Hindawi
Journal of Function Spaces
Volume 2020, Article ID 7061549, 11 pages
https://doi.org/10.1155/2020/7061549

Research Article

Hermite–Jensen–Mercer Type Inequalities for Caputo Fractional Derivatives

Jinchao Zhao,[1] Saad Ihsan Butt,[2] Jamshed Nasir,[2] Zhaobo Wang,[1] and Iskander Tlili[3,4]

[1]*College of Computer and Communication Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, China*
[2]*Comsat University Islamabad, Lahore Campus, Lahore, Pakistan*
[3]*Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[4]*Faculty of Applied Sciences, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

Correspondence should be addressed to Iskander Tlili; iskander.tlili@tdtu.edu.vn

Received 27 December 2019; Accepted 11 February 2020; Published 24 March 2020

Academic Editor: Alberto Fiorenza

In this article, certain Hermite–Jensen–Mercer type inequalities are proved via Caputo fractional derivatives. We established some new inequalities involving Caputo fractional derivatives, such as Hermite–Jensen–Mercer type inequalities, for differentiable mapping $h^{(n)}$ whose derivatives in the absolute values are convex.

1. Introduction

In recent years, inequality theory attracts many researchers due to its applications in our daily life and within the mathematics [1–9]. Let $0 < x_1 \le x_2 \le \cdots \le x_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be nonnegative weights such that $\sum_{r=1}^{n} \mu_r = 1$. The Jensen inequality [10] states that h is a convex function on the interval $[u, v]$; then,

$$h\left(\sum_{r=1}^{n} \mu_r x_r\right) = \left(\sum_{r=1}^{n} \mu_r h(x_r)\right), \tag{1}$$

where $\forall x_r \in [u, v]$ and all $\mu_r \in [0, 1]$, $(r = 1, 2, \ldots, n)$.

The Hermite–Hadamard inequality asserts that if a mapping $h\colon J \subseteq R \longrightarrow R$ is a convex function on J with $u, v \in J$, $u < v$, then

$$h\left(\frac{u+v}{2}\right) \le \frac{1}{v-u} = \int_u^v h(\tau)\,\mathrm{d}\tau \le \frac{h(u)+h(v)}{2}. \tag{2}$$

The reverse direction in the above inequality holds when h is concave.

Theorem 1 (see [11]). *If h is a convex function on $[u, v]$, then*

$$h\left(u + v - \sum_{r=1}^{n} \mu_r x_r\right) = h(u) + h(v) - \sum_{r=1}^{n} \mu_r h(x_r), \tag{3}$$

$$\forall x_r \in [u, v] \quad \text{and all } \mu_r \in [0, 1], (r = 1, 2, \ldots, n). \tag{4}$$

Inequality (3) is known as the Jensen–Mercer inequality. Recently, inequality (3) has been generalized, see ([12–15]). For more recent and related results connected with Jensen–Mercer inequality, see ([11, 16–18]).

The previous era of fractional calculus is as old as the history of differential calculus. Several fractional operators are introduced that generalize ordinary integrals. However, the fractional derivatives have some basic properties than the corresponding classical ones. On the contrary, besides the smooth requirement, the Caputo derivative does not coincide with the classical derivative [19]. Caputo fractional derivatives are introduced by the Italian mathematician Caputo in 1967. Since then, a lot of research involves Caputo fractional derivatives [20–22].

The Caputo fractional derivatives are defined as in [23–26].

Definition 1. Suppose $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, and $h \in C^n[u, v]$. The Caputo fractional derivatives of order α are defined as follows:

$$\left({}^{c}D_{u^{+}}^{\alpha}h\right)(z)=\frac{1}{\Gamma(n-\alpha)}\int_{u}^{z}\frac{h^{(n)}(\tau)}{(z-\tau)^{\alpha-n+1}}\mathrm{d}\tau;\quad z>u,$$

$$\left({}^{c}D_{v^{-}}^{\alpha}h\right)(z)=\frac{(-1)^{n}}{\Gamma(n-C)}\int_{z}^{v}\frac{h^{(n)}(\tau)}{(\tau-z)^{\alpha-n+1}}\mathrm{d}\tau;\quad z<v. \tag{5}$$

If $\alpha=n\in\{1,2,3,\ldots\}$ and usual derivatives of h of order n exist, then the Caputo fractional derivatives $({}^{c}D_{u^{+}}^{\alpha}h)(z)$ coincide with $h^{(n)}(z)$.

Specifically, we get

$$\left({}^{c}D_{u^{+}}^{0}h\right)(z)=\left({}^{c}D_{v^{-}}^{0}h\right)(z)=h(z), \tag{6}$$

where $n=1$ and $\alpha=0$.

In this article, by using the Jensen–Mercer inequality, we proved Hermite–Hadamard's inequalities for fractional integrals and established some Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are convex.

2. Hermite–Hadamard–Mercer Type Inequalities for Caputo Fractional Derivatives

By using Jensen–Mercer inequalities, Hermite–Hadamard type inequalities can be expressed in Caputo fractional derivatives as follows.

Theorem 2. *Suppose that a positive function h: $[u,v]\longrightarrow R$ with $0\le u<v$ and $h\in C^{n}[u,v]$. If $h^{(n)}$ is a convex function on $[u,v]$, then the following inequalities for Caputo fractional derivatives hold:*

$$\begin{aligned}h^{(n)}\left(u+v-\frac{x+y}{2}\right)&\le h^{(n)}(u)+h^{(n)}(v)\\&\quad-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\left\{\left({}^{c}D_{x^{+}}^{\alpha}h\right)(y)\right.\\&\quad\left.+(-1)^{n}\left({}^{c}D_{y^{-}}^{\alpha}h\right)(x)\right\}\\&\le h^{(n)}(u)+h^{(n)}(v)-h^{(n)}\left(\frac{x+y}{2}\right),\end{aligned} \tag{7}$$

$\forall x,y\in[u,v]$, $\alpha>0$, and $\Gamma(\cdot)$ is the gamma function.

Proof. Using the Jensen–Mercer inequality, we get

$$\begin{aligned}h^{(n)}\left(u+v-\frac{u_1+v_1}{2}\right)&\le h^{(n)}(u)+h^{(n)}(v)\\&\quad-\frac{h^{(n)}(u_1)+h^{(n)}(v_1)}{2},\quad\forall u_1,v_1\in[a,b].\end{aligned} \tag{8}$$

Now, by changing of variables $u_1=\tau x+(1-\tau)y$ and $v_1=(1-\tau)x+\tau y$, $\forall x,y\in[u,v]$, and $\tau\in[0,1]$ in (8), we get

$$\begin{aligned}h^{(n)}\left(u+v-\frac{x+y}{2}\right)&\le h^{(n)}(u)+h^{(n)}(v)\\&\quad-\frac{h^{(n)}(\tau x+(1-\tau)y)+h^{(n)}((1-\tau)x+\tau y)}{2}.\end{aligned} \tag{9}$$

Multiplying both sides by $\tau^{n-\alpha-1}$ and then integrating with respect to τ over $[0,1]$, we get

$$\begin{aligned}\frac{1}{n-\alpha}h^{(n)}\left(u+v-\frac{x+y}{2}\right)&\le\frac{1}{n-\alpha}\left\{h^{(n)}(u)+h^{(n)}(v)\right\}\\&\quad-\frac{1}{2}\left\{\int_{0}^{1}\tau^{n-\alpha-1}\left(h^{(n)}(\tau x+(1-\tau)y)\right.\right.\\&\quad\left.\left.+h^{(n)}((1-\tau)x+\tau y)\right)\mathrm{d}\tau\right\}.\end{aligned} \tag{10}$$

After simplification, we get

$$\begin{aligned}&h^{(n)}\left(u+v-\frac{x+y}{2}\right)\\&\le h^{(n)}(u)+h^{(n)}(v)-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\\&\quad\left\{\left({}^{c}D_{x^{+}}^{\alpha}h\right)(y)+(-1)^{n}\left({}^{c}D_{y^{-}}^{\alpha}h\right)(x)\right\},\end{aligned} \tag{11}$$

and so the first inequality of (2.1) is proved.

Now, for the proof of the second inequality of (2.1), we first note that if $h^{(n)}$ is a convex function, then for $\tau\in[0,1]$, it gives

$$\begin{aligned}h^{(n)}\left(\frac{x+y}{2}\right)&=h^{(n)}\left(\frac{\tau x+(1-\tau)y+(1-\tau)x+\tau y}{2}\right)\\&\le\frac{h^{(n)}(\tau x+(1-\tau)y)+h^{(n)}((1-\tau)x+\tau y)}{2}.\end{aligned} \tag{12}$$

Multiplying both sides by $\tau^{n-\alpha-1}$ and then integrating the resulting inequality with respect to τ over $[0,1]$, we get

$$\begin{aligned}\frac{1}{n-\alpha}h^{(n)}\left(\frac{x+y}{2}\right)&\le\frac{1}{2}\left\{\int_{0}^{1}\tau^{n-\alpha-1}\left(h^{(n)}(\tau x+(1-\tau)y)\right.\right.\\&\quad\left.\left.+h^{(n)}((1-\tau)x+\tau y)\right)\right\}\\h^{(n)}\left(\frac{x+y}{2}\right)&\le\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\left\{\left({}^{c}D_{x^{+}}^{\alpha}h\right)(y)+(-1)^{n}\left({}^{c}D_{y^{-}}^{\alpha}h\right)(x)\right\}.\end{aligned} \tag{13}$$

Multiplying by (-1) and adding $h^{(n)}(u)+h^{(n)}(v)$ both sides in (13), we get the second inequality of (2.1), which completes the proof. □

Theorem 3. *Suppose that a positive function h: $[u,v]\longrightarrow R$ with $0\le u<v$ and $h\in C^{n}[u,v]$. If $h^{(n)}$ is a convex function on*

$[u,v]$, *then the following inequalities for Caputo fractional derivatives hold:*

$$h^{(n)}\left(u+v-\frac{x+y}{2}\right)\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\times\left\{\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x)+(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right\}\leq h^{(n)}(u)+h^{(n)}(v)-\left(\frac{h^{(n)}(x)+h^{(n)}(y)}{2}\right),\tag{14}$$

where $\forall x,y\in[u,v]$, $\alpha>0$, *and* $\Gamma(\cdot)$ *is the gamma function.*

Proof. To prove the first part of inequality, we use convexity of $h^{(n)}$ to get

$$h^{(n)}\left(u+v-\frac{u_1+v_1}{2}\right)=h^{(n)}\left(\frac{u+v-u_1+u+v-v_1}{2}\right)\leq\frac{h^{(n)}(u+v-u_1)+h^{(n)}(u+v-v_1)}{2},\tag{15}$$

for all $u_1,v_1\in[u,v]$. Now by writing variables $u_1=(\tau/2)x+(2-\tau/2)y$ and $v_1=(2-\tau/2)x+(\tau/2)y$ for $x,y\in[u,v]$ and $\tau\in[0,1]$, we get

$$2h^{(n)}\left(u+v-\frac{x+y}{2}\right)\leq h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)+h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right).\tag{16}$$

Multiplying both sides of (16) by $\tau^{n-\alpha-1}$ and then integrating with respect to τ over $[0,1]$, we get

$$2h^{(n)}\left(u+v-\frac{x+y}{2}\right)\int_0^1\tau^{n-\alpha-1}\mathrm{d}\tau\leq\int_0^1\tau^{n-\alpha-1}\left(h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)+h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\right)\mathrm{d}\tau.\tag{17}$$

Further simplifying gives

$$h^{(n)}\left(u+v-\frac{x+y}{2}\right)\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x)+(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right\},\tag{18}$$

and so the first inequality of (14) is proved.

Now, for the proof of the second inequality of (14), we first note that if $h^{(n)}$ is a convex function, then by employing Jensen–Mercer inequality (3) for $\tau\in[0,1]$ gives

$$h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\leq h^{(n)}(u)+h^{(n)}(v)-\left[\frac{\tau}{2}h^{(n)}(x)+\frac{2-\tau}{2}h^{(n)}(y)\right],\tag{19}$$

$$h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\leq h^{(n)}(u)+h^{(n)}(v)-\left[\frac{2-\tau}{2}h^{(n)}(x)+\frac{\tau}{2}h^{(n)}(y)\right].\tag{20}$$

By adding the inequalities of (19) and (20), we get

$$h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)+h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\leq 2\left(h^{(n)}(u)+h^{(n)}(v)\right)-\left(h^{(n)}(x)+h^{(n)}(y)\right).\tag{21}$$

Multiplying both sides by $\tau^{n-\alpha-1}$ and then integrating over $\tau\in[0,1]$, we get

$$\int_0^1\tau^{n-\alpha-1}\left(h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)+h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\right)\mathrm{d}\tau\leq\left\{2\left(h^{(n)}(u)+h^{(n)}(v)\right)-\left(h^{(n)}(x)+h^{(n)}(y)\right)\right\}\int_0^1\tau^{n-\alpha-1}\mathrm{d}\tau.\tag{22}$$

After simplification, we get

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\times\left\{\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x)+(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right\}\leq\left(h^{(n)}(u)+h^{(n)}(v)\right)-\frac{h^{(n)}(x)+h^{(n)}(y)}{2}.\tag{23}$$

Now, concatenating (18) and (23), we get (14).

Now, we introduce some new lemmas involving Caputo fractional derivatives. □

Lemma 1. *Suppose that* $h\colon[u,v]\longrightarrow R$ *is a differentiable mapping on* (u,v) *with* $u<v$ *and* $h\in C^{n+1}[u,v]$, *then the following equality for Caputo fractional derivatives holds:*

$$\frac{h^{(n)}(u+v-x)+h^{(n)}(u+v-y)}{2}-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}$$
$$\times\left\{\left({}^{c}D^{\alpha}_{(u+v-y)^{+}}h\right)(u+v-x)+(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-x)^{-}}h\right)(u+v-y)\right\}$$
$$=\frac{y-x}{2}\int_{0}^{1}\left(\tau^{n-\alpha}-(1-\tau)^{n-\alpha}\right)h^{n+1}(u+v-(\tau x+(1-\tau)y))\,\mathrm{d}\tau, \tag{24}$$

where $\forall x, y\in[u,v]$, $\alpha>0$, $\tau\in[0,1]$, *and* $\Gamma(\cdot)$ *is the gamma function.*

Proof. It suffices to note that

$$I=\frac{y-x}{2}\{I_1-I_2\}, \tag{25}$$

where

$$\begin{aligned}
I_1&=\int_0^1\left(\tau^{n-\alpha}\right)h^{n+1}(u+v-(\tau x+(1-\tau)y))\,\mathrm{d}\tau\\
&=\frac{h^{(n)}(u+v-x)}{y-x}-\frac{n-\alpha}{y-x}\int_0^1\tau^{n-\alpha-1}h^{(n)}(u+v-(\tau x+(1-\tau)y))\,\mathrm{d}\tau\\
&=\frac{h^{(n)}(u+v-x)}{y-x}-\frac{\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left\{(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-x)^{-}}h\right)(u+v-y)\right\}\\
I_2&=\int_0^1(1-\tau)^{n-\alpha}f^{n+1}(u+v-(\tau x+(1-\tau)y))\,\mathrm{d}\tau\\
&=-\frac{h^{(n)}(u+v-y)}{y-x}+\frac{n-\alpha}{y-x}\int_0^1\int_0^1(1-\tau)^{n-\alpha-1}h^{(n)}(u+v-(\tau x+(1-\tau)y))\,\mathrm{d}\tau\\
&=-\frac{h^{(n)}(u+v-y)}{y-x}+\frac{\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left\{\left({}^{c}D^{\alpha}_{(u+v-y)^{+}}h\right)(u+v-x)\right\},
\end{aligned} \tag{26}$$

and replacing the values of I_1 and I_2 in (25), we get (24). □

Remark 1. If we take $x=u$ and $y=v$ in Lemma 1, then it reduces to Remark 2.5 in [25].

Lemma 2. *Suppose that* h: $[u,v]\longrightarrow R$ *is a differentiable mapping on* (u,v) *with* $u<v$ *and* $h\in C^{n+1}[u,v]$, *then the following equality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\\
&\left\{\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x)+(-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right\}\\
&=\frac{y-x}{4}\left[\int_0^1\tau^{n-\alpha}h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\mathrm{d}\tau-\int_0^1\tau^{n-\alpha}h^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\mathrm{d}\tau\right].
\end{aligned} \tag{27}$$

Proof. It suffices to note that

$$I=\frac{y-x}{4}\{I_1-I_2\}, \tag{28}$$

$$\begin{aligned}
I_1&=\int_0^1 t^{n-\alpha}h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\mathrm{d}\tau\\
&=\frac{2}{y-x}h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2(n-\alpha)}{y-x}\int_0^1\tau^{n-\alpha-1}h^{(n)}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\mathrm{d}\tau\\
&=\frac{2}{y-x}h^{(n)}\left(u+v-\frac{x+y}{2}\right)-(-1)^{n}\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y),
\end{aligned} \tag{29}$$

$$\begin{aligned} I_2 &= \int_0^1 \tau^{n-\alpha} h^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\mathrm{d}\tau \\ &= -\frac{2}{y-x}h^{(n)}\left(u+v-\frac{x+y}{2}\right)+\frac{2(n-\alpha)}{y-x}\int_0^1 \tau^{n-\alpha-1}h^{(n)}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\mathrm{d}\tau \\ &= -\frac{2}{y-x}h^{(n)}\left(u+v-\frac{x+y}{2}\right)+\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x), \end{aligned} \tag{30}$$

and combining (29) and (30) with (28), we get (27). □

Remark 2. If we take $x = u$ and $y = v$ in Lemma 2, then it reduces to Lemma 2 in [24].

Theorem 4. *Suppose that h: $[u,v] \longrightarrow R$ is a differentiable mapping on (u,v) with $u<v$ and $h \in C^{n+1}[u,v]$. If $|h^{n+1}|$ is convex function on $[u,v]$, then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned} &\left|\frac{h^{(n)}(u+v-x)+h^{(n)}(u+v-y)}{2}-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\left(\left({}^cD^{\alpha}_{(u+v-y)^+}f\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-x)^-}h\right)(u+v-y)\right)\right| \\ &\le \frac{y-x}{n-\alpha+1}\left(1-\frac{1}{2^{n-\alpha}}\right)\left\{\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|-\left(\frac{\left|h^{n+1}(x)\right|+h^{n+1}(y)}{2}\right)\right\}, \end{aligned} \tag{31}$$

where $\forall\, x, y \in [u,v], \alpha > 0,\ \tau \in [0,1]$, and $\Gamma(.)$ is the gamma function.

Proof. By using Lemma 1 and the Jensen–Mercer inequality, we get

$$\begin{aligned} &\left|\frac{h^{(n)}(u+v-x)+h^{(n)}(u+v-y)}{2}-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\left(\left({}^cD^{\alpha}_{(u+v-y)^+}h\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-x)^-}h\right)(u+v-y)\right)\right| \\ &\le \frac{y-x}{2}\int_0^1\left|\tau^{n-\alpha}-(1-\tau)^{n-\alpha}\right|\left|h^{n+1}(u+v-(\tau x+(1-\tau)y))\right|d\tau \\ &\le \frac{y-x}{2}\int_0^1\left|\tau^{n-\alpha}-(1-\tau)^{n-\alpha}\right|\left\{\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|-\left(\tau\left|h^{n+1}(x)\right|+(1-\tau)\left|h^{n+1}(y)\right|\right)\right\}d\tau \\ &\le \frac{y-x}{2}\left[I_1+I_2\right], \end{aligned} \tag{32}$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}}\left((1-\tau)^{n-\alpha}-\tau^{n-\alpha}\right)\left\{\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|-\left(\tau\left|h^{n+1}(x)\right|+(1-\tau)\left|h^{n+1}(y)\right|\right)\right\} \\ &= \left(\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|\right)\left(\frac{1}{n-\alpha+1}-\frac{2^{-n+\alpha}}{n-\alpha+1}\right) \\ &\quad -\left\{\left|h^{n+1}(x)\right|\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{2^{-n+\alpha-2}}{n-\alpha+2}\right)+\left|h^{n+1}(y)\right|\left(\frac{1}{(n-\alpha+2)}-\frac{2^{-n+\alpha-1}}{n-\alpha+1}\right)\right\} \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^{1} \left(\tau^{n-\alpha} - (1-\tau)^{n-\alpha}\right)\left\{\left|h^{n+1}(u)\right| + \left|h^{n+1}(v)\right| - \left(\tau\left|h^{n+1}(x)\right| + (1-\tau)\left|h^{n+1}(y)\right|\right)\right\} \\ &= \left(\left|h^{n+1}(u)\right| + \left|h^{n+1}(v)\right|\right)\left(\frac{1}{(n-\alpha+1)} - \frac{2^{-n+\alpha}}{n-\alpha+1}\right) \\ &\quad - \left\{\left|h^{n+1}(x)\right|\left(\frac{1}{(n-\alpha+2)} - \frac{2^{-n+\alpha-1}}{n-\alpha+1}\right) + \left|h^{n+1}(y)\right|\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{2^{-n+\alpha-1}}{n-\alpha+1}\right)\right\}, \end{aligned} \tag{33}$$

and putting the values of I_1 and I_2 in (32), we get (31). □

Remark 3. If we take $x = u$ and $y = v$ in Theorem 4, then it reduces to Corollary 2.7 in [25].

Theorem 5. *Suppose that h: $[u, v] \longrightarrow R$ is a differentiable mapping on (u, v) with $u < v$ and $h \in C^{n+1}[u, v]$. If $|h^{n+1}|$ is a convex function on $[u, v]$, then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned} &\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right) - \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x)\right.\right. \\ &\quad \left.\left. + (-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right]\right| \\ &\le \frac{y-x}{2(n-\alpha+1)}\left\{\left|h^{n+1}(u)\right| + \left|h^{n+1}(b)\right| - \left(\frac{\left|h^{n+1}(x)\right| + \left|h^{n+1}(y)\right|}{2}\right)\right\}, \end{aligned} \tag{34}$$

where $\forall x, y \in [u, v]$, $\alpha > 0$, $\tau \in [0, 1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. By using Lemma 2 and the Jensen–Mercer inequality, we get

$$\begin{aligned} &\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right) - \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{+}}h\right)(u+v-x) + (-1)^{n}\left({}^{c}D^{\alpha}_{(u+v-(x+y/2))^{-}}h\right)(u+v-y)\right]\right| \\ &\le \frac{y-x}{4}\left[\int_0^1 \tau^{n-\alpha}\left|h^{n+1}\left(u+v-\left(\frac{\tau}{2}x + \frac{2-\tau}{2}y\right)\right)\right|\mathrm{d}\tau\right. \\ &\quad \left. + \int_0^1 \tau^{n-\alpha}\left|h^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x + \frac{\tau}{2}y\right)\right)\right|\mathrm{d}\tau\right] \\ &\le \frac{y-x}{4}\left[\int_0^1 \tau^{n-\alpha}\left\{\left|h^{n+1}(u)\right| + \left|h^{n+1}(v)\right| - \left(\frac{\tau}{2}\left|h^{n+1}(x)\right| + \frac{(2-\tau)}{2}\left|h^{n+1}(y)\right|\right)\right\}\mathrm{d}\tau\right. \\ &\quad \left. + \int_0^1 \tau^{n-\alpha}\left\{\left|h^{n+1}(u)\right| + \left|h^{n+1}(v)\right| - \left(\frac{(2-\tau)}{2}\left|h^{n+1}(x)\right| + \frac{\tau}{2}\left|h^{n+1}(y)\right|\right)\right\}\mathrm{d}\tau\right] \\ &\le \frac{y-x}{2(n-\alpha+1)}\left\{\left|h^{n+1}(u)\right| + \left|h^{n+1}(v)\right| - \left(\frac{\left|h^{n+1}(x)\right| + \left|h^{n+1}(y)\right|}{2}\right)\right\}, \end{aligned} \tag{35}$$

which completes the proof. □

Theorem 6. *Suppose that $h\colon [u,v] \longrightarrow R$ is a differentiable mapping on (u,v) with $u<v$ and $h \in C^{n+1}[u,v]$. If $|h^{n+1}|^q$ is a convex function on $[u,v]$ and $q>1$, then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+v-y)\right]\right|\\&\le\frac{y-x}{16}\left(\frac{4}{np-\alpha p+1}\right)^{(1/p)}\left[4^{(1/q)}\cdot 2\left(\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|\right)-\left(3^{(1/q)}+1\right)\left(\left|h^{n+1}(x)\right|+\left|h^{n+1}(y)\right|\right)\right],\end{aligned}\tag{36}$$

where $\forall\, x,y\in[u,v]$, $\alpha>0$, $\tau\in[0,1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. By using Lemma 2 and applying the famous Hölder integral inequality, we get

$$\begin{aligned}&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)\right.\right.\\&\left.\left.+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+v-y)\right]\right|\\&\le\frac{y-x}{4}\left[\left(\int_0^1\tau^{(n-\alpha)p}\mathrm{d}\tau\right)^{(1/p)}\cdot\left(\int_0^1\left|h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right.\\&\left.+\left(\int_0^1\tau^{(n-\alpha)p}\mathrm{d}\tau\right)^{(1/p)}\cdot\left(\int_0^1\left|\tau^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right]\\&\le\frac{y-x}{4}\left(\frac{1}{np-\alpha p+1}\right)^{(1/p)}\\&\times\left[\left\{\int_0^1\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q-\left(\frac{\tau}{2}\left|h^{n+1}(x)\right|^q+\frac{2-\tau}{2}\left|h^{n+1}(y)\right|^q\right)\right)\mathrm{d}\tau\right\}^{(1/q)}\right.\\&\left.+\left\{\int_0^1\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q-\left(\frac{2-\tau}{2}\left|h^{n+1}(x)\right|^q+\frac{\tau}{2}\left|h^{n+1}(y)\right|^q\right)\right)\mathrm{d}\tau\right\}^{(1/q)}\right]\\&\le\frac{y-x}{4}\left(\frac{1}{np-\alpha p+1}\right)^{(1/p)}\left[\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q-\left(\frac{\left|h^{n+1}(x)\right|^q+3\left|h^{n+1}(y)\right|^q}{4}\right)\right)^{(1/q)}\right.\\&\left.+\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q-\left(\frac{3\left|h^{n+1}(x)\right|^q+\left|h^{n+1}(y)\right|^q}{4}\right)\right)^{(1/q)}\right].\end{aligned}\tag{37}$$

By applying Minkowski's inequality, we get

$$\begin{aligned}&\frac{y-x}{16}\left(\frac{4}{np-\alpha p+1}\right)^{(1/p)}\left\{\left\{4\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q\right)-\left(\left|h^{n+1}(x)\right|^q+3\left|h^{n+1}(y)\right|^q\right)\right\}^{(1/q)}\right.\\&\left.+\left\{4\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q\right)-\left(3\left|h^{n+1}(x)\right|^q+\left|h^{n+1}(y)\right|^q\right)\right\}^{(1/q)}\right\}\\&\le\frac{y-x}{16}\left(\frac{4}{np-\alpha p+1}\right)^{(1/p)}\left[4^{(1/q)}.2\left(\left|h^{n+1}(u)\right|+\left|h^{n+1}(v)\right|\right)-\left(3^{(1/q)}+1\right)\left(\left|h^{n+1}(x)\right|+\left|h^{n+1}(y)\right|\right)\right],\end{aligned}\tag{38}$$

which completes the proof. □

3. New Hölder's and Improved Iscan Inequalities

Theorem 7. *Suppose that $h\colon [u,v] \longrightarrow R$ is a differentiable mapping on (u,v) with $u<v$ and $h \in C^{n+1}[u,v]$. If $|h^{n+1}|^q$ is a convex function on $[u,v]$ and $q>1$, then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+v-y)\right]\right| \\
&\le \frac{y-x}{4}\left\{\left\{\left(\frac{1}{((n-\alpha)p+1)((n-\alpha)p+2)}\right)^{(1/p)}\cdot\left(\frac{1}{2}\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q\right)\right.\right.\right. \\
&\quad\left.-\left(\frac{1}{12}\left|h^{n+1}(x)\right|^q+\frac{5}{12}\left|h^{n+1}(y)\right|^q\right)\right)^{(1/q)}+\left(\frac{1}{((n-\alpha)p+2)}\right)^{(1/p)}\left(\frac{1}{2}\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q\right)\right. \\
&\quad\left.\left.-\left(\frac{1}{6}\left|h^{n+1}(x)\right|^q+\frac{1}{3}\left|h^{n+1}(y)\right|^q\right)\right)^{(1/q)}\right\}+\left\{\left(\frac{1}{((n-\alpha)p+1)((n-\alpha)p+2)}\right)^{(1/p)}\left(\frac{1}{2}\left(\left|h^{n+1}(u)\right|^q\right.\right.\right. \\
&\quad\left.+\left|h^{n+1}(v)\right|^q\right)-\left(\frac{5}{12}\left|h^{n+1}(x)\right|^q+\frac{1}{12}\left|h^{n+1}(y)\right|^q\right)\Bigg)^{(1/q)} \\
&\quad+\left(\frac{1}{((n-\alpha)p+2)}\right)^{(1/p)} \\
&\quad\left.\left.\times\left(\frac{1}{2}\left(\left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q\right)-\left(\frac{1}{3}\left|h^{n+1}(x)\right|^q+\frac{1}{6}\%\left|h^{n+1}(y)\right|^q\right)\right)^{(1/q)}\right\}\right\},
\end{aligned} \tag{39}$$

where $\forall\, x,y \in [u,v]$, $\alpha>0$, $\tau \in [0,1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. By using Lemma 2 with Jensen–Mercer inequality and applying the Hölder–Iscan integral inequality (Theorem 1.4 [27]), we get

$$\begin{aligned}
&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)\right.\right. \\
&\quad\left.\left.+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+v-y)\right]\right| \\
&\le \frac{y-x}{4}\left\{\left(\int_0^1(1-\tau)\tau^{np-\alpha p}\mathrm{d}\tau\right)^{(1/p)}\left(\int_0^1(1-\tau)\left|h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right. \\
&\quad\left.+\left(\int_0^1\tau^{np-\alpha p+1}\mathrm{d}\tau\right)^{(1/p)}\left(\int_0^1\tau\left|h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right\} \\
&\quad+\left\{\left(\int_0^1(1-\tau)\tau^{np-\alpha p}\mathrm{d}\tau\right)^{(1/p)}\left(\int_0^1(1-\tau)\left|f^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right. \\
&\quad\left.+\left(\int_0^1\tau^{np-\alpha p+1}\mathrm{d}\tau\right)^{(1/p)}\left(\int_0^1\tau\left|h^{n+1}\left(u+v-\left(\frac{2-\tau}{2}x+\frac{\tau}{2}y\right)\right)\right|^q\mathrm{d}\tau\right)^{(1/q)}\right\}.
\end{aligned} \tag{40}$$

By the convexity of $|h^{n+1}|^q$,

$$\left|h^{n+1}\left(u+v-\left(\frac{\tau}{2}x+\frac{2-\tau}{2}y\right)\right)\right|^q \le \left|h^{n+1}(u)\right|^q+\left|h^{n+1}(v)\right|^q-\left(\frac{\tau}{2}\left|h^{n+1}(x)\right|^q+\frac{2-\tau}{2}\left|h^{n+1}(y)\right|^q\right). \tag{41}$$

By using calculus tools, one can have required result. □

Theorem 8. *Suppose that $h\colon [u,v]\longrightarrow R$ is a differentiable mapping on (u,v) with $u<v$ and $h\in C^{n+1}[u,v]$. If $|h^{n+1}|^q$ is a convex function on $[u,v]$, $p>1$, and $q=(p/p-1)$, then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+v-y)\right]\right| \\
&\le\frac{y-x}{4}\left\{\left\{\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right)^{1-(1/q)}\right.\right. \\
&\quad\times\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+1)(n-\alpha+2)}-\left(\frac{\left|h^{n+1}(x)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+1)(n-\alpha+2)(n-\alpha+3)}\right)\right) \\
&\quad+\left(\frac{1}{(n-\alpha+2)}\right)^{1-(1/q)} \\
&\quad\left.\times\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+2)}-\left(\frac{\left|h^{n+1}(x)\right|^q}{2(n-\alpha+3)}+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}\right)\right)^{(1/q)}\right\} \\
&\quad+\left\{\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right)^{1-(1/q)}\right. \\
&\qquad\times\left(\frac{\left|h^{n+1}(a)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+1)(n-\alpha+2)}-\left(\frac{(n-\alpha+5)\left|h^{n+1}(x)\right|^q}{2(n-\alpha+1)(n-\alpha+2)(n-\alpha+3)}\right.\right. \\
&\qquad\left.\left.+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}\right)\right)^{(1/q)} \\
&\qquad+\left(\frac{1}{(n-\alpha+2)}\right)^{1-(1/q)} \\
&\qquad\left.\left.\times\left(\frac{\left|h^{n+1}(a)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+2)}-\left(\frac{(n-\alpha+4)\left|h^{n+1}(x)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+3)}\right)\right)^{(1/q)}\right\}\right\},
\end{aligned} \tag{42}$$

where $\forall x,y\in[u,v]$, $\alpha>0$, $\tau\in[0,1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. By using Lemma 2 with Jensen–Mercer inequality and applying the improved power-mean integral inequality (Theorem 1.5 [27]), we get

$$\begin{aligned}
&\left|h^{(n)}\left(u+v-\frac{x+y}{2}\right)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\left[\left({}^cD^{\alpha}_{(u+v-(x+y/2))^+}h\right)(u+v-x)+(-1)^n\left({}^cD^{\alpha}_{(u+v-(x+y/2))^-}h\right)(u+(v-y))\right]\right| \\
&\le\frac{y-x}{4}\left\{\left\{\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right)^{1-(1/q)}\times\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+1)(n-\alpha+2)}-\left(\frac{\left|h^{n+1}(x)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}\right.\right.\right. \\
&\quad\left.\left.+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+1)(n-\alpha+2)(n-\alpha+3)}\right)\right)^{(1/q)}+\left(\frac{1}{(n-\alpha+2)}\right)^{1-(1/q)}\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+2)}-\left(\frac{\left|h^{n+1}(x)\right|^q}{2(n-\alpha+3)}\right.\right.
\end{aligned}$$

$$+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}\Bigg)\Bigg)^{(1/q)}\Bigg\}+\Bigg\{\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right)^{1-(1/q)}\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+1)(n-\alpha+2)}-\right.$$
$$\left(\frac{(n-\alpha+5)\left|h^{n+1}(x)\right|^q}{2(n-\alpha+1)(n-\alpha+2)(n-\alpha+3)}+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}\right)^{(1/q)}+\left(\frac{1}{(n-\alpha+2)}\right)^{1-(1/q)} \tag{43}$$
$$\times\left(\frac{\left|h^{n+1}(u)^q\right|+\left|h^{n+1}(v)\right|^q}{(n-\alpha+2)}-\left(\frac{(n-\alpha+4)\left|h^{n+1}(x)\right|^q}{2(n-\alpha+2)(n-\alpha+3)}+\frac{\left|h^{n+1}(y)\right|^q}{2(n-\alpha+3)}\right)\right)^{(1/q)}\Bigg\}\Bigg\}.$$

After some simplifications, we get required result. □

4. Conclusion

In this paper, we presented Hermite–Hadamard–Mercer inequalities for convex functions via Caputo fractional derivatives. We also developed some new bounds using Hölder–Iscan and improved power-mean integral inequalities. Our results will attract attentions of many researchers working in the field of inequalities and enable them to think further for other generalized convex functions. One may think to extend these results for higher order convex functions. The works above can also build up for a convex function of two variables. For further directions, we refer to [28–33].

Data Availability

All data are included within this paper.

Conflicts of Interest

The authors of this paper declare that they have no conflicts of interest.

Acknowledgments

The research of Saad Ihsan Butt has been fully supported by the H.E.C. of Pakistan under NRPU project 7906.

References

[1] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.

[2] I. A. Baloch and S. S. Dragomir, "New inequalities based on harmonic log-convex functions," *Open Journal of Mathematical Analysis*, vol. 3, no. 2, pp. 103–105, 2019.

[3] S. M. Kang, G. Farid, W. Nazeer, and B. Tariq, "Hadamard and Fejér–Hadamard inequalities for extended generalized fractional integrals involving special functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 119, 2018.

[4] M. Z. Sarikaya and N. Alp, "On Hermite-Hadamard-Fejér type integral inequalities for generalized convex functions via local fractional integrals," *Open Journal of Mathematical Sciences*, vol. 3, no. 1, pp. 273–284, 2019.

[5] S. M. Kang, G. Farid, W. Nazeer, and S. Mehmood, ".(hm)-convex functions and associated fractional Hadamard and Fejér–Hadamard inequalities via an extended generalized Mittag-Leffler function," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–10, 2019.

[6] S. Kermausuor, "Simpson's type inequalities for strongly (s, m)-convex functions in the second sense and applications," *Open Journal of Mathematical Sciences*, vol. 3, no. 1, pp. 74–83, 2019.

[7] D. Uçar, V. F. Hatipog, and A. Akincali, "Fractional integral inequalities on time scales," *Open Journal of Mathematical Sciences*, vol. 2, no. 1, pp. 361–370, 2018.

[8] S. Kang, G. Abbas, G. Farid, and W. Nazeer, "A generalized Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results," *Mathematics*, vol. 6, no. 7, p. 122, 2018.

[9] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[10] J. E. Pečarić, D. S. Mitrinovic, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[11] A. Matkovic, J. Pečarić, and I. Perić, "A variant of Jensens inequality of Mercers type for operators with applications," *Linear Algebra and Its Applications*, vol. 418, pp. 551–564, 2006.

[12] S. Abramovich, J. Baric, and J. Pečarić, "A variant of Jessens inequality of Mercers type for superquadratic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 3, p. 62, 2008.

[13] J. Baric and A. Matkovic, "Bounds for the normalized Jensen Mercer functional," *Journal of Mathematical Inequalities*, vol. 3, no. 4, pp. 529–541, 2009.

[14] S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf, and Y. M. Chu, "New fractional estimates for hermite-hadamard-mercer type inequalities," *Frontiers in Applied Mathematics and Statistics*, vol. 6, no. 5, 2020.

[15] T. Furuta, H. Micic, J. Pecaric, and Y. Seo, *Mond Pecaric Method in Operator Inequalities*, Zagreb Element, Zagreb, Croatia, 2005.

[16] M. M. Ali and A. R. Khan, "Generalized integral Mercer's inequality and integral means," *Journal of Inequalities and Applications*, vol. 10, no. 1, pp. 60–76, 2019.

[17] M. Kian and M. S. Moslehian, "Refinements of the operator Jensen Mercer inequality," *Electronic Journal of Linear Algebra*, vol. 26, pp. 742–753, 2013.

[18] Y. J. Cho, M. Matic, and J. Pečarić, "Two mappings in connection to Jensen inequality," *Pan-American Mathematical Journal*, vol. 12, pp. 43–50, 2002.

[19] C. Li and W. Deng, "Remarks on fractional derivatives," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 777–784, 2007.

[20] A. Ardjouni, A. Lachouri, and A. Djoudi, "Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations," *Open Journal of Mathematical Analysis*, vol. 3, no. 2, pp. 106–111, 2019.

[21] D. Vivek, E. M. Elsayed, and K. Kanagarajan, "On the oscillation of fractional differential equations via ψ-Hilfer fractional derivative," *Engineering and Applied Science Letter*, vol. 2, no. 3, pp. 1–6, 2019.

[22] A. Ardjouni and A. Djoudi, "Positive solutions for nonlinear Caputo-Hadamard fractional differential equations with integral boundary conditions," *Open Journal of Mathematical Analysis*, vol. 3, no. 1, pp. 62–69, 2019.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Derivatial Equations*, North-Holland Mathematical Studies Elsevier, New York, NY, USA, 2006.

[24] G. Farid, S. Naqvi, and A. U. Rehman, "A version of the Hadamard inequality for Caputo fractional derivatives and related results," p. 20, RGMIA Research Report Collection, Melbourne, Australia, 2017.

[25] S. Naqvi, G. Farid, and B. Tariq, *Caputo Fractional Integral Inequalities via M Convex Function*, RGMIA Research Report Collection, Melbourne, Australia, 2014.

[26] L. N. Mishra, Q. U. Ain, G. Farid, and A. U. Rehman, ".k–Fractional integral inequalities for (h, m)–convex functions via Caputo k– fractional derivatives," *The Korean Journal of Mathematics*, vol. 27, no. 2, pp. 357–374, 2019.

[27] S. Ozcan and I. Imdat, "Some new Hermite-Hadamard type inequalities for s-convex functions and their applications," *Journal of Inequalities and Applications*, vol. 1, p. 201, 2019.

[28] B. Sambo and G. B. Meller, "On certain subclasses of p-valent functions with negative coefficients defined by a generalized differential operator," *Open Journal of Mathematical Analysis*, vol. 3, no. 2, pp. 32–41, 2019.

[29] A. Abdalmonem, O. Abdalrhman, and S. Tao, "Multilinear fractional integral with rough kernel on variable exponent Morrey-Herz spaces," *Open Journal of Mathematical Sciences*, vol. 3, no. 1, pp. 167–183, 2019.

[30] A. Abdalmonem, O. Abdalrhman, and H. E. Mohammed, "Commutators of fractional integral with variable kernel on variable exponent Herz-Morrey spaces," *Open Journal of Mathematical Analysis*, vol. 3, no. 1, pp. 19–29, 2019.

[31] A. Abdalmonem, O. Abdalrhman, and S. Tao, "Boundedness of Littlewood-Paley operators with variable kernel on weighted Herz spaces with variable exponent," *Engineering and Applied Science Letters*, vol. 1, no. 2, p. 1022, 2018.

[32] G. Farid, W. Nazeer, M. Saleem, S. Mehmood, and S. Kang, "Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications," *Mathematics*, vol. 6, no. 11, p. 248, 2018.

[33] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, "Hermite–Hadamard-type inequalities for functions whose derivatives are p-convex via fractional integrals," *Journal of Inequalities and Applications*, vol. 1, pp. 1–16, 2019.