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Research Article

Geometric Inequalities via a Symmetric Differential Operator Defined by Quantum Calculus in the Open Unit Disk

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The present investigation covenants with the concept of quantum calculus besides the convolution operation to impose a comprehensive symmetric q-differential operator defining new classes of analytic functions. We study the geometric representations with applications. The applications deliberated to indicate the certainty of resolutions of a category of symmetric differential equations type Briot-Bouquet.

1. Introduction

In this effort, we deal with the structure of q-calculus, which develops an interesting technique for calculations and organizes different classes of operators and specific transformations. The significance of q-calculus appeared in a huge number of applications including physical problems. The symmetric q-activation normally achieves q-difference equations (may involve derivative). A close connection between these operators and symmetries of q-symmetric operator is accordingly to be estimated (see [1–9]). In recent investigation, we deliver a process for deriving and interpreting from a symmetry possessions and infirm analogy with the traditional cases. By combining the q-calculus and the symmetric Salagean differential operator, we introduce a novel modified symmetric Salagean q-differential operator. Via employing this operator, we deliver new classes of analytic functions.

2. Preliminaries

This section gives out the mathematical processing to deliver the suggested SDOs and complex conformable operator for some classes of analytic functions in the open unit disk $\cup = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Let $\wedge$ be the category of smooth function elicited as pursue

$$\gamma(\xi) = \xi + \sum_{n=2}^{\infty} \gamma_n \xi^n, \quad \xi \in \cup. \tag{1}$$

A function $\gamma \in \wedge$ is known as a starlike with respect to $\xi = 0$ if the straight line segment combining the origin to all else point of γ embedding completely in $\gamma(\xi : |\xi| < 1)$. The aim is that each point of $\gamma(\xi : |\xi| < 1)$ must be manifested via (0,0). A univalent function ($\gamma \in Ⓢ;$) is indicated to be convex in $\cup$ if the linear slice combining two ends of $\gamma(\xi : |\xi| < 1)$ stays completely in $\gamma(\xi : |\xi| < 1)$. We denote these classes by $\mathcal{S}^*$ and $\mathcal{C}$ for starlike and convex, respectively. In addition, suppose that the category $\mathcal{P}$ involves all functions γ analytic in $\cup$ with a positive real part in $\cup$ achieving $\gamma(0) = 1$. Mathematically, $\gamma \in \mathcal{S}^*$ if and only if $\xi\gamma'(\xi)/\gamma(\xi) \in \mathcal{P}$, and $\gamma \in \mathcal{C}$ if and only if $1 + \xi\gamma''(\xi)/\gamma'(\xi) \in \mathcal{P}$; equivalently,

$$\Re\left(\xi\gamma'(\xi)/\gamma(\xi)\right) > 0, \tag{2}$$

for starlikeness and

$$1+\Re\left(\xi\gamma''(\xi)/\gamma'(\xi)\right)>0, \tag{3}$$

for convexity.

For two functions, γ_1 and γ_2 belong to the category $\wedge$ and are said to be subordinate, noting by $\gamma_1 \prec \gamma_2$, if we can find a Schwarz function $\top$ with $\top(0)=0$ and $|\top(\xi)|<1$ achieving $\gamma_1(\xi)=\gamma_2(\top(\xi)), \xi \in \cup$ (the detail can be located in [10]). Obviously, $\gamma_1(\xi) \prec \gamma_2(\xi)$ implies that $\gamma_1(0)=\gamma_2(0)$ and $\gamma_1(\cup) \subset \gamma_2(\cup)$. We employ next facts, one can find it in [11].

For every nonnegative integer n, the q-integer number n symbolized by $[n,q]$ is structured by

$$[n,q]=\frac{1-q^n}{1-q}, \tag{4}$$

where $[0,q]=0$, $[1,q]=1$, and $\lim_{q\to 1^-}[n,q]=n$. Consequently, the analytic function γ is written by the formula

$$\Delta_q\gamma(\xi)=\frac{\gamma(q\xi)-\gamma(\xi)}{q\xi-\xi}, \quad \xi \in \cup. \tag{5}$$

Clearly, we have $\Delta_q\xi^n=[n,q]\xi^{n-1}$. Consequently, for $\gamma \in \wedge$, we attain

$$\Delta_q\gamma(\xi)=\sum_{n=1}^{\infty}\gamma_n[n,q]\xi^{n-1}, \xi \in \cup, \gamma_1=1. \tag{6}$$

For $\gamma \in \wedge$, it realized that the Sàlàgean q-differential operator [12] has the formula

$$\begin{aligned} S_q^0\gamma(\xi)&=\gamma(\xi),\\ S_q^1\gamma(\xi)&=\xi\Delta_q\gamma(\xi),\\ &\cdots,\\ S_q^k\gamma(\xi)&=\xi\Delta_q\left(S_q^{k-1}\gamma(\xi)\right), \end{aligned} \tag{7}$$

such that k represents as a positive integer. A computation based on the definition of Δ_q implies that $S_q^k\gamma(\xi)=\gamma(\xi) * \Pi_q^k(\xi)$, where $*$ is the convolution product.

$$\Pi_q^k(\xi)=\xi+\sum_{n=2}^{\infty}[n,q]^k\xi^n, \tag{8}$$

and

$$S_q^k\gamma(\xi)=\xi+\sum_{n=2}^{\infty}[n,q]^k\gamma_n\xi^n. \tag{9}$$

Clearly,

$$\lim_{q\to 1^-}S_q^k\gamma(\xi)=\xi+\sum_{n=2}^{\infty}n^k\gamma_n\xi^n, \tag{10}$$

the well-known Salagean differential operator [6].

Consider the role $\gamma(\xi)$ and a constant $0\le\lambda\le 1$, we introduce the q-symmetric Salagean differential operator using the definition of Δ_q as follows:

$$\begin{aligned} \mathcal{S}_q^{\lambda,0}\gamma(\xi)&=\gamma(\xi),\\ \mathcal{S}_q^{\lambda,1}\gamma(\xi)&=\lambda\xi\Delta_q\gamma(\xi)-(1-\lambda)\xi\Delta_q\gamma(-\xi)\\ &=\xi\left(\lambda\Delta_q\gamma(\xi)-(1-\lambda)\Delta_q\gamma(-\xi)\right)\\ &=\lambda\left(\xi+\sum_{n=2}^{\infty}[n,q]\gamma_n\xi^n\right)-(1-\lambda)\\ &\quad\cdot\left(-\xi+\sum_{n=2}^{\infty}[n,q](-1)^n\gamma_n\xi^n\right)\\ &=\xi+\sum_{n=2}^{\infty}[n,q][(\lambda-(1-\lambda)(-1)^n)]\gamma_n\xi^n, \qquad (11)\\ \mathcal{S}_q^{\lambda,2}\gamma(\xi)&=\mathcal{S}_q^{\lambda,1}\left[\mathcal{S}_q^{\lambda,1}\gamma(\xi)\right]\\ &=\xi+\sum_{n=2}^{\infty}[n,q]^2[(\lambda-(1-\lambda)(-1)^n)]^2\gamma_n\xi^n,\\ &\vdots\\ \mathcal{S}_q^{\lambda,k}\gamma(\xi)&=\mathcal{S}_q^{\lambda,1}\left[\mathcal{S}_q^{\lambda,k-1}\gamma(\xi)\right]\\ &=\xi+\sum_{n=2}^{\infty}[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^n. \end{aligned}$$

Obviously, we indicate that when $\lambda=1(\mathcal{S}_q^{1,k}=S_q^k)$, we get the Salagean q-differential operator. We can call (11) as the symmetric Sàlàgean q-differential operator in $\cup$. Also, we have the following two limits.

$$\lim_{q\to 1^-}\mathcal{S}_q^{1,k}\gamma(\xi)=\xi+\sum_{n=2}^{\infty}n^k\gamma_n\xi^n, \tag{12}$$

which are represented to the well-known Salagean differential operator [6] and

$$\lim_{q\to 1^-}\mathcal{S}_q^{\lambda,k}\gamma(\xi)=\xi+\sum_{n=2}^{\infty}n^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^n, \tag{13}$$

the symmetric differential operator [8], respectively.

Depending on the definition of (11), we impose the recognizing classes:

Definition 1. A member $\gamma \in \wedge$ is called in the category $S_q^*(\lambda, k, \hbar)$ when

$$S_q^*(\lambda, k, \hbar) = \left\{ \gamma \in \wedge \frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)''}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} \prec \hbar(\xi), \hbar \in \mathscr{C} \right\}. \quad (14)$$

We obtain the following special cases:

(i) $\lim_{q\to 1^-} S_q^*(\lambda, k, \hbar) = S_1^*(\lambda, k, \hbar)$ ([8])

(ii) $S_q^*(\lambda, 0, (1 + A\xi/1 + B\xi))$ ([13–15])

(iii) $S_q^*(\lambda, 0, (2/1 + e^{-\xi}))$ ([16])

(iv) $S_q^*(\lambda, 0, (1 + \varepsilon^2\xi^2/1 - \varepsilon^2 - \varepsilon^2\xi^2)), \varepsilon = 1 - \sqrt{5}/2$ ([17, 18])

(v) $S_q^*(\lambda, 0, 1 + (\beta - \alpha/\pi)i \log (1 - \exp (2\pi i((1 - \alpha)/(\beta - \alpha)))\xi/1 - \xi))$ ([19])

(vi) $S_q^*(\lambda, 0, 1 + (2/\pi(1 - \alpha))i \log (1 - \exp (\pi i(1 - \alpha)^2\xi/1 - \xi))$ ([20])

(vii) $S_q^*(\lambda, 0, \sqrt{1 + \xi})$ ([21])

(viii) $S_q^*(\lambda, 0, \sin (\xi) + 1)$ ([22])

(ix) $S_q^*(\lambda, 0, \cos (\xi) + 1)$ ([23])

(x) $S_q^*(\lambda, 0, (1 + \xi/1 + ((1 - \nu)/\nu)\xi)^{1/\nu}), \nu \geq 0.5, \nu \geq 1$ ([24])

Definition 2. If $\gamma \in \wedge$, then $\gamma \in \mathbb{J}_q^{\lambda,\flat}(A, B, k)$ if and only if

$$1 + \frac{1}{\flat}\left(\frac{2\mathcal{S}_q^{\lambda,k+1}\gamma(\xi)}{\mathcal{S}_q^{\lambda,k}\mathrm{E}(\xi) - \mathcal{S}_q^{\lambda,k}\gamma(-\xi)}\right) \prec \frac{1 + A\xi}{1 + B\xi} (\xi \in \mathbb{U}, -1 \leq B < A \leq 1, k \in \mathbb{Z}_+, \flat \in \mathbb{C}\{0\}, \lambda \in [0, 1]). \quad (15)$$

(i) $\lambda = 1, q \longrightarrow 1^- \Rightarrow$ [25]

(ii) $\lambda = 1, B = 0, q \longrightarrow 1^- \Rightarrow$ [26]

(iii) $\lambda = 1, A = 1, B = -1, \# = 2, q \longrightarrow 1^- \Rightarrow$ [27]

(iv) $q \longrightarrow 1^- \Rightarrow$ [8]

In our investigation, we focus on the geometric presentation of the special classes $S_q^*(\lambda, k, \hbar)$ and $\mathbb{J}_q^{\lambda,\flat}(A, B, k)$ via utilizing the basis information given in [11].

Lemma 3. *Let $a \in \mathbb{C}$, let n be a positive integer and let*

$$\mathfrak{H}[a, n] = \left\{\gamma : \gamma(\xi) = a + \xi + \gamma_n\xi^n + \gamma_{n+1}\xi^{n+1} + \cdots\right\}. \quad (16)$$

(i) Suppose that $\wp \in \mathbb{R}$ such that $\Re(\gamma(\xi) + \wp\xi\gamma'(\xi)) > 0 \Rightarrow \Re(\gamma(\xi)) > 0$. In addition, let $\wp > 0$ and $\gamma \in \mathfrak{H}[1, n]$; thereafter, there are two fixed positive numbers $\ell > 0$ and $\flat > 0$ with $\flat = \flat(\wp, \ell, n)$ so that

$$\gamma(\xi) + \wp\xi\gamma'(\xi) \prec \left[\frac{1 + \xi}{1 - \xi}\right]^{\flat} \Rightarrow \gamma(\xi) \prec \left[\frac{1 + \xi}{1 - \xi}\right]^{\ell}. \quad (17)$$

(ii) Assume that $\nu \in [0, 1)$ and $\nu \in \mathfrak{H}[1, n]$, subsequently, there occurs a fixed number $\ell > 0$ with ℓ such that

$$\Re\left(\gamma^2(\xi) + 2\gamma(\xi).\xi\gamma'(\xi)\right) > \nu \Rightarrow \Re(\gamma(\xi)) > \ell. \quad (18)$$

(iii) Consider $\gamma \in \mathfrak{H}[a, n]$ with $\Re(a) > 0$ posterity

$$\Re\left(\gamma(\xi) + \xi\gamma'(\xi) + \xi^2\gamma''(\xi)\right) > 0, \quad (19)$$

or for $\theta : \mathbb{U} \longrightarrow \mathbb{R}$ such that

$$\Re\left(\gamma(\xi) + \theta(\xi)\frac{\xi\gamma'(\xi)}{\gamma(\xi)}\right) > 0, \quad (20)$$

thus $\Re(\gamma(\xi)) > 0$.

Remark 4. Concerning Lemma 3 (i), more information about ℓ and $\flat$ can be found in [11] (Theorem 3.1c, p. 73). About (ii), more information about ν and ℓ can be found in [11] (Theorem 3.1e or Corollary 3.1e.2, p. 77 and p. 79, respectively). And for (iii), one can see Theorem 4.1g in [11] (p. 198) and, for $\theta(\xi) = 1$, the inequality contains a Briot-Bouquet formula.

3. Main Results

Here, we focus in the geometric representations of the classes $S_q^*(\lambda, k, \hbar)$ and $\mathbb{J}_q^{\lambda,\flat}(A, B, k)$ and the outcomes of these classes.

Theorem 5. *For $\gamma \in \wedge$ if one of the recognizing determinations are indicated by*

(i) $\mathcal{S}_q^{\lambda,k}\gamma(\xi)$ is of bounded turning

(ii) γ *satisfies the subordination formula*

$$\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' \prec \left(\frac{1+\xi}{1-\xi}\right)^{\flat}, \flat > 0, \xi \in \cup; \tag{21}$$

(iii) γ *achieves the formula*

$$\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > \frac{\varsigma}{2}, \varsigma \in [0,1), \xi \in \cup, \tag{22}$$

(iv) ψ *satisfies the relation*

$$\Re\left(\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'' - \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 0, \tag{23}$$

(v) γ *admits the inequality*

$$\Re\left(\frac{z\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 1, \tag{24}$$

then $(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) \in \mathcal{P}(\sigma)$ *where* $0 \le \sigma < 1$.

Proof. Sort out a function ρ in the following construction:

$$\rho(\xi) = \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} \Rightarrow \xi\gamma'(\xi) + \rho(\xi) = \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'. \tag{25}$$

Via the main information, $\mathcal{S}_q^{\lambda,k}\gamma(\xi)$ is of constrained limit turning; it infers that $\Re(\xi\rho'(\xi) + \rho(\xi)) > 0$. Accordingly, via Lemma 3 (i), we acquire $\Re(\rho(\xi)) > 0$ which incomes the first requested statement of the theorem. In opinion of the additional information, we obligate the subsequent subordination relative

$$\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' = \xi\rho'(\xi) + \rho(\xi) \prec \left[\frac{1+\xi}{1-\xi}\right]^{\flat}. \tag{26}$$

Now, based on Lemma 3 (i), there exists a fixed positive number $\ell > 0$ satisfying $\flat = \flat(\ell)$ and the subordination inequality

$$\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^{\ell}. \tag{27}$$

This leads to the conclusion

$$\Re\left(\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > \sigma, \sigma \in [0,1). \tag{28}$$

Lastly, we assume the third fact, a direct reckoning reaches to

$$\Re\left(\rho^2(\xi) + 2\rho(\xi).\xi\rho'(\xi)\right) = 2\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) \ge \varsigma. \tag{29}$$

According to the virtue of Lemma 3 (ii), there occurs a fixed positive number, say $\ell > 0$ achieving $\Re(\rho(\xi)) > \ell$. Consequently, we obtain

$$\rho(\xi) = \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} \in \mathcal{P}(\sigma), \sigma \in [0,1). \tag{30}$$

Hence, via Equation (29), it indicates that

$$\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'\right) > 0. \tag{31}$$

So by Noshiro-Warschawski and Kaplan Propositions, $\mathcal{S}_q^{\lambda,k}\gamma(\xi) \in \circledS$ and of bounded turning in $\cup$.

Via the derivative (25) and operating the real, one can attain the real relation

$$\Re\left(\rho(\xi) + \xi\rho'(\xi) + \xi^2\rho''(\xi)\right) = \Re\left(\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'' - \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 0. \tag{32}$$

Hence, according to Lemma 3 (ii), one get $\Re(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) > 0$.

Via considering the logarithmic derivative on (25) and acting as areal part, we obtain the consequence conversation:

$$\Re\left(\rho(\xi) + \frac{\xi\rho'(\xi)}{\rho(\xi)} + \xi^2\rho''(\xi)\right) = \Re\left(\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)''}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} - 1\right) > 0. \tag{33}$$

Thus, according to Lemma 3 (iii), where $\theta(\xi) = 1$, we attain that $\Re(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) > 0$.

Theorem 6. *Suppose that* $\gamma \in S_q^*(\lambda, k, \hbar)$, *with* $\hbar(\xi) \in \mathcal{C}$. *Then*

$$\mathcal{S}_q^{\lambda,k} \prec \xi e^{\int_0^{\xi}\hbar(\eth(u))-1/u\,du}, \tag{34}$$

where $\eth \in \cup$ (analytic) satisfies that $\eth(0)=0$ and $|\eth(\xi)|<1$. In addition, for $|\xi|=\chi$, $\mathcal{S}_q^{\lambda,k}$ reckoning fulfills the formula statement

$$\exp\left(\int_0^1 \frac{\hbar(\eth(-\chi))-1}{\chi}d\chi\right) \le \left|\frac{\mathcal{S}_q^{\lambda,k}}{\xi}\right| \le \exp\left(\int_0^1 \frac{\hbar(\eth(\chi))-1}{\chi}d\chi\right). \tag{35}$$

Proof. We note that $\gamma \in S_q^*(\lambda,k,\hbar)$, then one can gain

$$\left(\frac{z\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}\right) \prec \hbar(\xi), \xi\in\cup, \tag{36}$$

this confirms that there occurs analytic function type Schwarz achieving the relations $\eth(0)=0, |\eth(\xi)|<1$ and

$$\left(\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}\right) = \hbar(\eth(\xi)), \xi\in\cup. \tag{37}$$

A calculation gives us

$$\left(\frac{\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}\right) - \frac{1}{\xi} = \frac{\hbar(\eth(\xi))-1}{\xi}. \tag{38}$$

Via integrating left and right parts, one can achieve

$$\log \mathcal{S}_q^{\lambda,k}\gamma(\xi) - \log\xi = \int_0^\xi \frac{\hbar(\eth(u))-1}{u}du. \tag{39}$$

Thus, we have

$$\log\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} = \int_0^\xi \frac{\hbar(\eth(u))-1}{u}du. \tag{40}$$

Via the definition and the properties of subordination, one can have

$$\mathcal{S}_q^{\lambda,k}\gamma(\xi) \prec \xi\exp\left(\int_0^\xi \frac{\hbar(\eth(u))-1}{u}du\right). \tag{41}$$

Furthermore, we deliver that $\hbar(z)$ maps the disk $0<|\xi|<\chi<1$ onto a convex symmetric domain corresponding to the real axis, that is

$$\hbar(-\chi|\xi|) \le \Re\left(\hbar\left(\eth(\chi\xi)\right)\right) \le \hbar(\chi|\xi|), \chi\in(0,1), \tag{42}$$

which implies that

$$\begin{aligned}&\hbar(-\chi)\le\hbar(-\chi|\xi|), \quad \hbar(\chi|\xi|)\le\hbar(\chi),\\ &\int_0^1 \frac{\hbar(\eth(-\chi|\xi|))-1}{\chi}d\chi \le \Re\left(\int_0^1\frac{\hbar(\eth(\chi))-1}{\chi}d\chi\right)\\ &\qquad \le \int_0^1 \frac{\hbar(\eth(\chi|\xi|))-1}{\chi}d\chi.\end{aligned} \tag{43}$$

Via applying Equation (40), one can indicate that

$$\begin{aligned}\int_0^1 \frac{h(\eth(-\chi|\xi|))-1}{\chi}d\chi &\le \log\left|\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right|\\ &\le \int_0^1\frac{\hbar(\eth(\chi|\xi|))-1}{\chi}d\chi,\end{aligned} \tag{44}$$

which leads to

$$\begin{aligned}\exp\left(\int_0^1 \frac{h(\eth(-\chi|\xi|))-1}{\chi}d\chi\right) &\le \left|\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right|\\ &\le \exp\left(\int_0^1\frac{\hbar(\eth(\chi|\xi|))-1}{\chi}d\chi\right).\end{aligned} \tag{45}$$

Hence, we have

$$\begin{aligned}\exp\left(\int_0^1 \frac{h(\eth(-\chi))-1}{\eta}\right)d\chi &\le \left|\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right|\\ &\le \exp\left(\int_0^1\frac{\hbar(\eth(\chi))-1}{\chi}\right)d\chi.\end{aligned} \tag{46}$$

Corollary 7 [8]. *If $q\longrightarrow 1$ in Theorem 6, then*

$$\mathcal{S}_1^{\lambda,k}\gamma(\xi) \prec \xi\exp\left(\int_0^\xi \frac{\hbar(\eth(u))-1}{u}du\right). \tag{47}$$

Theorem 8. *Suppose that $\gamma\in\mathbb{J}_q^{\lambda,\flat}(A,B,k)$, then the odd construction formula*

$$\mathfrak{M}(\xi) = \frac{1}{2}[\gamma(\xi)-\gamma(-\xi)], \quad \xi\in\cup, \tag{48}$$

fulfills the consequently subordination

$$\begin{aligned}&1+\frac{1}{\flat}\left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)}-1\right) \prec \frac{1+A\xi}{1+B\xi},\\ &\Re\left(\frac{\xi\mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)}\right) \ge \frac{1-\varrho^2}{1+\varrho^2}, \quad |\xi|=\varrho<1,\\ &(-1\le B<A\le 1, \xi\in\cup, k=1,2,\cdots,\lambda\in[0,1], \flat\in\mathbb{C}\setminus\{0\}).\end{aligned} \tag{49}$$

Proof. Let $\gamma \in \mathbb{J}_q^{\lambda,b}(A,B,k)$. Subsequently, we get that there occurs a function $\mathfrak{L} \in \mathbb{J}(A,B)$ with the formula

$$(\mathfrak{L}(\xi)-1).b=\left(\frac{2\mathcal{S}_q^{\lambda,k+1}\gamma(\xi)}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)-\mathcal{S}_q^{\lambda,k}\gamma(-\xi)}\right),$$
$$(\mathfrak{L}(-\xi)-1).b=\left(\frac{-2\mathcal{S}_q^{\lambda,k+1}\gamma(-\xi)}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)-\mathcal{S}_q^{\lambda,k}\gamma(-\xi)}\right). \tag{50}$$

This yields

$$1+\frac{1}{b}\left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)}-1\right)=\frac{\mathfrak{L}(\xi)+\mathfrak{L}(-\xi)}{2}. \tag{51}$$

In addition, since $\mathfrak{L}$ achieves the inequality

$$\mathfrak{L}(\xi)\prec\frac{1+A\,\xi}{1+B\,\xi}, \tag{52}$$

taking account that the fractional functional express $(1+A\xi)/(1+B\xi)$ is univalent and hence, consequently, we attain the relation

$$1+\frac{1}{b}\left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{53}$$

Furthermore, the expression $\mathfrak{M}(\xi)\in\mathcal{S}^*$, which leads to the inequality

$$\frac{\xi\mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)}\prec\frac{1-\xi^2}{1+\xi^2}, \tag{54}$$

that is, there occurs a Schwarz function $\kappa\in\cup$, $|\kappa(\xi)|\le|\xi|<1$, $\kappa(0)=0$ with the property

$$\Psi(\xi)\coloneqq\frac{z\mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)}\prec\frac{1-\kappa(\xi)^2}{1+\kappa(\xi)^2}, \tag{55}$$

which implies that there is ζ, $|\zeta|=r<1$ such that

$$\kappa^2(\zeta)=\frac{1-\Psi(\zeta)}{1+\Psi(\zeta)},\quad \zeta\in\cup. \tag{56}$$

An operation, one can indicate that

$$\left|\frac{1-\Psi(\zeta)}{1+\Psi(\zeta)}\right|=|\kappa(\zeta)|^2\le|\zeta|^2<1. \tag{57}$$

Thus, one can inform the recognizing inequality

$$\left|\Psi(z)-\frac{|\zeta|^4+1}{1-|\zeta|^4}\right|\le\frac{2|\zeta|^2}{\left(1-|\zeta|^4\right)}. \tag{58}$$

This implies the consequence result

$$\Re(\Psi(\xi))\ge\frac{1-\varrho^2}{1+\varrho^2}. \tag{59}$$

Next consequence outcomes of the above result can be located in [8, 25, 26] accordingly.

Corollary 9. *If $\lambda=1$ in Theorem 8, then*

$$1+\frac{1}{b}\left(\frac{\mathcal{S}_q^{1,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{1,k}\mathfrak{M}(\xi)}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{60}$$

Corollary 10. *If $\lambda=1$, $k=1$, and $q\longrightarrow1$ in Theorem 8, then*

$$1+\frac{1}{b}\left(\frac{\mathcal{S}_q^{1,2}\mathfrak{M}(\xi)}{\mathcal{S}_q^{1,1}\mathfrak{M}(\xi)}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{61}$$

Corollary 11. *If $q\longrightarrow1$ in Theorem 8, then*

$$1+\frac{1}{b}\left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{62}$$

4. Applications

We introduce an application of our outcomes based on finding the outcome of Briot-Bouquet equation (BBE) (see [11] for more information). This category of ODE is an association of ODE whose outcomes are formulas in the complex plane. Existence and uniqueness theorems include the utility of majors and minors (or subordination and superordination concepts) (see [28–31]). Investigation of rational first ODEs in the complex region implies the finding of new transcendental special functions which are now known as symmetric BBE

$$\beta\gamma(\xi)+(1-\beta)\frac{\xi\,(\gamma(\xi))'}{\gamma(\xi)}=\hbar(\xi),\quad \hbar(0)=\gamma(0),\beta\in[0,1]. \tag{63}$$

By employing the q-differential operator (11), we have the q-formula of BBE

$$\beta\gamma(\xi)+(1-\beta)\left(\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}\right)=\hbar(\xi),\quad \hbar(0)=\gamma(0),\xi\in\cup. \tag{64}$$

A simple result of (64) can be recognized at $\beta=1$. Thus, we investigate the status, $\gamma\in\wedge$ and $\beta=0$. The initial condition will be $\gamma(0)=\hbar(0)=0$.

Theorem 12. *Suppose that Equation (64) with $\beta = 0$ and $\gamma \in \wedge$ with nonnegative coefficients. If $\hbar(\xi), \xi \in \cup$ is convex in $\cup$, then there occurs a major solution achieving the inequality*

$$\mathcal{S}_q^{\lambda,k}\gamma(\xi) \prec \xi \exp\left(\int_0^{\xi} \frac{h(\eth(u))-1}{u}du\right), \tag{65}$$

such that $\eth(\xi)$ indicates analytic function in $\cup$, satisfying the relations $|\eth(\xi)| < 1$ and $\eth(0) = 0$.

Proof. In virtue of every conditions indicated by (64), and $\gamma(\xi) \in \wedge$, we realize the information

$$\begin{aligned} &\mathfrak{R}\left(\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}\right) > 0 \\ &\Leftrightarrow \mathfrak{R}\left(\frac{\xi + \sum_{n=2}^{\infty} n[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^n}{\xi + \sum_{n=2}^{\infty}[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^n}\right) > 0 \\ &\Leftrightarrow \mathfrak{R}\left(\frac{1 + \sum_{n=2}^{\infty} n[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^{n-1}}{1 + \sum_{n=2}^{\infty}[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n\xi^{n-1}}\right) > 0 \\ &\Leftrightarrow \left(\frac{1 + \sum_{n=2}^{\infty} n[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n}{1 + \sum_{n=2}^{\infty}[n,q]^k[(\lambda-(1-\lambda)(-1)^n)]^k\gamma_n}\right) > 0, \end{aligned} \tag{66}$$

where $\mathfrak{R}(\xi) \neq 0$. Moreover, by the definition of $\mathcal{S}_q^{\lambda,k}\gamma(\xi)$, we indicate that $(\mathcal{S}_q^{\lambda,k}\gamma)(0) = 0$. Consequently, yields that

$$\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} \in \mathcal{P} \Rightarrow \gamma(\xi) \in S_q^*(\lambda, k, \hbar). \tag{67}$$

Thus, via Theorem 6, we reach the desired outcome in (65).

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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