

Research Article

On Inverse Nodal Problem and Multiplicities of Eigenvalues of a Vectorial Sturm-Liouville Problem

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An m -dimensional vectorial inverse nodal Sturm-Liouville problem with eigenparameter-dependent boundary conditions is studied. We show that if there exists an infinite sequence $\{\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2)\}_{j=1}^{\infty}$ of eigenfunctions which are all vectorial functions of type (CZ), then the potential matrix $Q(x)$ and A are simultaneously diagonalizable by the same unitary matrix U . Subsequently, some multiplicity results of eigenvalues are obtained.

1. Introduction

Consider the following m -dimensional vectorial Sturm-Liouville problem with the eigenparameter-dependent boundary conditions:

$$\begin{cases} -\mathbf{y}'' + Q(x)\mathbf{y} = \lambda^2\mathbf{y}, & 0 \leq x \leq \pi, \\ \mathbf{y}(0) = \mathbf{0}, \\ A\mathbf{y}'(\pi) + \lambda\mathbf{y}(\pi) = \mathbf{0}, \end{cases} \quad (1)$$

where λ is the spectral parameter and $\mathbf{y} = (y_1, y_2, \dots, y_m)^T$ is an m -dimensional vectorial function. The potential matrix $Q(x)$ is an $m \times m$ real symmetric and nonnegative definite matrix-valued function which is defined and integrable on the interval $[0, \pi]$. A is an $m \times m$ nonsingular real symmetric matrix. Throughout this paper, we set $m \geq 2$ and agree that $\mathbf{0}$ denotes the m -dimensional zero vector.

Sturm-Liouville problems with eigenparameter in the boundary conditions arise upon separation of variables in the one-dimensional wave and heat equations for various physical applications [1]. There are some literatures on such scalar problems (see [2–5]) and vectorial problems (see [6–8]). We consider the inverse nodal problem of (1) firstly. It was McLaughlin who initiated the study of the inverse nodal

problem [9]. In the recent years, the inverse nodal problem of the Sturm-Liouville problem has been investigated a lot (see the monographs [10–17] and the references therein) and the vectorial inverse nodal problem is studied in [18, 19]. However, McLaughlin's uniqueness theorem does not hold for the vectorial Sturm-Liouville problems. To clarify our problem, the following definitions could be seen in [18, 19]. For the convenience of reader, we state here again. Let $\mathbf{y}(x)$ be an m -dimensional vectorial function defined on the interval $[0, \pi]$, if $\mathbf{y}(x_0) = \mathbf{0}$, $x_0 \in [0, \pi]$, x_0 will be called a nodal point (zero) of $\mathbf{y}(x)$. We say that $\mathbf{y}(x)$ is a vectorial function of type (CZ) (or $\mathbf{y}(x)$ has a common zero property) if all the isolated zeros of its components are nodal points. The matrix $Q(x)$ is called simultaneously diagonalizable if there is a constant unitary matrix U such that $U^*Q(x)U$ is a diagonal matrix-valued function.

In 1999, Shen and Shieh [18] studied the inverse nodal problem of the vectorial equation in (1) with Dirichlet boundary condition when dimension $m = 2$. They proved that if the problem has infinitely many eigenfunctions $\{\mathbf{y}_{n_j}(x)\}_{j=1}^{\infty}$ which are of type (CZ), then $Q(x)$ is simultaneously diagonalizable. It seems to be the first study of the vectorial inverse nodal problem. Cheng et al. generalized to the arbitrary separated boundary conditions in [19]. Chan [6] investigated some eigenvalue problems of (1). He proved

that the eigenvalues of the problem are all real. Based on the results of [6, 18, 19], in this study, using a different method from that in [18, 19], we consider the inverse nodal problem of (1). Firstly, we derived that the matrix $\int_0^x Q(t)dt$ is simultaneously diagonalizable which is equivalent to $Q(x)$ possessing the same property. That is, there exists a constant unitary matrix U such that $U^*Q(x)U = \text{diag}(\mu_1(x), \mu_2(x), \dots, \mu_m(x))$, where $\mu_i(x)(i = 1, 2, \dots, m)$ are eigenvalues of $Q(x)$. Meanwhile, if $Q(x)$ and A are simultaneously diagonalizable by the same unitary matrix U , we prove that the eigenfunctions of problem (1) are all of type (CZ) (see Corollary 3.2).

Next, we investigate the relationship between multiplicities of eigenvalues of problem (1) and matrix A . It is known that the eigenvalues of a scalar Sturm-Liouville problem with the separate boundary conditions are all simple. However, the multiplicities of the m -dimensional vectorial Sturm-Liouville problem may be among 1 to m . Such problems were studied in [20–22]. In [20], Shen and Shieh study the multiplicities of 2-dimensional vectorial Sturm-Liouville problem defined in $[0, 1]$. Suppose that $Q(x) = (q_{ij}(x))$ is a continuous 2×2 Jacobian matrix-valued function; it is proved that if $\int_0^1 q_{12}(x)dx \neq 0$, then the sufficiently large eigenvalues are simple. Subsequently, Kong [21] developed and improved the results in [20] to the case when $Q(x)$ is real symmetric. In 2007, Yang et al. [22] extended the result of [20, 21] to the Sturm-Liouville equation with a weighted function w , a leading coefficient function p , and general separated conditions.

From the estimation of eigenvalues (see Lemma 4), it is not difficult to see that the multiplicities of eigenvalues of problem (1) are determined firstly by the multiplicities of eigenvalues of matrix A no matter $Q(x)$ possesses any properties. We provide conditions on A and $Q(x)$ under which, with finitely many exceptions, the eigenvalues of vectorial problem (1) are simple.

This paper is divided into four sections. Following this Introduction, in Section 2, we investigate the asymptotic expression of the eigenvalues and state some other preliminary lemmas for the main theorems. In Section 3, we discuss the inverse nodal problem of (1). In Section 4, some results about multiplicities of eigenvalues of the 2-dimensional vectorial Sturm-Liouville problem (1) are given.

2. Preliminaries

Let $Y(x, \lambda^2)$ be the matrix solutions of equation

$$-Y'' + Q(x)Y = \lambda^2 Y, \quad x \in [0, \pi], \quad (2)$$

satisfying the initial conditions

$$\begin{aligned} Y(0, \lambda^2) &= 0_m, \\ Y'(0, \lambda^2) &= E_m, \end{aligned} \quad (3)$$

where E_m denotes the $m \times m$ identity matrix and 0_m denotes the $m \times m$ zero matrix. Denote

$$\Delta(\lambda) = \det \left(AY'(\pi, \lambda^2) + \lambda Y(\pi, \lambda^2) \right). \quad (4)$$

$\Delta(\lambda)$ will be called the *characteristic function* of the eigenvalues of problem (1). The *algebraic multiplicity* of an eigenvalue λ is the order of λ as a zero of $\Delta(\lambda)$. The *geometric multiplicity* of λ as an eigenvalue of the problem (1) is defined to be the number of linearly independent solutions of the boundary problem. Let $C(\lambda) = AY'(\pi, \lambda^2) + \lambda Y(\pi, \lambda^2)$.

Lemma 1. *The geometric multiplicity of λ as an eigenvalue of problem (1) is equal to $m - \text{Rank}(C(\lambda))$.*

From the above lemma, the geometric multiplicity of an eigenvalue λ_* of problem (1) is at most m , and the geometric multiplicity of λ_* is m if and only if $C(\lambda_*)$ is the zero matrix. It is known that $\Delta(\lambda)$ is an entire function of order one with respect to λ . What is more, when $Q(x)$ is nonnegative definite for any $x \in [0, \pi]$, the eigenvalues of problem (1) are all real [6]. Denote $\lambda = \sigma + i\tau, \sigma, \tau \in \mathbb{R}$. By variation of constants, we have

$$Y(x, \lambda^2) = \frac{\sin(\lambda x)}{\lambda} E_m + \frac{1}{\lambda} \int_0^x \sin \lambda(x-t) Q(t) Y(t, \lambda^2) dt. \quad (5)$$

Lemma 2 ([23]). *For fixed $x \in [0, \pi]$, the matrix functions $Y(x, \lambda^2)$ and $Y'(x, \lambda^2)$ are entire functions with respect to variable λ . When $|\lambda| \rightarrow \infty$, we have*

$$\begin{aligned} Y(x, \lambda^2) &= \frac{\sin(\lambda x)}{\lambda} E_m + O\left(\frac{e^{|\tau|x}}{|\lambda|^2}\right), \\ Y'(x, \lambda^2) &= \cos(\lambda x) E_m + O\left(\frac{e^{|\tau|x}}{|\lambda|}\right). \end{aligned} \quad (6)$$

Remark 3. Using the first relation in (6) to iterate equation (5), we find that

$$\begin{aligned} Y(x, \lambda^2) &= \frac{\sin(\lambda x)}{\lambda} E_m - \frac{\cos(\lambda x)}{2\lambda^2} \int_0^x Q(t) dt \\ &\quad + \frac{1}{2\lambda^2} \int_0^x \cos \lambda(x-2t) Q(t) dt + O\left(\frac{e^{|\tau|x}}{|\lambda|^3}\right). \end{aligned} \quad (7)$$

By Riemann-Lebesgue lemma,

$$\lim_{\lambda \rightarrow \infty} \int_0^x \cos \lambda(x-2t) Q(t) dt = 0. \quad (8)$$

Thus, when $|\lambda| \rightarrow \infty$, $Y(x, \lambda^2)$ has the following asymptotic expansion:

$$Y(x, \lambda^2) = \frac{\sin(\lambda x)}{\lambda} E_m - \frac{\cos(\lambda x)}{2\lambda^2} \int_0^x Q(t) dt + o\left(\frac{e^{|\tau|x}}{|\lambda|^2}\right). \quad (9)$$

Similarly,

$$Y'(x, \lambda^2) = \cos(\lambda x)E_m + \frac{\sin(\lambda x)}{2\lambda} \int_0^x Q(t)dt + o\left(\frac{e^{|\tau|x}}{|\lambda|}\right). \quad (10)$$

Lemma 4. Suppose that an eigenvalue $\lambda_{n,r}$ ($r = 1, 2, \dots, m$) is with algebraic multiplicity k ($1 \leq k \leq m$). When $|n| \rightarrow \infty$, $\lambda_{n,r}$ have the following asymptotic expansion:

$$\lambda_{n,r} = n - \frac{1}{\pi} \tan^{-1} a_r + O\left(\frac{1}{n^{1/k}}\right). \quad (11)$$

where a_r are eigenvalues of matrix A .

Proof. Since A is an $m \times m$ nonsingular real symmetric matrix, there exists a unitary matrix U such that

$$U^*AU = \text{diag}(a_1, a_2, \dots, a_m), \quad (12)$$

where $a_1 \leq a_2 \leq \dots \leq a_m$. By formula (6) in Lemma 2,

$$U^*C(\lambda)U = \text{diag}(a_1, a_2, \dots, a_m) \cos(\lambda\pi) + \sin(\lambda\pi)E_m + O\left(\frac{e^{|\tau|\pi}}{|\lambda|}\right). \quad (13)$$

Thus,

$$\begin{aligned} \Delta(\lambda) &= \det C(\lambda) = \det U^*C(\lambda)U \\ &= \prod_{r=1}^m (a_r \cos(\lambda\pi) + \sin(\lambda\pi)) + O\left(\frac{e^{m|\tau|\pi}}{|\lambda|}\right) \\ &= \prod_{r=1}^m R_r \sin(\lambda\pi + \theta_r) + O\left(\frac{e^{m|\tau|\pi}}{|\lambda|}\right), \end{aligned} \quad (14)$$

where $R_r = (a_r^2 + 1)^{1/2}$, $\theta_r = \tan^{-1} a_r$ and $|\theta_r| < \pi/2$. Additionally, $\theta_1 \leq \theta_2 \leq \dots \leq \theta_m$. Denote $\Delta_0(\lambda) = \prod_{r=1}^m R_r \sin(\lambda\pi + \theta_r)$ and $\delta(\lambda) = O(e^{m|\tau|\pi}/|\lambda|)$. Take a closed λ -curve $\Gamma_n = \Gamma_n^{(1)} \cup \Gamma_n^{(2)} \cup \Gamma_n^{(3)}$ ($n = 1, 2, \dots$):

$$\begin{aligned} \Gamma_n^{(1)} &= \left\{ \lambda = \sigma + i\tau \mid \sigma = n + \frac{1}{2} - \frac{\theta_1}{\pi}, 0 \leq |\tau| \leq n \right\}, \\ \Gamma_n^{(2)} &= \left\{ \lambda = \sigma + i\tau \mid \sigma = -\left(n + \frac{1}{2}\right) - \frac{\theta_m}{\pi}, 0 \leq |\tau| \leq n \right\}, \\ \Gamma_n^{(3)} &= \left\{ \lambda = \sigma + i\tau \mid -\left(n + \frac{1}{2}\right) - \frac{\theta_m}{\pi} \leq \sigma \leq n + \frac{1}{2} - \frac{\theta_1}{\pi}, |\tau| = n \right\}. \end{aligned} \quad (15)$$

On $\Gamma_n^{(1)}$ or $\Gamma_n^{(2)}$, for $r = 1, 2, \dots, m$,

$$\begin{aligned} |\sin(\lambda\pi + \theta_r)| &= \left| \frac{1}{2i} \left(e^{i(\sigma\pi + \theta_r)} e^{-\tau\pi} - e^{-i(\sigma\pi + \theta_r)} e^{\tau\pi} \right) \right| \\ &\geq \frac{e^{\tau\pi}}{2} \left(1 - \left| e^{2i(\sigma\pi + \theta_r)} e^{-2\tau\pi} \right| \right) > \frac{e^{|\tau|\pi}}{4}. \end{aligned} \quad (16)$$

On $\Gamma_n^{(3)}$, similarly, we have $|\sin(\lambda\pi + \theta_r)| > e^{|\tau|\pi}/4$ for $r = 1, 2, \dots, m$. Denote $M = \prod_{r=1}^m |R_r|$. On the contour Γ_n , since $|\lambda| \geq 1$, we have

$$|\Delta_0(\lambda)|_{\Gamma_n} = |\prod_{r=1}^m R_r \sin(\lambda\pi + \theta_r)|_{\Gamma_n} > M \cdot \frac{e^{m|\tau|\pi}}{4^m \cdot |\lambda|}, \quad (17)$$

that is, $|\Delta_0(\lambda)|_{\Gamma_n} > |\delta(\lambda)|_{\Gamma_n}$. Using Rouché theorem, the number of zeros of $\Delta(\lambda)$ inside Γ_n coincides with the number of zeros of $\Delta_0(\lambda)$; i.e., it equals to $(2n+1)m$ (counting the multiple zeros). Note that the zeros of $\Delta(\lambda)$ and $\Delta_0(\lambda)$ are all real. Now, we consider the zeros of $\Delta_0(\lambda)$ in Γ_n . They are

$$\lambda_{j,r}^0 = j - \frac{\theta_r}{\pi}, \quad j = 0, \pm 1, \dots, \pm n, \quad r = 1, 2, \dots, m. \quad (18)$$

Now, it is easy to see that the algebraic multiplicity of an eigenvalue λ of problem (1) is between 1 to m . Note that the eigenvalues of problem (1), in this case, stretch from $-\infty$ to $+\infty$. Consequently, we denote all the eigenvalues as $\lambda_{n,r}$ ($n = 0, \pm 1, \pm 2, \dots; r = 1, 2, \dots, m$). For sufficiently large n , $\lambda_{n,r}$ lies between Γ_{n-1} and Γ_n . Set $\lambda_{n,r} = \lambda_{n,r}^0 + \varepsilon_{n,r} = n - (\theta_r/\pi) + \varepsilon_{n,r}$. Moreover, denote $C_\delta^{n,r} := \{\lambda : |\lambda - \lambda_{n,r}^0| \leq \delta\}$, where $0 < \delta < \min_{1 \leq i,j \leq m, i \neq j} \{(1/\pi)|\theta_i - \theta_j|\}$. Applying the Rouché theorem again to the circle $C_\delta^{n,r}$, if $\lambda_{n,r}^0$ is k -multiple ($1 \leq k \leq m$), in $C_\delta^{n,r}$, there are k zeros of $\Delta(\lambda)$. For sufficiently small δ , it follows that $\varepsilon_{n,r} = o(1)$.

If $\lambda_{n,r}$ is k -multiple, substituting $\lambda_{n,r}$ into (14) and noting that $\tau = 0$, we get that $(\varepsilon_{n,r})^k = O(1/n)$. Consequently, $\varepsilon_{n,r} = O(1/n^{1/k})$.

Using the asymptotic expression of eigenvalues, we get that the asymptotic expression about the nodal points.

Lemma 5. Suppose that $\mathbf{y}_{n,r}(x, \lambda_{n,r}^2)$ is an eigenfunction of the vectorial problem (1) of type (CZ) corresponding to the eigenvalue $\lambda_{n,r}$, and $\{x_k^{n,r}\}$ is the nodal set of $\mathbf{y}_{n,r}(x, \lambda_{n,r}^2)$. Then,

(i) for sufficiently large $|n|$, $\mathbf{y}_{n,r}(x, \lambda_{n,r}^2)$ possesses $|n|$ nodal points $x_k^{n,r}$ in $(0, \pi)$

(ii) the nodal points have the following asymptotic expression:

$$x_k^{n,r} = \frac{(k+1)\pi}{|n|} + O\left(\frac{1}{n^2}\right), \quad k = 0, 1, \dots, |n|-1, \quad (19)$$

for sufficiently large $|n|$

(iii) $\{x_k^{n,r} : 0 \leq k \leq |n|-1\}_{|n|=1}^\infty$ is dense in $[0, \pi]$

Proof. The proof is similar to that of Lemma 2 in [18] and Theorem 2.3 in [19].

3. Inverse Nodal Problem

In this section, we discuss the inverse nodal problem of (1).

Theorem 6. Suppose that there exists a sequence $\{\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2)\}_{j=1}^\infty$ of eigenfunctions of problem (1) which are all vectorial functions of type (CZ), then $Q(x)$ and A are simultaneously diagonalizable by the same unitary matrix U .

Proof. Denote $\mathbf{v}_{n_j,r}$ a unit null vector of $C(\lambda_{n_j,r})$, where $Y(x, \lambda_{n_j,r}^2)$ is the solution of matrix initial problem (2)–(3) corresponding to $\lambda_{n_j,r}$. Then,

$$\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2) = Y(x, \lambda_{n_j,r}^2) \mathbf{v}_{n_j,r} \quad (20)$$

is an eigenfunction of vectorial problem (1) corresponding to $\lambda_{n_j,r}$. Assuming that $\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2)$ is a vectorial eigenfunction of type (CZ), by Lemma 5, we know that, $\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2)$ has $|n_j|$ nodal points $x_k^{n_j,r}$ in $(0, \pi)$. What is more, $\{x_k^{n_j,r} \mid k = 0, 1, \dots, |n_j| - 1\}_{j=1}^\infty$ is dense in $[0, \pi]$; thus for a fixed x_0 in $(0, \pi]$, there exists a sequence of nodal $x_k^{n_j,r}$ such that

$$\lim_{j \rightarrow \infty} x_k^{n_j,r} = x_0. \quad (21)$$

Since

$$\mathbf{y}_{n_j,r}(x_k^{n_j,r}, \lambda_{n_j,r}^2) = Y(x_k^{n_j,r}, \lambda_{n_j,r}^2) \mathbf{v}_{n_j,r} = \mathbf{0}, \quad (22)$$

by (9), we have

$$\begin{aligned} Y(x_k^{n_j,r}, \lambda_{n_j,r}^2) &= \frac{\sin(\lambda_{n_j,r} x_k^{n_j,r})}{\lambda_{n_j,r}} E_m \\ &\quad - \frac{\cos(\lambda_{n_j,r} x_k^{n_j,r})}{2\lambda_{n_j,r}^2} \int_0^{x_k^{n_j,r}} Q(t) dt \\ &\quad + o\left(\frac{1}{|\lambda_{n_j,r}|^2}\right). \end{aligned} \quad (23)$$

Thus,

$$\int_0^{x_k^{n_j,r}} Q(t) dt \cdot \mathbf{v}_{n_j,r} = \frac{2\lambda_{n_j,r} \sin(\lambda_{n_j,r} x_k^{n_j,r})}{\cos(\lambda_{n_j,r} x_k^{n_j,r})} E_m \cdot \mathbf{v}_{n_j,r} + o(1) \cdot \mathbf{v}_{n_j,r}. \quad (24)$$

Since $\|\mathbf{v}_{n_j,r}\| = 1$, $\mathbf{v}_{n_j,r}$ converges to some unit vector denoting as \mathbf{v} . Therefore, taking the limit as $j \rightarrow \infty$ for both sides of Equation (24), we obtain

$$\int_0^{x_0} Q(t) dt \cdot \mathbf{v} = c_1(x_0) \cdot \mathbf{v}, \quad (25)$$

where $c_1(x_0)$ is a scalar associated with x_0 . Since x_0 is arbitrary in $(0, \pi]$, we can choose $x_1 \neq x_0$ in $(0, \pi]$ and use the above argument for x_1 such that \mathbf{v} is also an eigenvector of $\int_0^{x_1} Q(t) dt$. Then, $\int_0^{x_0} Q(t) dt$ and $\int_0^{x_1} Q(t) dt$ have the same eigenvector \mathbf{v} . That is, \mathbf{v} is chosen independently of x .

Furthermore, using Schmidt orthogonalization, \mathbf{v} could expand into standard orthogonal basis denoting as $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$. Since $\int_0^x Q(t) dt$ is diagonalizable for all $x \in (0, \pi]$, $\mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m$ have to be eigenvectors for $\int_0^x Q(t) dt$. Thus, there are other scalars $c_i(x)(i = 2, 3, \dots, m)$ depending on x such that $\int_0^x Q(t) dt \cdot \mathbf{u}_i = c_i(x) \cdot \mathbf{u}_i$. Hence, we derive that $\int_0^x Q(t) dt$ is simultaneously diagonalizable. In fact, denote $U = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m]$, then we have

$$U^* \int_0^x Q(t) dt U = \text{diag}(c_1(x), c_2(x), \dots, c_m(x)). \quad (26)$$

Deriving the two sides of the above formula, we derived that matrix $Q(x)$ is also simultaneously diagonalizable by the same matrix U . That is,

$$U^* Q(x) U = \Lambda(x) = \text{diag}(\mu_1(x), \mu_2(x), \dots, \mu_m(x)), \quad (27)$$

where $\mu_i(x) = c'_i(x)(i = 1, 2, \dots, m)$.

Since $\mathbf{y}_{n_j,r}(x, \lambda_{n_j,r}^2)$ satisfies the boundary conditions, hence,

$$AY'(\pi, \lambda_{n_j,r}^2) \mathbf{v}_{n_j,r} + \lambda_{n_j,r} Y(\pi, \lambda_{n_j,r}^2) \mathbf{v}_{n_j,r} = \mathbf{0}. \quad (28)$$

By Lemma 2, we have

$$A\mathbf{v}_{n_j,r} = -\tan(\lambda_{n_j,r} \pi) \mathbf{v}_{n_j,r} + O\left(\frac{1}{n_j}\right). \quad (29)$$

Let $j \rightarrow \infty$, $\mathbf{v}_{n_j,r}$ tends to the same vector \mathbf{v} , and the sequence $\tan(\lambda_{n_j,r} \pi)$ has a limit $-c \in \mathbb{R}$. Thus, $A\mathbf{v} = c\mathbf{v}$. Subsequently, with a similar process, we conclude that A is diagonalizable by the same matrix U . That is,

$$U^* AU = \Lambda_A = \text{diag}(a_1, a_2, \dots, a_m). \quad (30)$$

The proof of the theorem is finished.

Theorem 7. Suppose that $Q(x)$ and A are simultaneously diagonalizable by the same unitary matrix U . Then,

- (1) the eigenfunctions of problem (1) are all of type (CZ)
- (2) the algebraic multiplicities and geometric multiplicities of eigenvalues of problem (1) are equal

Proof.

- (1) By the transformation $\mathbf{y} = U\mathbf{z}$, the problem (1) becomes

$$\begin{aligned}\mathbf{z}'' &= U^{-1}\mathbf{y}'' = U^{-1}(QU\mathbf{z} - \lambda^2U\mathbf{z}), \mathbf{z}(0) \\ &= \mathbf{0}, \Lambda_A\mathbf{z}'(\pi) + \lambda\mathbf{z}(\pi) = \mathbf{0},\end{aligned}\quad (31)$$

that is,

$$-\mathbf{z}'' + \Lambda(x)\mathbf{z} = \lambda^2\mathbf{z}, \mathbf{z}(0) = \mathbf{0}, \Lambda_A\mathbf{z}'(\pi) + \lambda\mathbf{z}(\pi) = \mathbf{0}. \quad (32)$$

Note that the two problems (1) and (32) have exactly the same eigenvalues and multiplicities. Let $\mathbf{z} = (z_1, z_2, \dots, z_m)^T$. The vectorial problem (32) is equivalent to the following scalar Sturm-Liouville problems:

$$\begin{cases} -z_i'' + \mu_i(x)z_i = \lambda^2 z_i, & x \in [0, \pi], \\ z_i(0) = 0, \\ a_i z_i'(\pi) + \lambda z_i(\pi) = 0, \\ i = 1, 2, \dots, m. \end{cases} \quad (33)$$

All the eigenvalues of problem (32) are the collections of eigenvalues of scalar problems in (33). If $\lambda_*^{(i)}$ is an eigenvalue of the i th problem in (33) and $z_{*,i}(x, \lambda_*^{(i)})$ is the eigenfunction corresponding to $\lambda_*^{(i)}$, then

$$\mathbf{y}_*(x, \lambda_*^{(i)}) = U(0, \dots, z_{*,i}(x, \lambda_*^{(i)}), \dots, 0)^T \quad (34)$$

is the eigenfunction of the original problem (1) corresponding to $\lambda_*^{(i)}$. Obviously, $\mathbf{y}_*(x, \lambda_*^{(i)})$ is a vectorial function of type (CZ).

- (2) Let $z_{0,i}(x, \lambda^2)$ ($i = 1, 2, \dots, m$) be the solutions of equations in (33) satisfying the initial conditions

$$z_{0,i}(0, \lambda^2) = 0, z_{0,i}'(0, \lambda^2) = 1 \quad (35)$$

We have

$$\begin{aligned}C(\lambda) &= \text{diag} \left(a_1 z_{0,1}'(\pi, \lambda^2) + \lambda z_{0,1}(\pi, \lambda^2), a_2 z_{0,2}'(\pi, \lambda^2) \right. \\ &\quad \left. + \lambda z_{0,2}(\pi, \lambda^2), \dots, a_m z_{0,m}'(\pi, \lambda^2) + \lambda z_{0,m}(\pi, \lambda^2) \right).\end{aligned}\quad (36)$$

Thus, $\Delta(\lambda) = \prod_{i=1}^m (a_i z_{0,i}'(\pi, \lambda^2) + \lambda z_{0,i}(\pi, \lambda^2))$. If λ_* is a k -multiple (geometric) eigenvalue, then $\text{Rank}(C(\lambda_*)) = m - k$. We may as well let the last k elements in the diagonal of $C(\lambda_*)$ be zero. Hence, λ_* is an eigenvalue of the last k problems in (33). Denote $\Delta_i(\lambda) = a_i z_{0,i}'(\pi, \lambda^2) + \lambda z_{0,i}(\pi, \lambda^2)$.

We have $\Delta_i(\lambda_*) = 0$ and $\Delta'_i(\lambda_*) \neq 0$ for $i = m - k + 1, \dots, m$. Thus, λ_* is also with algebraic multiplicity k .

4. Multiplicities of Eigenvalues

In this section, we first discuss the relationship between the multiplicities of eigenvalues of 2-dimensional vectorial Sturm-Liouville problem (1) and the matrix A in the boundary condition.

Lemma 8. *When $m = 2$, the algebraic multiplicity and geometric multiplicity of λ as an eigenvalue of the problem (1) is equal.*

Proof. See [6].

Theorem 9. *Let $m = 2$. If the eigenvalues of matrix A are all simple, then with finitely many exceptions, the eigenvalues of problem (1) are all simple too.*

Proof. Let $\lambda_{n,r}$ be an eigenvalue of problem (1). By Lemma 4,

$$\lambda_{n,r} = n - \frac{1}{\pi} \tan^{-1} a_r + o(1). \quad (37)$$

Suppose, to the contrary, that the multiplicities of the eigenvalues of problem (1) are 2. By Lemma 1, $\text{Rank}(C(\lambda)) = 0$. That is, $C(\lambda_{n,r}) = O_2$. By (13), we have

$$\cos(\lambda_{n,r}\pi) \text{diag}(a_1, a_2) + \sin(\lambda_{n,r}\pi)E_2 + O\left(\frac{e^{|\tau|}\pi}{|\lambda_{n,r}\pi|}\right) = O_2. \quad (38)$$

Calculating directly and noting that the eigenvalues are real, we get

$$\tan(\lambda_{n,r}\pi)E_2 = \text{diag}(a_1, a_2) + O\left(\frac{1}{n}\right). \quad (39)$$

Taking the limit as $n \rightarrow \infty$, (39) follows that $a_1 = a_2$ which contradicts with the facts that the eigenvalues of the matrix A are all simple.

The next theorem shows the relationship between multiplicities of eigenvalues and $Q(x)$ when $m = 2$ and $A = cE_2$ ($c \neq 0$).

Theorem 10. *Suppose that $A = cE_2$ ($c \neq 0$). If the eigenvalues of the matrix $\int_0^\pi Q(t)dt$ have no repeated eigenvalues, then, with finitely many exceptions, the eigenvalues of problem (1) are all simple.*

Proof. The proof of this conclusion is analogous to that of Theorem 9 and therefore is omitted. Note that formulas (9) and (10) are needed to calculate $C(\lambda)$ in this case.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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