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Research Article

Nikol'skii–Type Inequalities for Trigonometric Polynomials for Lorentz–Zygmund Spaces

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Nikol'skii–type inequalities, that is inequalities between different metrics of trigonometric polynomials on the torus $\boldsymbol{T}^d$ for the Lorentz–Zygmund spaces, are obtained. The results of previous paper "Nikol'skii inequalities for Lorentz–Zygmund spaces" are extended. Applications to approximation spaces in Lorentz–Zygmund spaces and to Besov spaces are given.

1. Introduction

The classical Nikol'skii (or Jackson–Nikol'skii) inequality for the trigonometric polynomials T_n on [0,1] of degree at most n can be written as [1, 2]

$$\left\|T_n\right\|_p \prec n^{((1/q)-(1/p))}\left\|T_n\right\|_q, \tag{1}$$

where $1 \leq q < p \leq \infty$ and $\|*\|_p$ is the usual norm on the Lebesgue spaces L_p. For $p = \infty$, this estimate has been proved by Jackson [3]. The proofs in [1–3] are based on Bernstein's inequality.

Inequalities between different (quasi-)norms of the same function are known as Nikol'skii–type (or Jackson–Nikol'skii–type) inequalities. They play a crucial role in many areas of mathematics, e.g., theory of approximation, theory of functions of several variables, and functional analysis (embedding theorems for Besov spaces).

Nessel and Wilmes [4] extended inequality (1) for $0 < q < p \leq \infty$. They also observed that in inequality (1) one may in fact take into account the spectrum of the function involved for $q \leq 2$ or certain gaps in the spectrum for $q > 2$. Sherstneva [5] extended inequality (1) in the Lorentz spaces $L_{p,b}$ and showed that they are exact relative to the order n. Moreover, she investigated the limiting case when Lorentz spaces on both sides have the same value of the main parameter. She proved that if $0 < p < \infty$ and $0 < b < c \leq \infty$, then

$$\left\|T_n\right\|_{p,b} \prec (1+\ln n)^{((1/b)-(1/c))}\left\|T_n\right\|_{p,c}. \tag{2}$$

Some Nikol'skii–type inequalities for the Lorentz–Zygmund spaces are considered in [6–8] and for the generalized Lorentz space in [9]. In other investigations, different sets of functions, domains, and measures were explored. For further information about these results and applications, we refer to [1–19] and references therein.

In [19], the results of [5] were improved in two directions. First, the functions T_n of the form $\sum_{k=1}^{n} c_k \varphi_k$ were considered, where $\{\varphi_k\}$ is an orthonormal system in $L_2(\mathrm{M})$ uniformly bounded in $L_\infty(\mathrm{M})$ (with $\mu(\mathrm{M}) < \infty$). No assumptions about smoothness of φ_k were made. Secondly, the inequalities (1) and (2) were extended to the Lorentz–Zygmund spaces. However, in [19], only the case $0 < q \leq 2$ is obtained.

The principal aim of this paper is to extend the results of [19] for the case $2 \leq q < \infty$ for the trigonometric polynomials. The technique we apply relies on the observation that the power of a trigonometric polynomial is also a trigonometric polynomial [4, 10, 17, 18]. This paper is organized as follows. Section 2 contains necessary notations and definitions. Main results of this contribution are Theorems 4, 6, 8, and 9. They are formulated and proved in Section 3. Note that Theorems 8, and 9 deal with the limiting case. In Section 4, we reformulate Theorems 4, 6, 8, and 9 for trigonometric polynomials of degree at most n. In Section 5, we give some applications to embeddings between approximation spaces in Lorentz–Zygmund spaces and between Besov spaces.

2. Preliminaries

We write $X \subset Y$ for two quasi-normed spaces X and Y to indicate that X is continuously embedded in Y. The notation $X \cong Y$ means that $X \subset Y$ and $Y \subset X$. If f and g are positive functions, we write $f \prec g$ if $f \leq Cg$, where the constant C is independent on all significant quantities. We put $f \approx g$ if $f \prec g$ and $g \prec f$. We adopt the convention that $(1/\infty) = 0$. We use the abbreviation $l(x) = 1 + |\ln x|$ and $ll(x) = l(l(x))$, $0 < x < \infty$.

Throughout the paper, let $\boldsymbol{T}^d = \{x = (x_1, \dots, x_d) \in \boldsymbol{R}^d : 0 \leq x_i \leq 1\}(d \in \mathbf{N})$ be the d-dimensional torus with Lebesgue measure. We consider (equivalence classes of) complex-valued measurable functions f on $\boldsymbol{T}^d$ and bounded complex-valued sequences $\{c_k\}$. As usual [20–23], $f^*(t)(t > 0)$ and $\{c_n^*\}(n \in \mathbf{N})$ are the nonincreasing rearrangements of a function f and of a sequence $\{c_k\}$, respectively. Because the measure of the torus $\boldsymbol{T}^d$ equals 1, $f^*(t) = 0$ if $t > 1$.

Definition 1. Let $0 < p, b \leq \infty$ and $-\infty < \alpha < \infty$. The Lorentz–Zygmund space $L_{p,b;\alpha} \equiv L_{p,b;\alpha}(\boldsymbol{T}^d)$ can be defined as follows:

$$L_{p,b;\alpha} := \left\{ f : \|f\|_{p,b;\alpha} := \left\| t^{\frac{1}{p}-\frac{1}{b}} l^{\alpha}(t) f^*(t) \right\|_b < \infty \right\}, \tag{3}$$

where $\|*\|_b$ is the usual (quasi-)norm on the Lebesgue space $L_b(0,1)$. Similarly,

$$l_{p,b;\alpha} := \left\{ \{c_k\} : \|\{c_k\}\|_{p,b;\alpha} := \left\| n^{\frac{1}{p}-\frac{1}{b}} l^{\alpha}(n) c_n^* \right\|_b < \infty \right\}, \tag{4}$$

where $\|*\|_b$ is the usual (quasi-)norm on the Lebesgue sequence space l_b.

We use the same notation $\|*\|_{p,b;\alpha}$ for both (quasi-)norms. If $\alpha = 0$, then $L_{p,b;\alpha}$ coincides with the Lorentz space $L_{p,b}$. If in addition $p = b$, then the space becomes the Lebesgue space L_p. The same is valid for sequence spaces, too. Note that $L_{\infty,b;\alpha}$ is not trivial if and only if either $\alpha < -1/b$ or $b = \infty$ and $\alpha = 0$. For the detailed information about these spaces we refer to [21–24]. It is known that all spaces $L_{p,b;\alpha}$ are complete. Moreover [23, Theorem 7.4], on $L_{p,b;\alpha}$ there exists a norm, equivalent to $\|*\|_{p,b;\alpha}$, iff $(p, b; \alpha) \in \Omega$, where

$$\begin{aligned} \Omega := \Big\{ & (p, b; \alpha) \in [1, \infty] \times [1, \infty] \times (-\infty, \infty) \,\big|\, p = b = 1, \\ & \alpha \geq 0; \text{ or } 1 < p < \infty; \\ & \quad \text{or } p = \infty, \alpha < -\frac{1}{b}; \text{ or } p = b = \infty, \alpha = 0 \Big\}. \end{aligned} \tag{5}$$

In this case, it is a Banach function space. For details see [20, 23]. In addition, let

$$\begin{aligned} \Gamma := \Big\{ & (p, b; \alpha) \,\big|\, 0 < p \leq \infty, 0 < b < \min(1, p), -\infty < \alpha < \infty; \\ & \text{or } 0 < b = p < 1, \ \alpha \geq 0; \text{ or } p = \infty, 0 < b < 1, \alpha < -\frac{1}{b} \Big\}. \end{aligned} \tag{6}$$

The following statement must surely be known (see Acknowledgments).

Lemma 2. *Let $(p, b; \alpha) \in \Gamma$. Then, the space $L_{p,b;\alpha}$ is b-normed, that is, there exists $C = C(p, b; \alpha) > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} g_n \right\|_{p,b;\alpha} < C \left(\sum_{n=0}^{\infty} \|g_n\|_{p,b;\alpha}^b \right)^{1/b}, \tag{7}$$

for each sequence $\{g_n\} \subset L_{p,b;\alpha} (n = 0, 1, 2, \dots)$ such that the series $\sum_{n=0}^{\infty} g_n$ converse in $L_{p,b;\alpha}$.

Proof. It is enough to prove that (7) holds for finite sums $\sum_{n=0}^{N} \dots$. Because $b < 1$, for $x \in \boldsymbol{T}^d$, we have

$$\left(\left| \sum_{n=0}^{N} g_n(x) \right| \right)^b \leq \left(\sum_{n=0}^{N} |g_n(x)| \right)^b \leq \sum_{n=0}^{N} |g_n(x)|^b. \tag{8}$$

Let f^{**} be as usual the maximal function of f^* given by $f^{**}(s) = (1/s)\int_0^s f^*(t)dt$. If $b < p$ or $b = p$, $\alpha \geq 0$, then [21, Theorem 9.1 and Theorem 9.3 (i)] $L_{p,b;\alpha} \subset L_b$. Hence, for each function $g \in L_{p,b;\alpha}$, it holds $|g|^b \in L_1$ and therefore $\left(|g|^b\right)^{**}$ is well-defined. In virtue of the monotonicity and subadditivity of f^{**} (see [20, Chapter 2, Proposition 3.2 and Theorem 3.4]), we conclude:

$$\left(\left| \sum_{n=0}^{N} g_n \right|^b \right)^{**} (s) \leq \left(\sum_{n=0}^{N} |g_n|^b \right)^{**} (s) \leq \sum_{n=0}^{N} \left(|g_n|^b \right)^{**} (s), \tag{9}$$

which together with [20, Chapter 2, (1.20)] yields that

$$\int_0^s \left(\left(\sum_{n=0}^{N} g_n \right)^* (t) \right)^b dt \leq \sum_{n=0}^{N} \int_0^s \left(g_n{}^*(t) \right)^b dt. \tag{10}$$

Applying now Hardy's lemma [20, Chapter 2, Proposition 3.6], we infer that

$$\int_0^1 \left(h(t) \left(\sum_{n=0}^{N} g_n \right)^* (t) \right)^b dt \leq \sum_{n=0}^{N} \int_0^1 \left(h(t) g_n^*(t) \right)^b dt = \sum_{n=0}^{N} \|h g_n^*\|_b^b, \tag{11}$$

for any nonnegative decreasing function h. Let $0 < p \leq \infty$, $0 < b < \min(1, p)$, $-\infty < \alpha < \infty$ or $0 < b = p < 1$, $\alpha > 0$. In this case, we can find a decreasing function h_0 such that $h_0(t) \approx t^{(1/p)-(1/b)} l^{\alpha}(t)$ (see, e.g., [25]). Therefore, by (11),

$$\left\| \sum_{n=0}^{N} g_n \right\|_{p,b;\alpha}^b \approx \left\| h_0(t) \left(\sum_{n=0}^{N} g_n \right)^* (t) \right\|_b^b \leq \sum_{n=0}^{N} \|h_0 g_n^*\|_b^b \approx \sum_{n=0}^{N} \|g_n\|_{p,b;\alpha}^b. \tag{12}$$

The case $0 < b = p < 1$, $\alpha = 0$ is trivial. This completes the proof. □

Lemma 3. *Let $0 < p, b \leq \infty$, $-\infty < \alpha < \infty$, $f \in L_{p,b;\alpha}$, and $k \in \boldsymbol{N}$. Then,*

$$\left\{ \|f\|_{p,b;\alpha} \right\}^k = \left\| |f|^k \right\|_{p/k, b/k; \alpha k}. \tag{13}$$

3. Main Results

Let $\Lambda \subset \boldsymbol{Z}^d$ be a finite set of lattice points. We denote the number of elements of the set Λ by $\#\Lambda$. Everywhere below

$T_\Lambda(x) = \sum_{m\in\Lambda} c_m e^{2\pi imx}\left(x\in \boldsymbol{T}^d, \{c_m\} \subset \boldsymbol{C}, c_m \neq 0\right)$. So, for the m th $\left(m \in \boldsymbol{Z}^d\right)$ Fourier coefficient of the function T_Λ it holds $\widehat{T}_\Lambda(m) = c_m$. Hence, $\operatorname{supp}\widehat{T}_\Lambda := \left\{m \in \boldsymbol{Z}^d : \widehat{T}_\Lambda(m) \neq 0\right\} = \Lambda$.

Our goal is to obtain Nikol'skii–type inequalities of the form

$$\|T_\Lambda\|_{p,b;\alpha} \le CG(\Lambda; p, b, \alpha, q, c, \beta)\|T_\Lambda\|_{q,c;\beta}, \tag{14}$$

where the constant C does not depend on Λ. Or, writing it shorter

$$\|T_\Lambda\|_{p,b;\alpha} \prec G(\Lambda; p, b, \alpha, q, c, \beta)\|T_\Lambda\|_{q,c;\beta}. \tag{15}$$

Obviously, only the situations $L_{q,c;\beta} \not\subset L_{p,b;\alpha}$ are of interest, otherwise $G = 1$.

For any triple $(q, c; \beta) \in (0, \infty] \times (0, \infty] \times (-\infty, \infty)$, we define a corresponding natural number $\rho = \rho(q, c; \beta)$ in the following way. Let $M := (0, 2) \times (0, \infty] \times (-\infty, \infty) \cup (2, 2; 0)$. For all triples $(q, c; \beta) \in M$, we set $\rho = 1$. For each other triple $(q, c; \beta)$, we define ρ as the smallest integer such that $((q/\rho), (c/\rho), \rho\beta) \in M$. We set $N(T_\Lambda, \rho) := \#\operatorname{supp}(T_\Lambda^\rho)^\wedge$. Note that $N(T_\Lambda, 1) = \#\Lambda$.

Theorem 4. *Let $0 < q < \infty$, $0 < c \le \infty$, $-\infty < \beta < \infty$, and $\rho = \rho(q, c; \beta)$. For the triple (p, b, α), we assume that either $q < p < \infty$, $0 < b \le \infty$, $-\infty < \alpha < \infty$, or $p = b = \infty$, $\alpha = 0$. Then*

$$\|T_\Lambda\|_{p,b;\alpha} \prec \left(N(T_\Lambda, \rho)\right)^{((1/q)-(1/p))}\left(l(N(T_\Lambda, \rho))\right)^{(\alpha-\beta)}\|T_\Lambda\|_{q,c;\beta}. \tag{16}$$

Proof. For the case $\rho = 1$, from [19, Theorem 1] we immediately have

$$\|T_\Lambda\|_{p,b;\alpha} \prec \#\Lambda^{((1/q)-(1/p))}(l(\#\Lambda))^{(\alpha-\beta)}\|T_\Lambda\|_{q,c;\beta}. \tag{17}$$

Consider the case $\rho > 1$. Note that for the triple $((q/\rho), (c/\rho), \rho\beta)$ the corresponding ρ-value is 1. Hence, due to (13) and (17) we have

$$\begin{aligned}\left\{\|T_\Lambda\|_{p,b;\alpha}\right\}^\rho &= \|T_\Lambda^\rho\|_{p/\rho,b/\rho;\alpha\rho} \prec \left(N(T_\Lambda, \rho)\right)^{\rho((1/q)-(1/p))} \\ &\quad \left(l(N(T_\Lambda, \rho))\right)^{\rho(\alpha-\beta)}\|T_\Lambda^\rho\|_{q/\rho,c/\rho;\beta\rho} \\ &= \left\{N(T_\Lambda, \rho)^{((1/q)-(1/p))}\left(l(N(T_\Lambda, \rho))\right)^{(\alpha-\beta)}\|T_\Lambda\|_{q,c;\beta}\right\}^\rho.\end{aligned} \tag{18}$$

This completes the proof. □

Remark 5. Theorem 4 covers Theorems 1 and 2 in [4]. See also [10].

Theorems 6, 8, and 9 can be proved following similar approach as proof of Theorem 4, using [19, Theorems 2, 3, and 4] respectively.

Theorem 6. *Let $0 < q < \infty$, $0 < b$, $c \le \infty$, $-\infty < \beta < \infty, -\infty < \alpha < -1/b$, and $\rho = \rho(q, c; \beta)$. Then*

$$\|T_\Lambda\|_{\infty,b;\alpha} \prec \left(N(T_\Lambda, \rho)\right)^{1/q}\left(l(N(T_\Lambda, \rho))\right)^{(\alpha+(1/b)-\beta)}\|T_\Lambda\|_{q,c;\beta}. \tag{19}$$

Remark 7. According to [21, Theorem 9.3], the conditions of Theorems 4 and 6 imply that $L_{q,c;\beta} \supset L_{p,b;\alpha}$.

Both Theorems 4 and 6 deal with the case $q < p$. The next two theorems examine the limiting case $q = p$.

Theorem 8. *Let $0 < q < \infty$, $0 < b \le c \le \infty$, $-\infty < \alpha, \beta < \infty$, and $\rho = \rho(q, c; \beta)$.*

(i) If $\alpha + (1/b) > \beta + (1/c)$, then,

$$\|T_\Lambda\|_{q,b;\alpha} \prec \left(l(N(T_\Lambda, \rho))\right)^{(\alpha+(1/b)-\beta-(1/c))}\|T_\Lambda\|_{q,c;\beta}. \tag{20}$$

(ii) If $\alpha + (1/b) = \beta + (1/c)$, then

$$\|T_\Lambda\|_{q,b;\alpha} \prec \left(ll(N(T_\Lambda, \rho))\right)^{((1/b)-(1/c))}\|T_\Lambda\|_{q,c;\beta}. \tag{21}$$

Theorem 9. *Let $0 < q < \infty, 0 < c \le b \le \infty, -\infty < \beta < \alpha < \infty$, and $\rho = \rho(q, c; \beta)$. Then*

$$\|T_\Lambda\|_{q,b;\alpha} \prec \left(l(N(T_\Lambda, \rho))\right)^{(\alpha-\beta)}\|T_\Lambda\|_{q,c;\beta}. \tag{22}$$

Remark 10. According to [22, Proposition 3.1], Theorem 8 (i) and Theorem 9 deal with comparable $\left(L_{q,b;\alpha} \subset L_{q,c;\beta}\right)$ as well as incomparable pairs of Lorentz–Zygmund spaces. Theorem 8 (ii) only handles incomparable pairs.

4. Corollaries for Trigonometric Polynomials of Degree at Most n

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n ($n = 1, 2, \ldots$), i. e.

$$\mathcal{T}_n := \left\{T_n(x) = \sum_{|m|\le n} c_m e^{2\pi imx} : c_m \in \boldsymbol{C}, m \in \boldsymbol{Z}^d, x \in \boldsymbol{T}^d\right\}, \tag{23}$$

where $|m| = |m_1| + \cdots + |m_d|$.

Corollary 11. *Let $0 < q < \infty$, $0 < c \le \infty$, and $-\infty < \beta < \infty$. For the triple (p, b, α), we assume that either $q < p < \infty$, $0 < b \le \infty$, $-\infty < \alpha < \infty$, or $p = b = \infty$, $\alpha = 0$. Then for all $T_n \in \mathcal{T}_n$.*

$$\|T_n\|_{p,b;\alpha} \prec n^{d((1/q)-(1/p))}(l(n))^{(\alpha-\beta)}\|T_n\|_{q,c;\beta}. \tag{24}$$

Proof. Because ρ in Theorem 4 depends only on q, c, and β, we have

$$N(T_n, \rho) = \#\operatorname{supp}(T_n^\rho)^\wedge \le (2\rho n + 1)^d \prec n^d. \tag{25}$$

For $\nu > 0$, $-\infty < \gamma < \infty$, the sequence $k^\nu(l(k))^\gamma$ is either increasing or equivalent to an increasing sequence (see, e.g., [25]). In both cases, because of (25)

$$(N(T_n,\rho))^{((1/q)-(1/p))}(l(N(T_n,\rho)))^{(\alpha-\beta)} \prec n^{d((1/q)-(1/p))}(l(n))^{(\alpha-\beta)}. \quad (26)$$

The last estimate and Theorem 4 imply (24). □

Analogously, Theorems 6, 8, and 9 imply next three corollaries.

Corollary 12. *Let $0<q<\infty$, $0<b,c\le\infty$, $-\infty<\beta<\infty$, and $-\infty<\alpha<-1/b$. Then for all $T_n\in\mathcal{T}_n$.*

$$\|T_n\|_{\infty,b;\alpha} \prec n^{d/q}(l(n))^{(\alpha+(1/b)-\beta)}\|T_n\|_{q,c;\beta}. \quad (27)$$

Corollary 13. *Let $0<q<\infty$, $0<b\le c\le\infty$, and $-\infty<\alpha,\beta<\infty$.*

(i) If $\alpha+(1/b)>\beta+(1/c)$, then for all $T_n\in\mathcal{T}_n$,

$$\|T_n\|_{q,b;\alpha} \prec (l(n))^{(\alpha+(1/b)-\beta-(1/c))}\|T_n\|_{q,c;\beta}. \quad (28)$$

(ii) If $\alpha+(1/b)=\beta+(1/c)$, then for all $T_n\in\mathcal{T}_n$,

$$\|T_n\|_{q,b;\alpha} \prec (ll(n))^{((1/b)-(1/c))}\|T_n\|_{q,c;\beta}. \quad (29)$$

Corollary 14. *Let $0<q<\infty$, $0<c\le b\le\infty$, and $-\infty<\beta<\alpha<\infty$. Then for all $T_n\in\mathcal{T}_n$.*

$$\|T_n\|_{q,b;\alpha} \prec (l(n))^{(\alpha-\beta)}\|T_n\|_{q,c;\beta}. \quad (30)$$

Remark 15. Corollary 13(i) covers [7, (3.16)], [8, Lemma 5.4], and [9, Corollary 2]. Corollary 14 covers [7, Lemma 3.4]. In [26, Lemma 3] a variant of (29) was obtained. Inequality given in [26, Lemma 2] provides a limiting counterpart (i.e., $\alpha=\beta$) of (30). Both Lemmas in [26] involve generalized Lorentz–Zygmund spaces with two iterations of logarithm.

5. Applications

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n described above. The sequence $(\mathcal{T}_n)$ $(n=1,2,\ldots)$ allows construction of an approximation family in all Lorentz–Zygmund spaces, which produces approximation spaces. In Section 5.1 we start with some necessary definitions and auxiliary results dealing with approximation spaces. In Section 5.2 we present some corollaries of the statements from Section 4 dealing with embeddings of the approximation spaces $(L_{p,b;\alpha})_u^{(\sigma,\gamma)}$ into Lorentz–Zygmund spaces and between these approximation spaces. In Sections 5.3 and 5.4 these corollaries will be reformulated in terms of Besov spaces.

5.1. Basic Approximation Constructions. A quasi-norm on a quasi-Banach space X is denoted by $\|*\|_X$. Let X be a quasi-Banach space and $C_X(\ge 1)$ be the constant from the triangle inequality in the space X. A sequence $(G_n)(n=0,1,2,\ldots)$ of subsets of X is called an approximation family in X if the following conditions are satisfied:

(1) $\{0\}=G_0\subset G_1\subset G_2\subset\ldots\subset X$,

(2) $\lambda G_n\subseteq G_n$ for all scalars λ and $n=1,2,\ldots$,

(3) $G_m+G_n\subseteq G_{m+n}$ for $m,n=1,2,\ldots$.

For $f\in X$ and $n=1,2,\ldots$, the nth approximation number is defined by

$$E_n(f)\equiv E_n(f;X)=\inf\{\|f-g\|_X : g\in G_{n-1}\}. \quad (31)$$

Let $\sigma\ge 0$, $0<u\le\infty$, and $-\infty<\gamma<\infty$. The approximation space $X_u^{(\sigma,\gamma)}\equiv(X,G_n)_u^{(\sigma,\gamma)}$ is formed by all those $f\in X$, for which $\{E_n(f)\}\in l_{(1/\sigma),u;\gamma}$ with the quasi-norm $\|f\|_{X_u^{(\sigma,\gamma)}}:=\|\{E_n(f)\}\|_{(1/\sigma),u;\gamma}$. Note that $X_u^{(0,\gamma)}$ coincides with X if $\gamma<-1/u$. Approximation spaces were investigated in many works. For more information about such spaces, we refer to [8, 27–30] and references therein. We will use some common statements from the approximation theory. We begin with the two representation theorems.

Theorem 16 [8, 27, 28]. *Let $\sigma>0$. An element $f\in X$ belongs to $X_u^{(\sigma,\gamma)}$ if and only if there is a representation*

$$f=\sum_{n=0}^{\infty} g_n, (\textit{convergence} \quad \textit{in} \quad X), g_n\in G_{2^n}, \quad (32)$$

with

$$\sum_{n=0}^{\infty}(2^{n\sigma}(1+n)^{\gamma}\|g_n\|_X)^u<\infty. \quad (33)$$

Moreover,

$$\|f\|_{X_u^{(\sigma,\gamma)}}^{rep}:=\inf\left(\sum_{n=0}^{\infty}(2^{n\sigma}(1+n)^{\gamma}\|g_n\|_X)^u\right)^{1/u}, \quad (34)$$

where the infimum is taken over all possible representations (32), defines an equivalent quasi-norm on $X_u^{(\sigma,\gamma)}$ with equivalence constants depending only on σ,u,γ, and C_X. The usual modification shall be made when $u=\infty$.

Theorem 17 ([29], Theorem 1). *Let $0<u\le\infty$ and $\gamma>-1/u$. Denote $\mu_n=2^{2^n}$ $(n=0,1,2,\ldots)$. An element $f\in X$ belongs to $X_u^{(0,\gamma)}$ if and only if there is a representation*

$$f=\sum_{n=0}^{\infty} g_n, (\textit{convergence} \quad \textit{in} \quad X), g_n\in G_{\mu_n}, \quad (35)$$

with

$$\sum_{n=0}^{\infty}\left(2^{n(\gamma+(1/u))}\|g_n\|_X\right)^u<\infty. \quad (36)$$

Moreover,

$$\|f\|_{X_u^{(0,\gamma)}}^{rep}:=\inf\left(\sum_{n=0}^{\infty}\left(2^{n(\gamma+(1/u))}\|g_n\|_X\right)^u\right)^{1/u}, \quad (37)$$

where the infimum is taken over all possible representations (35), defines an equivalent quasi-norm on $X_u^{(0,\gamma)}$ with equivalence constants depending only on u, γ, and C_X. The usual modification shall be made when $u = \infty$.

These representation theorems are useful to prove the following two lemmas.

Lemma 18 ([8, 30]). *Let X, Y be quasi-Banach spaces that are continuously embedded in a Hausdorff topological vector space. Let (G_n) be an approximation family such that $G_n \subseteq X \cap Y (n = 1, 2, \ldots)$. Assume that there are constants c, $\beta > 0$ such that*

$$\|g\|_Y \leq c(l(n))^{\beta}\|g\|_X, \quad g \in G_n, \quad n = 1, 2, \ldots. \tag{38}$$

Then, for $0 < u \leq \infty$ and $\gamma > -1/u$, we have

$$X_u^{(0,\gamma+\beta)} \subset Y_u^{(0,\gamma)}. \tag{39}$$

Lemma 19 (Cf. [8, Lemma 2.1], [28, Theorem 4.3], and [27, Theorem 3.4]). *Let X,Y be quasi-Banach spaces which are continuously embedded in a Hausdorff topological vector space. Let (G_n) be an approximation family such that $G_n \subseteq X \cap Y (n = 1, 2, \ldots)$. Assume that there are constants $c > 0$, $\delta \geq 0$, and $-\infty < \beta < \infty$ such that*

$$\|g\|_Y \leq cn^{\delta}(l(n))^{\beta}\|g\|_X, \quad g \in G_n, \quad n = 1, 2, \ldots. \tag{40}$$

Then, for $\sigma > 0$, $0 < u \leq \infty$, and $-\infty < \gamma < \infty$, we have

$$X_u^{(\sigma+\delta,\gamma+\beta)} \subset Y_u^{(\sigma,\gamma)}. \tag{41}$$

We omit the proof since it can be carried out as in [8, Lemma 2.1]. We will also use some interpolation formulae for approximation spaces. Let $0 \leq \theta \leq 1$, $0 < b \leq \infty$, and $-\infty < \alpha < \infty$. By $(^*,^*)_{\theta,b}$, we denote the classical real interpolation functor [20, 31] and by $(^*,^*)_{\theta,b;\alpha}$ the real interpolation functor involving logarithmic factor for ordered couples with integration over (0,1) (see, e.g., [30, 32, 33]). Note that $(^*,^*)_{\theta,b} = (^*,^*)_{\theta,b;0}$.

Lemma 20 ([30, Proposition 2.7]). *Suppose that $\sigma > 0$, $0 < u$, $q \leq \infty$, $0 < \theta < 1$, and $-\infty < \gamma < \infty$, then*

$$\left(X, X_u^{(\sigma,0)}\right)_{\theta,q;\gamma} \cong X_q^{(\theta\sigma,\gamma)}. \tag{42}$$

The reiteration relation (see, e.g., [32]) leads to the following result.

Corollary 21. *Suppose that $\sigma > 0$, $0 < u$, $q \leq \infty$, $0 < \theta < 1$, and $-\infty < \gamma, \delta < \infty$, then*

$$\left(X, X_u^{(\sigma,\delta)}\right)_{\theta,q;\gamma} \cong X_q^{(\theta\sigma,\theta\delta+\gamma)}. \tag{43}$$

5.2. Approximation Spaces in Lorentz–Zygmund Spaces. Everywhere below, we consider the following sequence of subsets: $G_0 = \{0\}$, $G_n = \mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n $(n = 1, 2, \ldots)$. It builds an approximation family in all nontrivial Lorentz–Zygmund spaces. Now we will apply Corollaries 11–14 to the approximation spaces $\left(L_{p,b;\alpha}\right)_u^{(\sigma,\gamma)}$. First, we will investigate embeddings of these approximation spaces into Lorentz–Zygmund spaces. Next, we will investigate embeddings between different approximation spaces $\left(L_{*,*;*}\right)_*^{(\sigma,*)}$ for $\sigma > 0$ and between different limiting approximation spaces $\left(L_{*,*;*}\right)_*^{(0,*)}$. From Corollary 11, we get the following result.

Corollary 22. *Let $0 < q < p < \infty$, $0 < b$, $c \leq \infty$, $-\infty < \alpha$, $\beta < \infty$. Then*

$$\left(L_{q,c;\beta}\right)_b^{(d((1/q)-(1/p)),\alpha-\beta)} \subset L_{p,b;\alpha}. \tag{44}$$

Proof. By Theorem 16, given any $f \in \left(L_{q,c;\beta}\right)_1^{((d/q),-\beta)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$, (convergence in $L_{q,c;\beta}$), $g_n \in \mathcal{T}_{2^n}$ such that

$$\sum_{n=0}^{\infty} 2^{nd/q}(1+n)^{-\beta}\|g_n\|_{q,c;\beta} \prec \|f\|_{\left(L_{q,c;\beta}\right)_1^{((d/q),-\beta)}}. \tag{45}$$

Because all $g_n \in \mathcal{T}_{2^n} \subset L_{q,c;\beta} \bigcap L_{\infty}$, it is not hard to show that the series $\sum_{n=0}^{\infty} g_n$ converges to f in L_{∞}. Since L_{∞} is a Banach space, using Corollary 11 and (45), we derive that

$$\|f\|_{\infty} \leq \sum_{n=0}^{\infty} \|g_n\|_{\infty} \prec \sum_{n=0}^{\infty} 2^{nd/q}(1+n)^{-\beta}\|g_n\|_{q,c;\beta} \prec \|f\|_{\left(L_{q,c;\beta}\right)_1^{((d/q),-\beta)}}. \tag{46}$$

Therefore,

$$\left(L_{q,c;\beta}\right)_1^{((d/q),-\beta)} \subset L_{\infty}. \tag{47}$$

Since $0 < q < p < \infty$, we can define θ and γ by formulae

$$\frac{1}{p} = \frac{1-\theta}{q} \quad \text{and} \quad \alpha = (1-\theta)\beta + \gamma. \tag{48}$$

From Corollary 21, we have

$$\left(L_{q,c;\beta}, \left(L_{q,c;\beta}\right)_1^{((d/q),-\beta)}\right)_{\theta,b;\gamma} \cong \left(L_{q,c;\beta}\right)_b^{(d((1/q)-(1/p)),\alpha-\beta)}. \tag{49}$$

Furthermore, it is known [32, Theorem 7] that

$$\left(L_{q,c;\beta}, L_{\infty}\right)_{\theta,b;\gamma} \cong L_{p,b;\alpha}. \tag{50}$$

Using now (47) and [32, Theorem 2], we obtain (44). ☐

From Corollary 12, we immediately get the following result.

Corollary 23. *Let $0 < q < \infty$, $0 < c \leq \infty$, $-\infty < \beta < \infty$, $0 < b \leq \infty$, and $\alpha < -1/b$. Then*

$$\left(L_{q,c;\beta}\right)_{\min(b,1)}^{((d/q),\alpha+(1/b)-\beta)} \subset L_{\infty,b;\alpha}. \tag{51}$$

Proof. Consider the case $1 \leq b \leq \infty$. By Theorem 16, given any $f \in \left(L_{q,c;\beta}\right)_1^{((d/q),\alpha+(1/b)-\beta)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$, (convergence in $L_{q,c;\beta}$), $g_n \in \mathcal{T}_{2^n}$ such that

$$\sum_{n=0}^{\infty} 2^{nd/q}(1+n)^{(\alpha+(1/b)-\beta)} \|g_n\|_{q,c;\beta} \prec \|f\|_{(L_{q,c;\beta})_1^{((d/q),\alpha+(1/b)-\beta)}}. \quad (52)$$

Because all $g_n \in \mathcal{T}_{2^n} \subset L_{q,c;\beta} \bigcap L_{\infty,b;\alpha}$, it is not hard to show that the series $\sum_{n=0}^{\infty} g_n$ converges to f in $L_{\infty,b;\alpha}$. Since $(\infty, b, \alpha) \in \Omega$, using Corollary 12 and (52), we derive that

$$\begin{aligned}\|f\|_{\infty,b;\alpha} &\prec \sum_{n=0}^{\infty} \|g_n\|_{\infty,b;\alpha} \prec \sum_{n=0}^{\infty} 2^{nd/q}(1+n)^{(\alpha+(1/b)-\beta)} \|g_n\|_{q,c;\beta} \\ &\prec \|f\|_{(L_{q,c;\beta})_1^{((1/q),\alpha+(1/b)-\beta)}}. \end{aligned} \quad (53)$$

For the case $0 < b < 1$, by Theorem 16, given any $f \in (L_{q,c;\beta})_b^{((d/q),\alpha+(1/b)-\beta)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$, (convergence in $L_{q,c;\beta}$), $g_n \in \mathcal{T}_{2^n}$ such that

$$\left(\sum_{n=0}^{\infty} \left(2^{nd/q}(1+n)^{(\alpha+(1/b)-\beta)} \|g_n\|_{q,c;\beta}\right)^b\right)^{1/b} \prec \|f\|_{(L_{q,c;\beta})_b^{((d/q),\alpha+(1/b)-\beta)}}. \quad (54)$$

Due to Lemma 2, Corollary 12 and (54), we obtain

$$\begin{aligned}\|f\|_{\infty,b;\alpha} &\prec \left(\sum_{n=0}^{\infty} \|g_n\|_{\infty,b;\alpha}^b\right)^{1/b} \\ &\prec \left(\sum_{n=0}^{\infty} \left(2^{nd/q}(1+n)^{(\alpha+(1/b)-\beta)} \|g_n\|_{q,c;\beta}\right)^b\right)^{1/b} \\ &\prec \|f\|_{(L_{q,c;\beta})_b^{((1/q),\alpha+(1/b)-\beta)}}. \end{aligned} \quad (55)$$

This completes the proof. □

Using some real interpolation technique formulae we obtain the following result, which complements Corollary 23.

Corollary 24. *Let $0 < q < \infty$, $0 < c$, $b \le \infty$, $-\infty < \beta < \infty$, and $\alpha < -1/b$. Then*

$$(L_{q,c;\beta})_b^{((d/q),\alpha+(1/\min(b.1))-\beta)} \subset L_{\infty,b;\alpha}. \quad (56)$$

Proof. Using Lemma 20, we get

$$\left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((2d/q),0)}\right)_{(1/2),1;-\beta} \cong (L_{q,c;\beta})_1^{((d/q),-\beta)}. \quad (57)$$

We consider $\left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((d/q),-\beta)}\right)_{1,b;\alpha}$. From (57), [32, Theorem 6, and Lemma 4], and Lemma 20, we have

$$\begin{aligned}&\left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((d/q),-\beta)}\right)_{1,b;\alpha} \\ &\cong \left(L_{q,c;\beta}, \left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((2d/q),0)}\right)_{(1/2),1;-\beta}\right)_{1,b;\alpha} \\ &\supset \left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((2d/q),0)}\right)_{(1/2),b;\alpha+(1/\min(b.1))-\beta} \\ &\cong (L_{q,c;\beta})_b^{((d/q),\alpha+(1/\min(b.1))-\beta)}. \end{aligned} \quad (58)$$

To complete the proof, we have only to use (47) and [32, Corollary 7]:

$$\left(L_{q,c;\beta}, (L_{q,c;\beta})_1^{((d/q),-\beta)}\right)_{1,b;\alpha} \subset (L_{q,c;\beta}, L_{\infty})_{1,b;\alpha} \cong L_{\infty,b;\alpha}. \quad (59)$$

□

Remark 25. Recall that the spaces $X_1^{((d/q),\alpha+(1/b)-\beta)}$ and $X_b^{((d/q),\alpha+1-\beta)}$ are formed by all those $f \in X$ for which $\{E_n(f)\}$ belongs to $l_{(q/d),1;\alpha+(1/b)-\beta}$ and $l_{(q/d),b;\alpha+1-\beta}$, respectively. However, for $b > 1$, these sequence spaces are incomparable [24]. Note that for $b \le 1$, formulae (51) and (56) coincide.

Corollary 26. *Let $(p, b, \alpha) \in \Omega \cup \Gamma$, $p < \infty$, $b \le c \le \infty$, and $-\infty < \beta < \infty$. If $\alpha + (1/b) > \beta + (1/c)$ then*

$$(L_{p,c;\beta})_{\min(b,1)}^{(0,\alpha+(1/b)-\beta-(1/c)-(1/\min(b.1)))} \subset L_{p,b;\alpha}. \quad (60)$$

Proof. Let $(p, b, \alpha) \in \Omega$. In this case $b \ge 1$. By Theorem 17, given any $f \in (L_{p,c;\beta})_1^{(0,\alpha+(1/b)-\beta-(1/c)-1)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$, (convergence in $L_{p,c;\beta}$), $g_n \in \mathcal{T}_{2^{2^n}}$ such that

$$\sum_{n=0}^{\infty} 2^{n(\alpha+(1/b)-\beta-(1/c))} \|g_n\|_{p,c;\beta} \prec \|f\|_{(L_{p,c;\beta})_1^{(0,\alpha+(1/b)-\beta-(1/c)-1)}}. \quad (61)$$

Because all $g_n \in \mathcal{T}_{2^{2^n}} \subset L_{p,b;\alpha} \bigcap L_{p,c;\beta}$, it is not hard to show that the series $\sum_{n=0}^{\infty} g_n$ converges to f in $L_{p,b;\alpha}$. Since $(p, b, \alpha) \in \Omega$, using Corollary 13(i), we derive that

$$\begin{aligned}\|f\|_{p,b;\alpha} &\prec \sum_{n=0}^{\infty} \|g_n\|_{p,b;\alpha} \prec \sum_{n=0}^{\infty} 2^{n(\alpha+(1/b)-\beta-(1/c))} \|g_n\|_{p,c;\beta} \\ &\prec \|f\|_{(L_{p,c;\beta})_1^{(0,\alpha+(1/b)-\beta-(1/c)-1)}}. \end{aligned} \quad (62)$$

Let now$(p, b, \alpha) \in \Gamma$. In this case, $b < 1$. By Theorem 17, given any $f \in (L_{p,c;\beta})_b^{(0,\alpha-\beta-(1/c))}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$, (convergence in $L_{p,c;\beta}$), $g_n \in \mathcal{T}_{2^{2^n}}$ such that

$$\left(\sum_{n=0}^{\infty} \left(2^{n(\alpha+(1/b)-\beta-(1/c))} \|g_n\|_{p,c;\beta}\right)^b\right)^{(1/b)} \prec \|f\|_{(L_{q,c;\beta})_b^{(0,\alpha-\beta-(1/c))}}. \quad (63)$$

Due to Lemma 2 and Corollary 13(i), we obtain

$$\begin{aligned}\|f\|_{p,b;\alpha} &\prec \left(\sum_{n=0}^{\infty} \|g_n\|_{p,b;\alpha}^b\right)^{(1/b)} \\ &\prec \left(\sum_{n=0}^{\infty} \left(2^{n(\alpha+(1/b)-\beta-(1/c))} \|g_n\|_{q,c;\beta}\right)^b\right)^{(1/b)} \\ &\prec \|f\|_{(L_{q,c;\beta})_b^{(0,\alpha-\beta-(1/c))}}. \end{aligned} \quad (64)$$

This completes the proof. □

The following result can be proved analogously based on Corollary 14.

Corollary 27. *Let $(p, b, \alpha) \in \Omega \cup \Gamma$, $p < \infty$, $0 < c \le b$, and $-\infty < \beta < \alpha < \infty$. Then*

$$(L_{p,c;\beta})_{\min(b,1)}^{(0,\alpha-\beta-(1/\min(b.1)))} \subset L_{p,b;\alpha}. \quad (65)$$

The next four corollaries investigate embeddings between different approximation spaces $(L_{*,*;*})_*^{(\sigma,*)}$ for $\sigma > 0$. They follow immediately from Corollaries 11, 12, 13(i), and 14 and from Lemma 19.

Corollary 28. *Let* $0 < q < \infty$, $0 < c \leq \infty$, $-\infty < \beta < \infty$. *For the triple* (p, b, α), *we assume that either* $q < p < \infty$, $0 < b \leq \infty$, $-\infty < \alpha < \infty$, *or* $p = b = \infty$, $\alpha = 0$. *Then, for* $\sigma > 0$, $0 < u \leq \infty$, *and* $-\infty < \gamma < \infty$, *we have*

$$\left(L_{q,c;\beta}\right)_u^{(\sigma+d((1/q)-(1/p)),\gamma+\alpha-\beta)} \subset \left(L_{p,b;\alpha}\right)_u^{(\sigma,\gamma)}. \quad (66)$$

Corollary 29. *Let* $0 < q < \infty$, $0 < c$, $b \leq \infty$, $-\infty < \beta < \infty$, *and* $\alpha < -1/b$. *Then, for* $\sigma > 0$, $0 < u \leq \infty$, *and* $-\infty < \gamma < \infty$, *we have*

$$\left(L_{q,c;\beta}\right)_u^{(\sigma+(d/q),\gamma+\alpha+(1/b)-\beta)} \subset \left(L_{\infty,b;\alpha}\right)_u^{(\sigma,\gamma)}. \quad (67)$$

Corollary 30. *Let* $0 < p < \infty$, $0 < b < c \leq \infty$, $-\infty < \alpha$, $\beta < \infty$, *and* $\alpha + (1/b) > \beta + (1/c)$. *Then, for* $\sigma > 0$, $0 < u \leq \infty$, *and* $-\infty < \gamma < \infty$, *we have*

$$\left(L_{p,c;\beta}\right)_u^{(\sigma,\gamma+\alpha+(1/b)-\beta-(1/c))} \subset \left(L_{p,b;\alpha}\right)_u^{(\sigma,\gamma)}. \quad (68)$$

Corollary 31. *Let* $0 < p < \infty$, $0 < c \leq b \leq \infty$, *and* $-\infty < \beta < \alpha < \infty$. *Then, for* $\sigma > 0$, $0 < u \leq \infty$, *and* $-\infty < \gamma < \infty$, *we have*

$$\left(L_{p,c;\beta}\right)_u^{(\sigma,\gamma+\alpha-\beta)} \subset \left(L_{p,b;\alpha}\right)_u^{(\sigma,\gamma)}. \quad (69)$$

The two next corollaries deal with embeddings between different limiting approximation spaces $(L_{*,*;*})_*^{(0,*)}$. They follow immediately from Lemma 18 and Corollaries 13(i) and 14, respectively.

Corollary 32. *Let* $0 < p < \infty$, $0 < b < c \leq \infty$, $-\infty < \alpha$, $\beta < \infty$, *and* $\alpha + (1/b) > \beta + (1/c)$. *Then, for* $0<u\leq\infty$ *and* $\gamma > -1/u$, *we have*

$$\left(L_{p,c;\beta}\right)_u^{(0,\gamma+\alpha+(1/b)-\beta-(1/c))} \subset \left(L_{p,b;\alpha}\right)_u^{(0,\gamma)}. \quad (70)$$

Corollary 33. *Let* $0 < p < \infty$, $0 < c \leq b \leq \infty$, *and* $-\infty < \beta < \alpha < \infty$. *Then, for* $0 < u \leq \infty$ *and* $\gamma > -1/u$, *we have*

$$\left(L_{p,c;\beta}\right)_u^{(0,\gamma+\alpha-\beta)} \subset \left(L_{p,b;\alpha}\right)_u^{(0,\gamma)}. \quad (71)$$

5.3. Embeddings of Besov Spaces $B_{p,u}^{\sigma,\gamma}$. Here we give some applications of previous results to embeddings of Besov spaces into Lorentz–Zygmund spaces and between Besov spaces.

There are several definitions of Besov spaces. The Besov space $B_{p,u}^{\sigma,\gamma} \equiv B_{p,u}^{\sigma,\gamma}(\boldsymbol{T}^d)(\sigma \geq 0, 0 < p, u \leq \infty, -\infty < \gamma < \infty)$ is based on L_p and has classical smoothness σ and additional logarithmic smoothness with exponent γ. It is formed by all those $f \in L_p$ such that

$$\|f\|_{B_{p,u}^{\sigma,\gamma}} = \|f\|_p + \left(\int_0^1 \left(t^{-\sigma} l^{\gamma}(t) \omega_k(f,t)_p\right)^u \frac{dt}{t}\right)^{1/u} < \infty, \quad (72)$$

with an obvious modification when $u = \infty$. Here $\omega_k(f,t)_p$ $(k > \sigma, k \in \boldsymbol{N})$ is the modulus of smoothness of order k with respect to the quasi-norm on L_p. It makes sense to consider the spaces $B_{p,u}^{0,\gamma}$ with zero classical smoothness only for $\gamma \geq -1/u$. All we need for our application is the characterization of the Besov spaces by approximation. An important result in approximation theory states that

$$B_{p,u}^{\sigma,\gamma} \cong \left(L_p\right)_u^{(\sigma,\gamma)}. \quad (73)$$

For details see, e.g., [8, 30] and the references given there. Using this characterization with Corollaries 22–24 and 26–28, we obtain the following embeddings. Due to similarity, we will give only the proof of Corollary 37.

Corollary 34 (Cf. [8, Theorem 5.2]). *Let* $0 < q < p < \infty$, $0 < b \leq \infty$, *and* $-\infty < \alpha < \infty$. *Then*

$$B_{q,b}^{d((1/q)-(1/p)),\alpha} \subset L_{p,b;\alpha}. \quad (74)$$

Corollary 35. *Let* $0 < q < \infty$, $0 < b \leq \infty$, *and* $\alpha < -1/b$. *Then*

$$B_{q,\min(b,1)}^{(d/q),\alpha+(1/b)} \subset L_{\infty,b;\alpha}. \quad (75)$$

Corollary 36 (Cf. [8, Theorem 5.3]). *Let* $0<q<\infty$, $0<b\leq\infty$, *and* $\alpha <-1/b$. *Then*

$$B_{q,b}^{(d/q),\alpha+(1/\min(b,1))} \subset L_{\infty,b;\alpha}. \quad (76)$$

Corollary 37 (Cf. [8, Theorems 5.5, 5.6, and 5.7]). *Let* $(p, b, \alpha) \in \Omega \cup \Gamma$, $b \leq p < \infty$, *and* $\alpha > (1/p) - (1/b)$. *Then*

$$B_{p,\min(b,1)}^{0,\alpha+(1/b)-(1/p)-(1/\min(b,1))} \subset L_{p,b;\alpha}. \quad (77)$$

Proof. Using the characterization (73) and Corollary 26, we obtain

$$B_{p,\min(b,1)}^{0,\alpha+(1/b)-(1/p)-(1/\min(b,1))} \cong \left(L_p\right)_{\min(b,1)}^{(0,\alpha+(1/b)-(1/p)-(1/\min(b,1)))} \subset L_{p,b;\alpha}. \quad (78)$$

□

Corollary 38. *Let* $\alpha > 0$. *In addition, let* $0 < p = b \leq 1$, *or* $1 < p < \infty$ *and* $p \leq b \leq \infty$. *Then*

$$B_{p,\min(b,1)}^{0,\alpha-(1/\min(b,1))} \subset L_{p,b;\alpha}. \quad (79)$$

Corollary 39 (Cf. [34, Corollary 5.3 (i)] and [31, Theorem 2.8.1]). *Let* $0 < q < p \leq \infty$. *Then, for* $\sigma > 0$, $0 < u \leq \infty$, *and* $-\infty < \gamma < \infty$, *we have*

$$B_{q,u}^{\sigma+d((1/q)-(1/p)),\gamma} \subset B_{p,u}^{\sigma,\gamma}. \quad (80)$$

5.4. Embeddings of Besov-Type Spaces $B_{(p,b;\alpha),u}^{0,\gamma}$. In [7], the Besov-type spaces $B_{(p,b;\alpha),u}^{0,\gamma} \equiv B_{(p,b;\alpha),u}^{0,\gamma}(\boldsymbol{T}^d)$ (with zero classical smoothness) based on $L_{p,b;\alpha}$ are introduced, so that $B_{(p,p;0),u}^{0,\gamma} = B_{p,u}^{0,\gamma}$. The space $B_{(p,b;\alpha),u}^{0,\gamma}$ $(0 < u < \infty, 1 < p < \infty, 1 \leq b \leq \infty, -\infty < \alpha < \infty, \gamma > -1/u.)$ is formed by all those $f \in L_{p,b;\alpha}$ such that

$$\|f\|_{B_{(p,b;\alpha),u}^{0,\gamma}} = \|f\|_{p,b;\alpha} + \left(\int_0^1 \left(l^{\gamma}(t) \omega_k(f,t)_{p,b;\alpha}\right)^u \frac{dt}{t}\right)^{(1/u)} < \infty, \quad (81)$$

with an obvious modification when $u = \infty$. Here $\omega_k(f,t)_{p,b;\alpha} (k \in \mathbf{N})$ is the modulus of smoothness of order k with respect to the quasi-norm on $L_{p,b;\alpha}$. The definition is independent of k. The spaces $B^{0,\gamma}_{(p,b;\alpha),u}$ have the following characterization by approximation [7, (3.7)]:

$$B^{0,\gamma}_{(p,b;\alpha),u} \cong \left(L_{p,b;\alpha}\right)^{(0,\gamma)}_u. \tag{82}$$

Using this characterization and Corollaries 26, 27, 32, and 33, we obtain the following embeddings.

Corollary 40 (Cf. [7, (3.9)]). *Let $1 < p < \infty$, $1 \le c \le \infty$, $0 < b \le c$, and $-\infty < \alpha$, $\beta < \infty$. If $\alpha + (1/b) > \beta + (1/c)$, then*

$$B^{0,\alpha+(1/b)-\beta-(1/c)-(1/\min(b,1))}_{(p,c;\beta),\min(b,1)} \subset L_{p,b;\alpha}. \tag{83}$$

Corollary 41 (Cf. [7, (3.8)]). *Let $1 < p < \infty$, $1 \le c \le b \le \infty$, and $-\infty < \beta < \alpha < \infty$. Then*

$$B^{0,\alpha-\beta-1}_{(p,c;\beta),1} \subset L_{p,b;\alpha}. \tag{84}$$

Corollary 42. *Let $1 < p < \infty$, $1 \le b \le c \le \infty$, $-\infty < \alpha$, $\beta < \infty$, and $\alpha + (1/b) > \beta + (1/c)$. Then, for $0 < u \le \infty$ and $\gamma > -1/u$, we have*

$$B^{0,\gamma+\alpha+(1/b)-\beta-(1/c)}_{(p,c;\beta),u} \subset B^{0,\gamma}_{(p,b;\alpha),u}. \tag{85}$$

Corollary 43. *Let $1 < p < \infty$, $1 \le c \le b \le \infty$, and $-\infty < \beta < \alpha < \infty$. Then, for $0 < u \le \infty$ and $\gamma > -1/u$, we have*

$$B^{0,\gamma+\alpha-\beta}_{(p,c;\beta),u} \subset B^{0,\gamma}_{(p,b;\alpha),u}. \tag{86}$$

Remark 44. Some other limiting embeddings between Besov spaces based on generalized Lorentz–Zygmund spaces with two iterations of logarithm were obtained in [26, Theorem 2].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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