Research Article

New Sequential Fractional Differential Equations with Mixed-Type Boundary Conditions

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In this paper, we introduce new sequential fractional differential equations with mixed-type boundary conditions

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( C_{-}^{D^{q}} + k^{q} C_{-}^{D^{q-1}} \right) u(t) = f(t, u(t), C_{-}^{D^{q-1}} u(t)), \quad t \in (0, 1), \\
\alpha_{i} u(0) + \beta_{i} u(1) + \gamma_{i} u'(\eta) = \epsilon_{i}, \quad \eta \in (0, 1), \\
\alpha_{i} u'(0) + \beta_{i} u''(1) + \gamma_{i} u''(\eta) = \epsilon_{i},
\end{array} \right.
\end{align*}
\]

where \( q \in (1, 2] \) is a real number, \( k, r > 0, \alpha_{i}, \beta_{i}, \gamma_{i}, \epsilon_{i} \in \mathbb{R}, \) \( i = 1, 2 \), \( C_{-}^{D^{q}} \) is the Caputo fractional derivative, and the boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases. Our approach to treat the above problem is based upon standard tools of fixed point theory and some new inequalities of norm form. Some existence results are obtained and well illustrated through the aid of examples.

1. Introduction

In this paper, we focus on sequential fractional differential equations with mixed-type boundary conditions.

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( C_{-}^{D^{q}} + k^{q} C_{-}^{D^{q-1}} \right) u(t) = f(t, u(t), C_{-}^{D^{q-1}} u(t)), \quad t \in (0, 1), \\
\alpha_{i} u(0) + \beta_{i} u(1) + \gamma_{i} u'(\eta) = \epsilon_{i}, \quad \eta \in (0, 1), \\
\alpha_{i} u'(0) + \beta_{i} u''(1) + \gamma_{i} u''(\eta) = \epsilon_{i},
\end{array} \right.
\end{align*}
\]

where \( q \in (1, 2] \) is a real number and \( k, r > 0, \alpha_{i}, \beta_{i}, \gamma_{i}, \epsilon_{i} \in \mathbb{R}, \) \( i = 1, 2 \), \( C_{-}^{D^{q}} \) is the Caputo fractional derivative of order \( q \). The nonlinearity term \( f \) contains the unknown function and its lower order fractional derivatives. The new boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases which can be regarded as the linear combination of the values of the unknown function and its first derivatives at the end points of interval, and the Riemann-Liouville fractional integral value of the unknown function and its first derivatives at an interior point of interval.

Fractional differential equations have attracted significant attention for their wide application in many fields of engineering and applied sciences (see [1–10]). Sequential fractional differential equations as an important branch have also received wide attention; for instance, see [11–16]. Motivated by the HIV infection model and its application background in [12], the existence and uniqueness of solutions for the following sequential fractional differential system are obtained by means of Leray-Schauder’s alternative and Banach’s contraction principle

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( C_{-}^{D^{q}} + \lambda^{q} C_{-}^{D^{q-1}} \right) u(t) = f_{1}(t, u(t), v(t)), \quad t \in (0, 1), \\
\left( C_{-}^{D^{q}} + \lambda^{q} C_{-}^{D^{q-1}} \right) v(t) = f_{2}(t, u(t), v(t)), \quad t \in (0, 1), \\
u(0) = u'(0) = 0, u(1) = av(\xi), \\
v(0) = v'(0) = 0, v(1) = bu(\eta),
\end{array} \right.
\end{align*}
\]
where $\alpha_i > 0$ ($i = 1, 2$) is a parameter; $2 < p, q \leq 3$, $\mathcal{D}^p$, and $\mathcal{D}^q$ are the Caputo fractional derivatives; and the nonlinearity terms $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the given continuous function.

Antiperiodic boundary conditions arise in the mathematical problems of certain physical phenomena and processes. Recently, many scholars paid attention to solvability for fractional differential equations involving antiperiodic boundary conditions (see [17–21]). For example, in [21], the authors considered the nonlinear antiperiodic boundary value problems

\begin{align*}
\left\{ \begin{array}{ll}
\mathcal{D}^q f(t, u(t)), & q \in (2, 3), \quad t \in (0, T), \\
\alpha_i u(0) + \gamma_i u(T) = a_i, & \alpha_i u'(0) + \gamma_i u'(T) = b_i, \quad i = 1, 2,
\end{array} \right.
\end{align*}

where $\mathcal{D}^q$ is the Caputo fractional derivatives of order $q$, $\alpha_i, \gamma_i$ ($i = 1, 2, 3$), $a, b, c \in \mathbb{R}$, $k > 0$, $f$ is a continuous function.

Integral boundary conditions are believed to be more reasonable than the local boundary conditions, which can describe modeling of blood flow, cellular systems, population dynamics, heat transmission, etc. There are a number of results about fractional differential equations and partial differential equations with integral boundary condition; we refer the reader to see [17, 20, 22–43] and the references cited therein. In [20], the authors discussed the following fractional differential equation with integral boundary conditions given by

\begin{align*}
\left\{ \begin{array}{ll}
\mathcal{D}^{\alpha + \beta} x(t) = f(t, x(t)), & t \in (0, 1), \\
x(0) = 0, x(1) = \mu \int_0^1 x(s) ds, \mathcal{D}^\alpha x(0) + \mathcal{D}^\beta x(1) = 0,
\end{array} \right.
\end{align*}

where $\mathcal{D}^\alpha$ and $\mathcal{D}^\beta$ are the Caputo fractional derivatives; $0 < \alpha < 1$, $1 < \beta \leq 2$, $k > 0$, and $\mu > 0$ are real numbers; and $f$ is a given continuous function.

Observing the results of the above literature, an interesting and important question is whether antiperiodic and integral boundary conditions can be unified in a system. If we have unified the conditions, how can we obtain the existence of the solutions? Through a literature search, the sequential fractional differential equation (1) has not been given up to now.

Now in this paper, we shall discuss the problem (1) by using the standard tools of fixed point theory and some new inequalities of norm form.

## 2. Preliminary and Lemmas

In this paper, we provide some necessary definitions and lemmas of the Caputo fractional calculus; for more information, see the books [1–3].
have a unique solution
\[ u(t) = \frac{\Delta_1 - \Delta e^{-kt}}{\Delta_2 \Delta_3} \Phi(h(1), h(\eta)) + \frac{1}{\Delta_2} \psi(h(1), h(\eta)) \]
\[ + \int_0^t e^{-k(t-s)} I_t^{r-1} h(s) ds, \]
(9)

where
\[ \Delta_1 = \alpha + \beta_1 e^{-k} + \gamma_1 \int_0^1 \frac{(\eta-s)^{r-1}}{r} e^{-k} ds, \]
\[ \Delta_2 = \alpha + \beta_1 + \frac{\gamma_1 r}{r(r+1)}, \]
\[ \Delta_3 = -\alpha_2 - \alpha e^{-k} + \gamma_2 \int_0^1 \frac{(\eta-s)^{r-1}}{r} e^{-k} ds, \]
\[ \Delta_4 = -\alpha_2 - \alpha e^{-k} - \gamma_2 \int_0^1 \frac{(\eta-s)^{r-1}}{r} e^{-k} ds. \]

Applying the boundary condition (8) in (11) and (12), we obtain
\[ \Delta_1 A_0 + \Delta_2 A_1 = \Psi(h(1), h(\eta)), \quad \Delta_3 A_0 + \Phi(h(1), h(\eta)) = 0. \]
(13)

A simultaneous solution of equation (13) leads to
\[ A_0 = -\frac{1}{\Delta_3} \Phi(h(1), h(\eta)), \quad A_1 = \frac{\Delta_1}{\Delta_2 \Delta_3} \Phi(h(1), h(\eta)) \]
\[ + \frac{1}{\Delta_2} \psi(h(1), h(\eta)). \]
(14)

Substituting \( A_0 \) and \( A_1 \) to (11), we obtain the desired solution in (9). The converse of the lemma follows by direct computation. The proof is completed.

**Remark 5.** Caputo fractional differentiating (11) with respect to \( t \), we obtain
\[ C^{\alpha} D_t^{\alpha} u(t) = \frac{k A_0}{I(2-q)} \int_0^t (t-s)^{1-q} e^{-ks} ds + \frac{1}{I(2-q)} \int_0^t (t-s)^{1-q} h(s) ds - \frac{k}{I(2-q)} \]
\[ \cdot \int_0^t (t-s)^{1-q} \left( \int_0^1 e^{-k(r-s)} I_t^{r-1} h(m) dm \right) ds, \]
where \( A_0 \) is defined as (14).

Set \( C[0,1] \) is all the continuous functions on \([0,1], C_{g-1}[0,1] = \{ u \in C[0,1] : C^{\alpha} D_t^{\alpha} u \in C[0,1] \} \). Let \( E = (C_{g-1}[0,1], ||\cdot||_{g-1}) \) denotes the Banach space endowed with the norm defined by \( ||u||_{g-1} = ||u|| + ||C^{\alpha} D_t^{\alpha} u|| = \sup_0^{2g-1} |u(t)| + \sup_0^{2g-1} |C^{\alpha} D_t^{\alpha} u(t)| \). For the convenience of the proofs in the next main results, we first give the bounds for integrals arising from the sequel, which are very important for us to establish the existence of solutions for problem (1).

**Lemma 6.** Suppose that \( h \in C([0,1], \mathbb{R}). \) Then, we have

(i) \( |\Phi(h(1), h(\eta))| \leq \frac{|\beta_2| (2 - e^{-k}) / I(q) + (|\gamma_2|/I(q)) (2 - e^{-k}) / I(q) |h| + |\epsilon_2|}{I(q)}. \)

(ii) \( |\Psi(h(1), h(\eta))| \leq \frac{|\beta_2| (1 - e^{-k}) / k I(q) + (|\gamma_2|/k I(q)) (1 - e^{-k}) / k I(q) |h| + |\epsilon_2|}{I(q)}. \)

**Proof.**

(i) Obviously, we have
\[ ||P_t^{r-1} h(s)|| \leq \frac{1}{I(q-1)} \int_0^1 (s - \omega)^{r-2} d\omega ||h|| \]
\[ = \frac{s^q}{I(q)} ||h||, \]
\[ ||P_t^{r-1} h(1)|| \leq \frac{1}{I(q)} ||h||, \]
\[ ||\int_0^1 e^{-k(r-s)} I_t^{r-1} h(m) dm|| \leq \frac{s^q}{I(q)} \int_0^1 e^{-k(r-s)} ||h|| \]
\[ = \frac{s^q}{I(q)} e^{-k} ||h||, \]
\[ \int_0^1 e^{-k(r-s)} I_t^{r-1} h(s) ds \leq \frac{1}{I(q)} \int_0^1 e^{-k(r-s)} ||h|| \]
\[ = \frac{1 - e^{-k}}{k I(q)} ||h||, \]
\[ \int_0^1 \frac{(\eta-s)^{r-1}}{r} I_t^{r-1} h(s) ds \leq \frac{\eta^q}{I(q)} \int_0^1 \frac{(\eta-s)^{r-1}}{r} ||h|| \]
\[ \leq \frac{\eta^q}{I(q)} ||h||. \]
(16)
Finally, we have

$$\left| \int_{0}^{\eta} \frac{(\eta-s)^{-1}}{I(r)} \left( \int_{0}^{t} e^{-k(i-m)} I_{r}^{-1} h(m) \, dm \right) \, ds \right| \leq \frac{\eta^{r} (1 - e^{-k\eta})}{k I(q) I(r + 1)} \|h\|. \tag{17}$$

Hence,

$$|\Phi(h(1), h(\eta))| \leq \left| \beta_2 I_{r}^{-1} h(1) \right| + k \left| \frac{\beta_2}{I(q)} \left( \int_{0}^{t} e^{-k(i-m)} I_{r}^{-1} h(m) \, dm \right) \, ds \right|$$

$$+ \left| \frac{\gamma_{2}}{I(q)} \left( \int_{0}^{t} e^{-k(i-m)} I_{r}^{-1} h(m) \, dm \right) \, ds \right|$$

$$+ k \left| \int_{0}^{\eta} \frac{(\eta-s)^{-1}}{I(r)} \left( \int_{0}^{t} e^{-k(i-m)} I_{r}^{-1} h(m) \, dm \right) \, ds \right| \tag{18}$$

$$+ |e_2| \leq \left\{ \left| \beta_2 \right| + \left| \beta_2 \right| \left( 1 - e^{-k} \right) \right\} \frac{\eta^{r} (1 - e^{-k\eta})}{k I(q) I(r + 1)} \|h\| + |e_2|.$$
Proof. We first define a ball in $E$ as $B_R = \{ u | u \in E, \| u \|_{q-1} \leq R \}$, where

$$R \geq \max \left\{ (3L_3a_{11})^{1/(1-\sigma_1)}, (3L_3a_{22})^{1/(1-\sigma_1)}, 3(3L_3a_{13} + L_4) \right\}. \quad (28)$$

Then, we show that $T : B_R \rightarrow B_{R^*}$. For $\forall u \in B_R$, using Lemma 6 and the condition (H1), we have

$$\| (f(t),u(t),C_{D^{q-1}}u(t)) \| \leq \frac{k}{I(3-q)} \left| \Phi (f(1,u(1),C_{D^{q-1}}u(1))) \right| + \frac{1}{I(3-q)} \left| f(\eta,u(\eta),C_{D^{q-1}}u(\eta)) \right| + \frac{1}{I(3-q)} \left| f(s,u(s),C_{D^{q-1}}u(s)) \right| \leq \frac{kL_3(3-q) + 2}{I(3-q)} \left\{ a_{11} R_{q_1} + a_{12} R_{q_2} + a_{13} \right\} + \frac{k|\eta|}{I(3-q)}. \quad (30)$$

From (29) and (30), we obtain

$$\| f \|_{q-1} \leq L_3 a_{11} R_{q_1} + L_3 a_{12} R_{q_2} + L_4 \leq \frac{1}{3} R + \frac{1}{3} R = R. \quad (31)$$

This means $F : B_R \rightarrow B_{R^*}$. From the formula (19), it is easy to know that operators $F(u)(t),\ C_{D^{q-1}}F(u)(t)$ are continuous on $[0,1]$. Now, we show that operator $F$ is equicontinuous. Set

$$M = \max_{t \in [0,1]} \{ |f(t,u(t),C_{D^{q-1}}u(t))| \}, \ \forall u \in B_R. \quad (32)$$

Let $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$\| (\Phi u)(t_2) - (\Phi u)(t_1) \| \leq \frac{|e^{k(t_2-t)} - e^{k(t_1-t)}|}{|\Delta_3|} \left| \left( f(1,u(1),C_{D^{q-1}}u(1)) \right) \right| + \frac{1}{|\Delta_3|} \left| f(\eta,u(\eta),C_{D^{q-1}}u(\eta)) \right| \cdot \left| f(t_1 - t, u(s), C_{D^{q-1}}u(s)) \right| ds \leq \frac{L_3 M}{|\Delta_3|} \left( e^{k(t_2-t)} - e^{k(t_1-t)} \right) + \frac{M}{|\Delta_3|} \int_{t_1}^{t_2} e^{k(t_2-t)} \left| f(t_1 - t, u(s), C_{D^{q-1}}u(s)) \right| ds \leq \frac{L_3 M}{|\Delta_3|} \left( e^{k(t_2-t)} - e^{k(t_1-t)} \right) + \frac{M}{|\Delta_3|} \int_{t_1}^{t_2} e^{k(t_2-t)} \left[ 2 \left( 1 - e^{-k(t_1-t)} \right) + (e^{-k(t_1-t)} - e^{-k(t_2-t)}) \right], \quad (33)$$

For convenience, we introduce the following symbols:

$$L_1 = \frac{|\beta_1|}{I(q)} (2 - e^{-k}) + \frac{|\gamma_1|}{k I(q)} (2 - e^{-b}), \quad L_2 = \frac{|\beta_2|}{I(q)} (1 - e^{-k}) + \frac{|\gamma_2|}{k I(q)} (1 - e^{-b}), \quad L_3 = \frac{(|\Delta_1| + |\Delta_2|)|L_1| + |\Delta_3| L_2}{|\Delta_2| |\Delta_3|} + \frac{1}{k I(q)} (2 - e^{-k}) L_1 + \frac{k I(q)}{I(q)} L_2,$$

$$L_4 = \frac{(|\Delta_1| + |\Delta_2|)|\eta| + |\Delta_3 e_1|}{|\Delta_2| |\Delta_3|} + \frac{k|\eta|}{k I(q)} (1 - e^{-k}) e_1.$$

(27)
\[
\left| (C^{D_{t_1}^{-1}} F)(u)(t_2) - (C^{D_{t_1}^{-1}} F)(u)(t_1) \right| \\
\leq \frac{k}{\Gamma(2-q)} \left| \Phi(f(1, u(1), C^{D_{t_1}^{-1}} u(1)), f(\eta, u(\eta), C^{D_{t_1}^{-1}} u(\eta))) \right| \times \left| \int_0^{t_1} (t_2 - s)^{1-q} e^{-k s} ds \right| \\
- \int_0^{t_1} (t_1 - s)^{1-q} e^{-k s} ds + \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t_2 - s)^{1-q} e^{-k s} ds \\
+ \frac{k}{\Gamma(2-q)} \int_0^{t_1} (t_2 - s)^{1-q} e^{-k s} ds \\
+ \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t_2 - s)^{1-q} e^{-k s} ds \\
\leq \frac{k L_1 M + k e_3}{\Gamma(2-q)} \left\{ \int_0^{t_1} (t_2 - s)^{1-q} e^{-k s} ds + \int_0^{t_1} (t_1 - s)^{1-q} e^{-k s} ds \right\}.
\]

In (33) and (34), letting \( t_1 \to t_2 \), then,
\[
\| (F u)(t_2) - (F u)(t_1) \|_0 = 0, \| (C^{D_{t_1}^{-1}} F u)(t_2) - (C^{D_{t_1}^{-1}} F u)(t_1) \|_0 = 0.
\]

That is, \( t_1 \to t_2 \),
\[
\| (F u)(t_2) - (F u)(t_2) \|_{q-1} \to 0.
\]

Therefore, \( F(B_R) \) is an equicontinuous set. Furthermore, it is uniformly bounded because of \( F(B_R) \subset B_R \). Applying the Arzelà-Ascoli theorem, we can infer that \( F \) is a completely continuous operator.

Consider \( V = \{ u \in B_R \mid u = \mu F u, 0 < \mu < 1 \} \) and show that \( V \) is bounded. For \( u \in V \), we know \( \| u \|_{q-1} < \| (F u) \|_{q-1} \leq R \). By Lemma 7, problem (1) has at least one solution in \( BR \).

**Theorem 10.** Suppose that \( (H0) \) and \( (H2) \) hold. If \( 3(L_1 a_{23} + L_4) \leq \min \{ 3(L_1 a_{21})^{1/(1-r_1)}, 3(L_3 a_{22})^{1/(1-r_2)} \} \), then problem (1) has at least one solution.

**Proof.** The proof is similar to Theorem 9. We just need to make sure that \( R \) satisfies \( 3(L_1 a_{23} + L_4) \leq R \leq \min \{ 3(L_1 a_{21})^{1/(1-r_1)}, 3(L_3 a_{22})^{1/(1-r_2)} \} \) in \( B_R \).

**Theorem 11.** Suppose that \( (H0) \) and \( (H3) \) hold. If \( L_3(a_{31} + a_{32}) < 1 \), then problem (1) has at least one solution.

**Proof.** The proof is similar to Theorem 9, we omit it.

**Theorem 12.** Suppose that \( (H0) \) and \( (H4) \) hold. If \( \max \{ (3L_1 a_{41})^{1/(1-r_1)}, 3(L_3 a_{43} + L_4) \} \leq (3L_3 a_{42})^{1/(1-r_2)} \), then problem (1) has at least one solution.

**Proof.** The proof is similar to Theorem 10, we omit it.

**Theorem 13.** Suppose that \( (H0) \) and \( (H5) \) hold. If \( L_3(a_{51} + a_{52}) < 1 \), then problem (1) has a unique solution.

**Proof.** Define \( \sup_{t \in [0,1]} |f(t, 0, 0)| = M < \infty \), such that \( r \geq (L_3 M + L_4)/(1 - L_3(a_{51} + a_{52})) \).

First, we show that \( T(B_R) \subset B_R \), where \( B_R = \{ u \in E : \| u \|_{p-1} \leq r \} \). For \( u \in B_R \), by direct calculation, we have
\[
|\Phi f(t) | \leq \frac{|A_1| + |A_2|}{|A_2|} |\phi(f(1, u(1), C^{D_{t_1}^{-1}} u(1))) - f(1, 0, 0)| \\
+ f(1, 0, 0), f(\eta, u(\eta), C^{D_{t_1}^{-1}} u(\eta)) - f(\eta, 0, 0) \\
+ f(\eta, 0, 0) | + \frac{1}{|A_3|} |\Psi(f(1, u(1), C^{D_{t_1}^{-1}} u(1)))) - f(1, 0, 0) + f(1, 0, 0), f(\eta, u(\eta), C^{D_{t_1}^{-1}} u(\eta)) \\
- f(\eta, 0, 0) + f(\eta, 0, 0)| + \int_0^\infty e^{-(s-t)} p^{-1} \\
\times \left| f(s, u(s), C^{D_{t_1}^{-1}} u(s)) - f(s, 0, 0) + f(s, 0, 0) | ds \right| \\
\leq \frac{|A_1| + |A_2|}{|A_2|} \left| (L_1 a_{51} |u| + a_{52} |C^{D_{t_1}^{-1}} u| + M) + |e_3| \right| \\
+ \frac{1}{|A_3|} \left| (L_1 a_{51} |u| + a_{52} |C^{D_{t_1}^{-1}} u| + M) + |e_3| \right| \\
+ \frac{1}{|A_2|} \left| (L_2 a_{51} |u| + a_{52} |C^{D_{t_1}^{-1}} u| + M) + |e_3| \right| \\
\leq \frac{|A_1| + |A_2|}{|A_2|} \left| (L_1 a_{51} |u| + a_{52} |C^{D_{t_1}^{-1}} u| + M) + |e_3| \right| \\
+ \frac{1}{|A_2|} \left| (A_1 |u| + |A_2| |e_3| + |A_3| |e_3|) \right|.
\]

Therefore, \( \frac{(C^{D_{t_1}^{-1}} F u)(t)}{k} \leq 1 \).
Combining (37) and (38), we obtain

\[
\|(|(F_u)|_{a_1} - |(F_v)|_{a_1}) \| \leq L_3(a_{51} + a_{52}) \| u \|_{a_1} + L_3M + L_4 \leq r. \tag{39}
\]

Now, for any \( u, v \in B_r \), we have

\[
|(|(F_u)| - (F_v)|)| \leq |A_1| + |A_2| \Phi(f(1, u_{11}, C^{D-1}u_{11}(1)),
\Phi(1, v_{11}, C^{D-1}v_{11}(1))) + |A_3|\Phi(f(1, u_{12}, C^{D-1}u_{12}(1)),
\Phi(1, v_{12}, C^{D-1}v_{12}(1))) + \left(1 + |A_2|\right)\Psi(f(1, u_{11}, C^{D-1}u_{11}(1)),
\Phi(1, v_{11}, C^{D-1}v_{11}(1)) + |A_3|\Psi(f(1, u_{12}, C^{D-1}u_{12}(1)),
\Phi(1, v_{12}, C^{D-1}v_{12}(1)) + |A_2|\right) \cdot \left(1 + \frac{1}{kT(q)}\right)
\leq \left[ (|A_1| + |A_2|)L_1 + |A_3|L_2 + \frac{1}{kT(q)} \right]
\cdot \left( a_{51} \| u - v \|_{a_1} + a_{52} \| C^{D-1}u - C^{D-1}v \|_{a_1} \right)
\leq \left[ (|A_1| + |A_2|)L_1 + |A_3|L_2 + \frac{1}{kT(q)} \right]
\cdot \left( a_{51} + a_{52} \right) \| u - v \|_{a_1}. \tag{40}
\]

In order to illustrate our main results, we consider the following sequential fractional differential equations:

\[
\begin{align*}
&\left\{ \begin{array}{l}
(C^{D-1.5} + C^{D-0.5}u(t)) = f(t, u(t), C^{D-0.5}u(t)), \quad t \in (0, 1), \\
0.1u(0) + 0.2u(1) - f^2u(0.5) = 0.01, \\
0.2u'(0) + 0.1u'(1) + 2f^2u'(0.5) = 0.02,
\end{array} \right.
\end{align*}
\tag{43}
\]

where \( q = 1.5, k = 1, r = 2, \eta = 0.5, \alpha_1 = 0.1, \beta_1 = 0.2, \gamma_1 = -1, \epsilon_1 = 0.01, \alpha_2 = 0.2, \beta_2 = 0.1, \gamma_1 = 2, \) and \( \epsilon_2 = 0.02. \)

**Example 1.** Let \( f(t, u(t), C^{D-0.5}u(t)) = 0.01u(t)^{0.2} + 0.02(C^{D-0.5}u(t))^{0.1} + 0.15(1 - t)^2. \) So we have

\[
\|f(t, u(t), C^{D-0.5}u(t))\| \leq 0.01\|u(t)\|^{0.2} + 0.02\|C^{D-0.5}u(t)\|^{0.1} + 0.15, \tag{44}
\]

where \( a_{11} = 0.01, a_{12} = 0.02, a_{13} = 0.15, \alpha_1 = 0.2, \) and \( \sigma_3 = 0.1. \)

Theorem 9 implies that problem (43) has at least one solution.

**Example 2.** Let \( f(t, u(t), C^{D-0.5}u(t)) = 0.01u(t)^{1.2} + 0.02(C^{D-0.5}u(t))^{1.1} + 0.15(1 - t)^2. \) By direct calculation, we obtain that

\[
\begin{align*}
\Delta_1 &= 0.093438, \Delta_2 = 0.246808, \Delta_3 \\
&= -0.407603, L_1 = 0.323142, L_2 \\
&= 0.161275, L_3 = 5.785852, L_4 = 0.130729. \tag{45}
\end{align*}
\]

In the meantime, we have

\[
\|f(t, u(t), C^{D-0.5}u(t))\| \leq 0.01\|u(t)\|^{1.2} + 0.02\|C^{D-0.5}u(t)\|^{1.1} + 0.01, \tag{46}
\]

where \( a_{21} = 0.01, a_{22} = 0.22, a_{23} = 0.01, \) \( r_1 = 1.2, \) and \( r_2 = 1.1. \)

So we obtain \( 3(L_1a_{22} + L_4) = 0.565763 < \min \{ (3L_1a_{21})^{1/(1-r_1)}, (3L_3a_{23})^{1/(1-r_2)} \} = 1.111599. \) Theorem 10 implies that problem (43) has at least one solution.

**Example 3.** Let \( f(t, u(t), C^{D-0.5}u(t)) = 0.01u(t) + 0.02C^{D-0.5}u(t) + 0.01(1 - t)^2. \) So we have

\[
\begin{align*}
&\|f(t, u(t), C^{D-0.5}u(t)) - f(t, v(t), C^{D-0.5}v(t))\| \\
&\leq 0.01\|u(t) - v(t)\| + 0.02\|C^{D-0.5}u(t) - C^{D-0.5}v(t)\|. \tag{47}
\end{align*}
\]

where \( a_{31} = 0.01 \) and \( a_{32} = 0.02. \) Combining with the calculation result of Example 2, we can obtain that \( L_3(a_{31} + a_{32}) = 0.173576 < 1. \) Hence, Theorem 13 implies that problem (43) has a unique solution.
4. Conclusions

We have obtained some existence results for new sequential fractional differential equations by using some nonlinear growth conditions

$$|f(t, u(v))| \leq b_{11}|u|^{\theta_1} + b_{12}|v|^{\theta_2} + b_{13}, \quad \theta_1, \theta_2 \in (0, +\infty),$$

which is different from the existing linear condition. Obviously, these results are easy to verify and apply (see Example 1 and Example 3).

On the other hand, we note that our results contain some special types of results by fixing the parameters in the given problem (1). For instance, let $a_1 = 1$, $b_1 = 1$, $y_1 = 0$, $a_2 = 1$, $b_2 = 1$, and $y_2 = 0$, then the results of this paper are the following sequential fractional differential equations with the boundary value conditions of the form $u(0) + u(1) = e_1$ and $u'(0) + u'(1) = e_2$. Further, letting $a_1 = 0$, $b_1 = 1$, $y_1 = -1$, $e_1 = 0$, $a_2 = 0$, $b_2 = 1$, $y_2 = -1$, and $e_2 = 0$, we obtain the results for the boundary conditions $u(1) = I^\gamma u(\eta)$ and $u'(1) = I^\epsilon u'(\eta)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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