, 5)$ "IM," then $f \equiv g$.

2. Some Lemmas

In this section, we will give some necessary lemmas, where the third lemma is motivated by the ideas of [20–22].

Lemma 1 (see [4], Theorem 1). Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda \geq 0$. Then,

(i) If $R_0 = +\infty$, then $m_0(r, f'/f) = O((\log(r/r_0))$ for $R \in (1, \infty)$ except for a set $\Delta_r$ such that $\int_{\Delta_r} dr < +\infty$.

(ii) If $R_0 < +\infty$, then $m_0(r, f'/f) = O((T_0(r, f)/R_0 - r))$ for $r \in (1, R_0)$ except for a set $\Delta'_r$ such that $\int_{\Delta'_r} dr/(R_0 - r)^{k_1} < +\infty$.

Lemma 2 (see [18], Theorem 2.3). Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be $q$ distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then,

$$(q - 2)T_0(r, f) < \sum_{j=1}^{q} \tilde{N}_0(r, \frac{1}{f - a_j}) + S(r, f).$$

Inspired by the ideas of [20–22], we propose the following lemma and give the proof.

Lemma 3. Let $f$ and $g$ be two transcendental meromorphic functions on $\mathbb{A} = \{z : (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and let $a_i (i = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\mathbb{C}$. If $f \equiv g$, then
\[
\mathcal{N}_0^E(r, a_i) \leq \sum_{j=1, j \neq i}^5 \mathcal{N}_0^D(r, a_j) + S(r, f) + S(r, g),
\]
(9)

where \( \mathcal{N}_0^D(r, a_i) \) is the reduced counting function of the common (different) zeros of \( f \) and \( g \) on \( \mathbb{A} \) \((i = 1, 2, 3, 4, 5)\).

**Proof.** Without loss of generality, we suppose that \( a_1 = 0 \), \( a_2 = 1 \), \( a_4 = \infty \), \( a_4 = a \), and \( a_5 = b \), in which \( a, b \) are two distinct complex numbers such that \( a, b \neq 0, 1, \infty \). Otherwise, a Möbius transformation as

\[
\frac{f - a}{a - a_1} \frac{a_2 - a}{a - a_1},
\]
(10)

will be done. Then, set

\[
h = h_1 - h_2,
\]
(11)

where

\[
h_1 = \frac{\left( f - g \right) f' g'}{f (f - 1)(g - 1)(g - a)}
\]
(12)

and

\[
h_2 = \frac{\left( f - g \right) f' g'}{g (g - 1)(f - 1)(f - a)}
\]

Noting that \( h_1 \) can be expressed by

\[
\frac{f'}{f - 1} \left[ g' + \frac{g'}{a(a - 1)(g - a)} - \frac{g'}{(a - 1)(g - 1)} \right] + \frac{1}{a - 1} \left( \frac{f'}{f - 1} \right) \left( \frac{g'}{g - a} - \frac{g'}{g - 1} \right)
\]
(13)

By Lemma 1, we have \( m_0(r, h_1) = S(r, f) + S(r, g) \). Similarly, we get \( m_0(r, h_2) = S(r, f) + S(r, g) \), and thus,

\[
m_0(r, h) = S(r, f) + S(r, g),
\]
(14)

holds.

Next, we will estimate the counting function \( \mathcal{N}_0(r, h) \). A simple computation yields

\[
h = (f - g)^2 \frac{f' g'}{f (f - 1)(f - a)g(g - 1)(g - a)}
\]
(15)

Then, it is easy to see that the poles of \( h \) only come from the zeros of \( f, g, f - 1, g - 1, f - a, g - a \) and the poles of \( f \) and \( g \) on \( \mathbb{A} \). Now, let \( z_0 \) be a common zero of \( f \) and \( g \) on \( \mathbb{A} \) with multiplicity \( p \) and \( q \), respectively. Without loss of generality, assume that \( p \geq q \). Then, it follows that \( z_0 \) is a zero of \( (f - g)^2 \) with multiplicity at least \( 2q \) and that \( z_0 \) is a pole of

\[
\frac{f' g'}{f (f - 1)(f - a)g(g - 1)(g - a)}
\]
(16)

with multiplicity 2. We consequently know that \( z_0 \) is not a pole of \( h \), and hence the poles of \( h \) cannot occur at the common zeros of \( f \) and \( g \). By similar methods, we can conclude that the poles of \( h \) cannot occur at the common zeros of \( f - 1 \) and \( g - 1 \), the common zeros of \( f - a \) and \( g - a \), and the common poles of \( f \) and \( g \), so the poles of \( h \) only come from the different zeros of \( f, g, f - 1, g - 1, f - a, g - a \) and the different poles of \( f \) and \( g \) on \( \mathbb{A} \). In order to analyze these different zeros and different poles, we distinguish the following distinct cases.

**Case 1:** let \( z_1 \) be a zero of \( f \) which is not a zero of \( g \). Then, by using the equation (15), we find that \( z_1 \) is a pole of \( h \) with multiplicity at most 1 if \( g(z_1) \neq 0, \infty, a \); otherwise, \( z_1 \) is a pole of \( h \) with multiplicity at most 2.

**Case 2:** let \( z_2 \) be a zero of \( f - 1 \) which is not a zero of \( g - 1 \). It is clear that \( z_2 \) is a pole of \( h \) with multiplicity at most 2.

**Case 3:** let \( z_3 \) be a pole of \( f \), which is not a pole of \( g \). It is clear that \( z_3 \) is a pole of \( h \) with multiplicity at most 1 if \( g(z_3) \neq 0, 1, a \); otherwise, \( z_3 \) is a pole of \( h \) with multiplicity at most 2.

**Case 4:** let \( z_4 \) be a zero of \( f - a \), which is not a zero of \( g - a \). It is clear that \( z_4 \) is a pole of \( h \) with multiplicity at most 1 if \( g(z_4) \neq 0, 1, \infty \); otherwise, \( z_4 \) is a pole of \( h \) with multiplicity at most 2.

**Case 5:** let \( z_5 \) be a zero of \( g \), which is not a zero of \( f \). It is clear that \( z_5 \) is a pole of \( h \) with multiplicity at most 1 if \( f(z_5) \neq 0, \infty, a \); otherwise, \( z_5 \) is a pole of \( h \) with multiplicity at most 2.

**Case 6:** let \( z_6 \) be a zero of \( g - 1 \), which is not a zero of \( f - 1 \). It is clear that \( z_6 \) is a pole of \( h \) with multiplicity at most 1 if \( f(z_6) \neq 0, 1, \infty \); otherwise, \( z_6 \) is a pole of \( h \) with multiplicity at most 2.

**Case 7:** let \( z_7 \) be a pole of \( g \), which is not a pole of \( f \). It is clear that \( z_7 \) is a pole of \( h \) with multiplicity at most 1 if \( f(z_7) \neq 0, 1, a \); otherwise, \( z_7 \) is a pole of \( h \) with multiplicity at most 2.

**Case 8:** let \( z_8 \) be a zero of \( g - a \), which is not a zero of \( f - a \). It is clear that \( z_8 \) is a pole of \( h \) with multiplicity at most 1 if \( f(z_8) \neq 0, 1, \infty \); otherwise, \( z_8 \) is a pole of \( h \) with multiplicity at most 2.

In view of these cases, we obtain

\[
\mathcal{N}_0(r, h) \leq \mathcal{N}_0^D(r, 0) + \mathcal{N}_0^D(r, 1) + \mathcal{N}_0^D(r, \infty) + \mathcal{N}_0^D(r, a),
\]
(17)

which means

\[
\mathcal{N}_0(r, h) \leq \sum_{i=1}^5 \mathcal{N}_0^D(r, a_i).
\]
(18)

Combining (14) with (18), we get
3. The Proof of Theorem 4

On the contrary, we suppose that \( f \equiv g \), and then it follows from Lemma 3 that

\[
\mathcal{N}_0^E(r, a_i) \leq \sum_{j=1, j\neq i}^5 \mathcal{N}_0^D(r, a_j) + S(r, f) + S(r, g),
\]

for \( i = 1, 2, 3, 4, 5 \). Thus, noting that

\[
\mathcal{N}_0\left(r, \frac{1}{f - a_i}\right) + \mathcal{N}_0\left(r, \frac{1}{g - a_j}\right) = 2\mathcal{N}_0^E(r, a_i) + \mathcal{N}_0^D(r, a_j),
\]

for \( i = 1, 2, 3, 4, 5 \), we know

\[
\mathcal{N}_0\left(r, \frac{1}{f - a_i}\right) + \mathcal{N}_0\left(r, \frac{1}{g - a_j}\right) \\
\leq \mathcal{N}_0^D(r, a_i) + \sum_{j=1, j\neq i}^5 2\mathcal{N}_0^D(r, a_j) + S(r, f) + S(r, g).
\]

This yields that

\[
\mathcal{N}_0\left(r, \frac{1}{f - a_i}\right) + \mathcal{N}_0\left(r, \frac{1}{g - a_j}\right) \\
\leq \sum_{j=1}^5 2\mathcal{N}_0^D(r, a_j) - \mathcal{N}_0^D(r, a_i) + S(r, f) + S(r, g),
\]

for \( i = 1, 2, 3, 4, 5 \), which further yields that

\[
\sum_{i=1}^5 \mathcal{N}_0\left(r, \frac{1}{f - a_i}\right) + \sum_{i=1}^5 \mathcal{N}_0\left(r, \frac{1}{g - a_i}\right) \\
\leq 9 \sum_{i=1}^5 \mathcal{N}_0^D(r, a_i) + S(r, f) + S(r, g).
\]

On the other hand, by utilizing Lemma 2, we find

\[
3T_0(r, f) < \sum_{i=1}^5 \mathcal{N}_0\left(r, \frac{1}{f - a_i}\right) + S(r, f),
\]

\[
3T_0(r, g) < \sum_{i=1}^5 \mathcal{N}_0\left(r, \frac{1}{g - a_i}\right) + S(r, f).
\]

Therefore, it follows from (27) that

\[
3T_0(r, f) + 3T_0(r, g) \leq 9 \sum_{i=1}^5 \mathcal{N}_0^D(r, a_i) + S(r, f) + S(r, g),
\]

which means

\[
\frac{1}{3} \sum_{i=1}^5 T_0(r, f) + T_0(r, g) + \frac{S(r, f) + S(r, g)}{T_0(r, f) + T_0(r, g)}.
\]

Consequently, from (30), we have

\[
\sum_{i=1}^5 \delta^D(a_j) \leq \frac{14}{3},
\]

which is a contradiction. Hence, the proof is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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