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Research Article

Pseudodifferential Operators on Weighted Hardy Spaces

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We study two sufficient conditions for the boundedness of a class of pseudodifferential operators T with symbols in the Hölmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$ on weighted Hardy spaces $H^1_\omega(\mathbb{R}^n)$, where ω belongs to Muckenhoupt class A_p. The first one is an estimate from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$. We get a better range of admissible p and m. The second one is a weighted version bounded for the operators T on $H^1_\omega(\mathbb{R}^n)$, and it is an addition to the literature.

1. Introduction

The purpose of this paper is to study some sufficient conditions for the boundedness of pseudodifferential operators T on weighted Hardy space $H^1_\omega(\mathbb{R}^n)$, where the operators T have symbols in the Hölmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$. As in [1], for $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, a symbol $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|} \tag{1}$$

holds for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, where $C_{\alpha,\beta}$ is independent of x and ξ. We now assume that the symbol $a(x,\xi)$ is smooth in both the spatial variable x and the frequency variable ξ.

Given $f \in C_0^\infty(\mathbb{R}^n)$, the pseudodifferential operator $T \in \mathscr{L}^m_{\rho,\delta}$ associated with the symbol $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is given by

$$Tf(x) = \int_{\mathbb{R}^n} a(x,\xi) e^{2\pi i x\cdot\xi} \widehat{f}(\xi) d\xi, \tag{2}$$

where $\widehat{f}$ denotes the Fourier transform of f. Moreover, we can express T by a kernel $K(x,y)$ as (see, e.g., [2])

$$Tf(x) = \int K(x,y) f(y) dy. \tag{3}$$

Pseudodifferential operators play an important role in the theory of partial differential equations. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ coincide with the Lebesgue spaces $L^p(\mathbb{R}^n)$ when $p > 1$. The L^p and weighted L^p boundedness of the operator $T \in \mathscr{L}^m_{\rho,\delta}$ have been extensively studied. We refer to [1, 2, 3, 4] for the L^p bounds and [5, 6, 7, 8] for the weighted L^p bounds.

For $p \in (0,1]$, there is an estimate from $L^1(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ for the pseudodifferential operator $T \in \mathscr{L}^m_{\rho,\delta}$ (cf. [5]). As known, the Hardy space $H^p(\mathbb{R}^n)$ is an advantageous substitute for $L^p(\mathbb{R}^n)$. The behavior of the pseudodifferential operator T on $H^p(\mathbb{R}^n)$ has attracted a lot of interest. For example, Alvarez and Hounie [5] have found that the pseudodifferential operator T with symbol in $S^m_{\rho,\delta}(\mathbb{R}^n)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where $0 < \rho \le 1$, $0 \le \delta < 1$ and $m \le -(n/2)(1-\rho) + \min\{0, (n(\rho-\delta)/2)\}$. Hounie and Kapp [9] have shown that the operator T with $0 \le \delta \le \rho < 1$ and $m = -(n(1-\rho))/2$ is bounded from the local Hardy space $h^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Yabuta [10] has proved the operator T involving a modulus of continuity $w(t)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

The bounds of the pseudodifferential operator T from the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ into the weighted Lebesgue space $L^1_\omega(\mathbb{R}^n)$ have also been studied. Yabuta [11] has found that the operator T is bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$, where $T \in \mathscr{L}^0_{1,\delta}$ and $\omega \in A_1$. In view of this, it is natural to look for a wide range of operator T in $\mathscr{L}^m_{\rho,\delta}$ to study the bounds on the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$.

In this paper, we establish two estimates for the pseudodifferential operator T with symbols in $S^m_{\rho,\delta}(\mathbb{R}^n)$. The first one is an estimate from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$. We extend the result in Yabuta [11] to $\omega \in A_p$ with $1 \le p < 1 + (\varepsilon/n)$ and the operator T with $0 < \rho \le 1$, $0 \le \delta < 1$, and $-(n+1) < m \le -(n+1)(1-\rho)$. Our first main result is stated as follows.

Theorem 1. *Let $\varepsilon = min\{1, (1+m+n/\rho)\}$, $p \in [1, 1+(\varepsilon/n))$, $\omega \in A_p$ and $T \in \mathscr{L}^m_{\rho,\delta}$ with $0<\rho\le 1$, $0\le\delta<1$. If $-(n+1) < m \le -(n+1)(1-\rho)$, then T is bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$, i.e., there exists a constant $C>0$ such that*

$$\|Tf\|_{L^1_\omega(\mathbb{R}^n)} \le C\|f\|_{H^1_\omega(\mathbb{R}^n)}. \tag{4}$$

The second one is an estimate on weighted Hardy spaces $H^1_\omega(\mathbb{R}^n)$ for the pseudodifferential operator T. It is well known that under certain conditions of m, ρ, δ, the operator T is bounded on $h^1(\mathbb{R}^n)$ (cf. [9, 12]). Alvarez and Hounie [5] have found that the pseudodifferential operator T is bounded on $H^1(\mathbb{R}^n)$, where $T \in \mathscr{L}^m_{\rho,\delta}$ with $0<\rho\le 1$, $0\le\delta<1$, and $m \le -\lambda + \min\{0, (n(\rho-\delta)/2)\}$ for some $((n(1-\rho))/2) \le \lambda < n/2$. It is natural to look for a weighted version estimate on $H^1_\omega(\mathbb{R}^n)$. We now state our second main result.

Theorem 2. *Let $\mu = ((1+n+m)/\rho) - n$, $p \in [1, (1+\mu/n))$, $\omega \in A_p$, and $T \in \mathscr{L}^m_{\rho,\delta}$ with $0<\rho\le 1$, $0\le\delta<1$. Assume $\rho n - (n+1) < m \le -(n+1)(1-\rho)$ and $T^*1 = 0$. Then, T is bounded on $H^1_\omega(\mathbb{R}^n)$; i.e., there exists a constant $C>0$ such that*

$$\|Tf\|_{H^1_\omega(\mathbb{R}^n)} \le C\|f\|_{H^1_\omega(\mathbb{R}^n)}. \tag{5}$$

The remainder of this paper is organized as follows. In Section 2, we present some definitions and well-known results we use later. The aim of Section 3 is to set up the estimate from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$ for pseudodifferential operators T in $\mathscr{L}^m_{\rho,\delta}$. We develop a method to handle $||Ta||_{L^1_\omega(2^{k+1}B/2^kB)}$ (see Proposition 1). The aim of Section 4 is to establish the estimate on weighted Hardy spaces $H^1_\omega(\mathbb{R}^n)$ for pseudodifferential operators T in $\mathscr{L}^m_{\rho,\delta}$.

Most of the notations we use are standard. C denotes a constant that may change from line to line and we write $a \lesssim b$ as shorthand for $a \le Cb$. If $a \lesssim b$ and $b \lesssim a$, we mean $a \sim b$. For a measurable set A, $|A|$ denotes the Lebesgue measure of A and χ_A the characteristic function. B will always denote a ball, and $tB\,(t>0)$ denotes the ball B dilated by t.

2. Notations and Auxiliary Lemma

In this section, we first present an auxiliary lemma about the pseudodifferential operator T associated with the kernel $K(x,y)$. Let $\mathscr{S}(\mathbb{R}^n)$ be the class of Schwartz functions and $\mathscr{S}'(\mathbb{R}^n)$ be its dual space. The space of C^∞-function with compact support is denoted by $C^\infty_0(\mathbb{R}^n)$. Pseudodifferential operators are bounded from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$ and so possess distribution kernels $K(x,y) \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Then, the following formula for the kernel is useful (cf. Proposition 3.1 in [9]; see also [5]).

Lemma 1. *Let $a(x,\xi) \in \mathscr{S}^m_{\rho,\delta}(\mathbb{R}^n)$ with $0<\rho\le 1$, $0\le\delta<1$, and associate with the pseudodifferential operator $T \in \mathscr{L}^m_{\rho,\delta}$. Then, the distribution kernel $K(x,y)$ of T is smooth away from the diagonal $\{(x,x)\colon x \in \mathbb{R}^n\}$ and is given by*

$$K(x,y) = \lim_{\varepsilon\longrightarrow 0}\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} a(x,\xi)\psi(\varepsilon\xi)\mathrm{d}\xi, \tag{6}$$

where $\psi \in C^\infty_0(\mathbb{R}^n)$ satisfies $\psi(\xi) = 1$ for $|\xi| \le 1$ and the limit is taken in $\mathscr{S}'(\mathbb{R}^n)$ and independent of the choice of ψ. If $M \in \mathbb{N}$ and $(M+m+n) > 0$, $K(x,y)$ satisfies the estimates

$$\sup_{|\alpha+\beta|=M}\left|D^\alpha_x D^\beta_y K(x,y)\right| \le C_M \frac{1}{|x-y|^{((M+m+n)/\rho)}}, x \ne y. \tag{7}$$

Moreover, for any multi-index $\alpha, \beta \in \mathbb{N}^n$ and $N \in \mathbb{N}$,

$$\sup_{|x-y|\ge 1/2}\left|x-y\,|^N\left|D^\alpha_x D^\beta_y K(x,y)\right| \le C_{\alpha,\beta,N}. \tag{8}$$

The following useful L^p_ω bound for the pseudodifferential operator T is obtained by Michalowski et al. [7].

Lemma 2. *Let $T \in \mathscr{L}^m_{\rho,\delta}$, $0\le\delta<1$, $0<\rho\le 1$, and $m < -n(1-\rho)$. Then, for each $p \in (1,\infty)$ and $\omega \in A_p$, there exists a constant $C>0$ such that*

$$\|Tf\|_{L^p_\omega(\mathbb{R}^n)} \le C\|f\|_{L^p_\omega(\mathbb{R}^n)}. \tag{9}$$

Remark 1. Obviously, the L^2_ω bounds of pseudodifferential operators T are established automatically.

Remark 2. $m \le -(n+1)(1-\rho)$ satisfies the condition of m in Lemma 2.

The following useful L^2 bound of the pseudodifferential operator $T \in \mathscr{L}^m_{\rho,\delta}$ is obtained by Alvarez and Hounie [5].

Lemma 3. *Let $T \in \mathscr{L}^m_{\rho,\delta}$, $0\le\delta<1, 0<\rho\le 1$, and $m < \min\{0, (n(\rho-\delta)/2)\}$. Then, the operator T is bounded on $L^2(\mathbb{R}^n)$.*

Remark 3. The range of m, p here is $m < -n(1-\rho)|(1/p) - (1/2)| + \min\{0, (n(\rho-\delta)/2)\}$ and $1<p<\infty$, respectively.

Remark 4. Obviously, $m \le -(n+1)(1-\rho)$ satisfies the condition of m in Lemma 3.

Let $p \in [1,\infty)$. A nonnegative locally integrable function ω belongs to Muckenhoupt class A_p, if there exists a constant $C>0$, such that for all balls $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B (\omega(x))^{-(1/p-1)}\mathrm{d}x\right)^{p-1} \le C \text{ if } 1<p<\infty,$$
$$\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x \le C\operatorname{essinf}_{x\in B}\omega(x) \quad \text{if} \quad p=1. \tag{10}$$

We denote $A_\infty = \cup_{p\ge 1} A_p$.

It is well known that $\omega \in A_p$ implies $\omega \in A_q$ for all $q>p$. Also, if $\omega \in A_p$, then $\omega \in A_q$ for some $q \in [1,p)$. We thus write $q_\omega = \inf\{p \ge 1\colon \omega \in A_p\}$ to denote the critical index of ω. For a measurable set E, we denote $\omega(E) = \int_E \omega(x)\mathrm{d}x$. The

following lemma provides a way to compare $|E|$ and $\omega(E)$ of a set E (see [13]).

Lemma 4. *Let $\omega \in A_p$ and $p \geq 1$. Then, there exists a constant $C > 0$ such that*

$$C\left(\frac{|E|}{|B|}\right)^p \leq \frac{\omega(E)}{\omega(B)}, \tag{11}$$

for all balls B and measurable subsets $E \subset B$.

Given a weight function ω on $\mathbb{R}^n$, we denote by $L^p_\omega(\mathbb{R}^n)$ the weighted Lebesgue space of all functions f satisfying

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x\right)^{(1/p)} < \infty. \tag{12}$$

When $p = \infty$, $L^\infty_\omega(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$. Analogous to the classical Hardy space, the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ can be defined in terms of maximal functions.

Definition 1. Let $\omega \in A_\infty$. The weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ is defined by

$$H^1_\omega(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) \colon \phi^*(f)(x) = \sup_{t>0}\left|\phi_t * f(x)\right| \in L^1_\omega(\mathbb{R}^n)\right\}, \tag{13}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a fixed function with $\int \phi \mathrm{d}x \neq 0$ and $\phi_t(y) = (1/t^n)\phi(y/t)$ for any $t > 0$. Moreover, we define $\|f\|_{H^1_\omega(\mathbb{R}^n)} = \|\phi^*(f)\|_{L^1_\omega(\mathbb{R}^n)}$.

Remark 5. Definition 1 is independent of the choice of ϕ (see [14]).

Definition 2. Let ω be a weight with the critical index q_ω. An $(1,\infty)$-atom with respect to ω is a function a satisfying

$$\operatorname{supp}(a) \subset B \quad \|a\|_{L^\infty} \leq \omega(B)^{-1}, \tag{14}$$

and $\int a(x)x^\alpha \mathrm{d}x = 0$ for every multi-index α with $|\alpha| \leq [n(q_\omega - 1)]$.

The Hardy space $H^1_\omega(\mathbb{R}^n)$ is a linear space spanned by all of $(1,\infty)$-atoms with respect to ω. Namely, $f \in H^1_\omega(\mathbb{R}^n)$ if and only if f can be written as (see [13])

$$f = \sum_j \lambda_j a_j, \tag{15}$$

in the sense of $\mathcal{S}'$, where each a_j is an $(1,\infty)$-atom with respect to ω and λ_j satisfies

$$\sum_j \left|\lambda_j\right| < \infty. \tag{16}$$

Moreover, $\|f\|_{H^1_\omega(\mathbb{R}^n)} = \inf\left\{\sum_{j=1}^\infty |\lambda_j| \colon f = \sum_{j=1}^\infty \lambda_j a_j\right\}$.

Definition 3. Let T be a pseudodifferential operator in $\mathcal{L}^m_{\rho,\delta}$. We say $T^*1 = 0$ if $\int_{\mathbb{R}^n} Ta(x)\mathrm{d}x = 0$ for all $a \in L^\infty(\mathbb{R}^n)$ with compact support and $\int_{\mathbb{R}^n} a(x)\mathrm{d}x = 0$.

3. The Proof of Theorem 1

In this section, we prove that the pseudodifferential operators T in $\mathcal{L}^m_{\rho,\delta}$ are bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$.

Proposition 1. *Let $\omega \in A_p$, $p \in [1,\infty)$ and $\varepsilon = \min\{1, (1 + m + n/\rho)\}$. Assume pseudodifferential operator $T \in \mathcal{L}^m_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$, and $-(n+1) < m \leq -(n+1)(1-\rho)$. Then, there exists a constant $C > 0$ such that*

$$\|Ta\|_{L^1_\omega(2^{k+1}B\backslash 2^k B)} \leq C2^{-k(\varepsilon + n(1-p))}, \tag{17}$$

holds for all $(1,\infty)$-atoms a with respect to ω, where $\operatorname{supp}(a) \subset B = B(x_0, r)$.

Proof. Inspired by the proof of Lemma 3.2 in [15], we consider two cases about the radius r. □

Case 1. When $r \geq 1$. For every $x \in 2^{k+1}B\backslash 2^k B$ and $y \in B(x_0, r)$, we have

$$|x - y| \geq \left|x - x_0\right| - \left|y - x_0\right| \geq 2^k r - r \geq 1. \tag{18}$$

Hence, by (8) and properties of $(1,\infty)$-atoms with respect to ω, we have

$$\begin{aligned} |Ta(x)| &= \left|\int_{\mathbb{R}^n} K(x,y)a(y)\mathrm{d}y\right| \leq \int_B |K(x,y)||a(y)|\mathrm{d}y \\ &\leq C\int_B \frac{1}{|x-y|^{n+1}}|a(y)|\mathrm{d}y \leq C\frac{1}{\left|x - x_0\right|^{n+1}}\int_B |a(y)|\mathrm{d}y \\ &\leq C2^{-k}\frac{|B|}{\left|2^k B\right|}\omega(B)^{-1}, \end{aligned} \tag{19}$$

for all $x \in 2^{k+1}B\backslash 2^k B$. Thus,

$$\|Ta\|_{L^1_\omega((2^{k+1}B)/2^k B)} \leq C2^{-k}\frac{|B|}{\left|2^k B\right|}\frac{\omega\left(2^{k+1}B\right)}{\omega(B)} \leq C2^{-k(\varepsilon + n(1-p))}. \tag{20}$$

Case 2. When $0 < r < 1$. For every $x \in ((2^{k+1}B)/2^k B)$ and $y \in B(x_0, r)$, by moments condition, we have

$$|Ta(x)| = \left|\int_{\mathbb{R}^n} K(x,y)a(y)\mathrm{d}y\right| \leq \int_B \left|K(x,y) - K(x,x_0)\right||a(y)|\mathrm{d}y. \tag{21}$$

By the mean value theorem, $1 + m + n > 0$, and (7), we have

$$|Ta(x)| \leq C\int_B \frac{\left|y - x_0\right|}{\left|x - x_0\right|^{(1+m+n)/\rho}}|a(y)| \leq C\frac{r}{\left(2^k r\right)^{(1+m+n)/\rho}}\frac{|B|}{\omega(B)}, \tag{22}$$

where we take $M = 1$ and use the fact that $|x - \xi| \sim |x - x_0|$ if $\xi \in B(x_0, r)$. Let us now consider two subcases.

Subcase 1. If $(2^k - 1)r \geq 1$, then, for any $y \in B(x_0, r)$ and $x \in 2^{k+1}B\backslash 2^k B$,

$$|x - y| \geq \left|x - x_0\right| - \left|y - x_0\right| \geq \left(2^k - 1\right)r \geq 1. \tag{23}$$

Similar to the case $r \geq 1$, we get

$$|Ta(x)| \le C\frac{|B|}{|2^kB|}\frac{r}{(2^kr)^{(1+m+n/\rho)}}\omega(B)^{-1}. \tag{24}$$

Since $0<r<1$ and $2^kr>1$, we have

$$\frac{r}{(2^kr)^{(1+m+n/\rho)}} \le \begin{cases} 2^{-k(1+m+n/\rho)}, & \dfrac{1+m+n}{\rho} \le 1; \\ 2^{-k}, & \dfrac{1+m+n}{\rho} \ge 1. \end{cases} \tag{25}$$

Noting $\varepsilon = \min\{1, (1+m+n/\rho)\}$, it easy to see $|Ta(x)| \le C2^{-\varepsilon k}(|B|/|2^kB|)\omega(B)^{-1}$. This implies (17).

Subcase 2. If $(2^k-1)r<1$. Since $m \le -(n+1)(1-\rho)$, (22) yields

$$\|Ta\|_{L^1_\omega(2^{k+1}B\setminus 2^kB)} \le \frac{Cr}{(2^kr|B|)^{(1+m+n/\rho)}}\frac{\omega(2^{k+1}B)}{\omega(B)} \le C2^{-k(\varepsilon+n(1-p))}. \tag{26}$$

In view of (20) and (26), we finish the proof of Proposition 1.

Proof. The proof of Theorem 1 is motivated by the atomic decomposition for $H^1_\omega(\mathbb{R}^n)$. Let $f \in H^1_\omega(\mathbb{R}^n)$. We obtain an atomic decomposition of f satisfying (15) and (16). So, to prove that the pseudodifferential operators T are bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$, it suffices to show that for each $(1,\infty)$-atom a with respect to ω, we have $Ta_{L^1_\omega(\mathbb{R}^n)} \le C$. Recall that an $(1,\infty)$-atom a with respect to ω is a function satisfying

$$\begin{gathered} \operatorname{sup} p(a) \subset B, \\ \|a\|_{L^\infty} \le \omega(B)^{-1}, \\ \int a(x)dx = 0, \end{gathered} \tag{27}$$

for some ball $B = B(x_0, r)$.

Now, let a be such an atom and write

$$\int |Ta|\omega = \int_{2B}|Ta|\omega + \int_{(2B)^c}|Ta|\omega = I_1 + I_2. \tag{28}$$

It is easy to estimate the term I_1. Using Hölder inequality and L^2_ω-boundedness for the pseudodifferential operator T (see Remark 1), we get

$$\begin{aligned} I_1 &\le \left(\int_{2B}|Ta|^2\omega\right)^{1/2}\left(\int_{2B}\omega\right)^{1/2} \le C\|Ta\|_{L^2_\omega(\mathbb{R}^n)}\omega(2B)^{1/2} \\ &\le C\|a\|_{L^2_\omega(\mathbb{R}^n)}\omega(2B)^{1/2} \le C\omega(B)^{-1/2}\omega(B)^{1/2} \le C, \end{aligned} \tag{29}$$

where C is independent of a.

For the term I_2, we write

$$I_2 = \int_{(2B)^c}|Ta|\omega \le \sum_{k=1}^{\infty}\int_{(2^{k+1}B\setminus 2^kB)}|Ta|\omega = \sum_{k=1}^{\infty}\|Ta\|_{L^1_\omega(2^{k+1}B\setminus 2^kB)}. \tag{30}$$

By Proposition 1, we get

$$I_2 \le \sum_{k=1}^{\infty} C2^{-k(\varepsilon+n(1-p))} \le C, \tag{31}$$

since $1 \le p < 1 + (\varepsilon/n)$. Combing (29) and (31), we finish the proof of Theorem 1. □

4. The Proof of Theorem 2

In this section, we establish the weighted norm inequality on weighted Hardy spaces $H^1_\omega(\mathbb{R}^n)$ for pseudodifferential operators T in $\mathscr{L}^m_{\rho,\delta}$.

Proof. Without loss of generality, we assume $1 \le p < 1 + (\mu/n)$, where $\mu = (1+m+n/\rho) - n$. Fix $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\phi(x)dx \ne 0$. By (15), it is sufficient to show that for each $(1,\infty)$-atom a with respect to ω, $\|(Ta)^*\|_{L^1_\omega(\mathbb{R}^n)} \le C$ with C independent of a. In order to do this, one can suppose $\operatorname{supp}(a) \subset B = B(x_0, r)$ and write

$$\begin{aligned} \|(Ta)^*\|_{L^1_\omega(\mathbb{R}^n)} &= \int_{|x-x_0|\le 4r}|(Ta)^*(x)|\omega(x)dx \\ &\quad + \int_{|x-x_0|>4r}|(Ta)^*(x)|\omega(x)dx \\ &= I_1 + I_2. \end{aligned} \tag{32}$$

For the term I_1, by Hölder inequality, L^2_ω-boundedness of the maximal function $(Ta)^*$, L^2_ω-boundedness of the pseudodifferential operator T, and (14), we get

$$\begin{aligned} I_1 &\le \left(\int_{|x-x_0|\le 4r}|(Ta)^*(x)|^2\omega(x)dx\right)^{1/2}\left(\int_{|x-x_0|\le 4r}\omega(x)dx\right)^{1/2} \\ &\le C\|Ta\|_{L^2_\omega(\mathbb{R}^n)}\omega(B(x_0,4r))^{1/2} \le C\|a\|_{L^2_\omega(\mathbb{R}^n)}\omega(B(x_0,4r))^{1/2} \le C, \end{aligned} \tag{33}$$

where C is independent of a.

To estimate I_2, we first estimate $(Ta)^*(x)$ for $|x-x_0| > 4r$. For any $t>0$, since $T^*1 = 0$(see Definition 3), we have

$$\begin{aligned} |Ta * \phi_t(x)| &= \left|\int_{\mathbb{R}^n}Ta(y)\frac{1}{t^n}\phi\left(\frac{x-y}{t}\right)dy\right| \\ &= \left|\int_{\mathbb{R}^n}Ta(y)\frac{1}{t^n}\left(\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right)\right)dy\right| \\ &\le \frac{1}{t^n}\int_{|y-x_0|<2r}|Ta(y)|\left|\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right)\right|dy \\ &\quad + \frac{1}{t^n}\int_{2r\le|y-x_0|<(|x-x_0|/2)}\cdots + \frac{1}{t^n}\int_{|y-x_0|\ge(|x-x_0|/2)}\cdots \\ &= E_1 + E_2 + E_3. \end{aligned} \tag{34}$$

For the term E_1, by the mean value theorem and Hölder's inequality, we have

$$E_1 \le \frac{1}{t^{n+1}}\|Ta\|_{L^2(\mathbb{R}^n)} \times \left(\int_{|y-x_0|<2r}\left|\nabla\phi\left(\frac{x-x_0-\gamma(y-x_0)}{t}\right)\right|^2|y-x_0|^2\mathrm{d}y\right)^{1/2} \le C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}, \tag{35}$$

where $\gamma \in (0,1)$ depends on x, y, and x_0, and $\nabla = ((\partial/\partial x_1), \ldots, (\partial/\partial x_n))$. Here, we use the inequalities

$$\frac{|x-x_0-\gamma(y-x_0)|^{n+1}}{t^{n+1}}\left|\nabla\phi\left(\frac{x-x_0-\gamma(y-x_0)}{t}\right)\right| \le C, \tag{36}$$

and $|x-x_0-\gamma(y-x_0)| \ge |x-x_0| - \gamma|y-x_0| \ge |x-x_0| - |y-x_0| \ge |x-x_0|/2$, and L^2-boundedness of the pseudo-differential operator T (see Lemma 3).

To estimate E_2 and E_3, we first estimate $Ta(y)$ when $|y-x_0| > 2r$ and consider two cases about r.

Case 3. If $r > 1$, then for every $y \in \mathbb{R}^n \backslash B(x_0, 2r)$ and $z \in B$, we have $|y-z| \ge |y-x_0| - |z-x_0| \ge r \ge 1$. Hence, by (8), we have

$$\begin{aligned}|Ta(y)| &= \left|\int_{\mathbb{R}^n} K(y,z)a(z)\mathrm{d}z\right| \le \int_B |K(y,z)||a(z)|\mathrm{d}z \\ &\le C\int_B \frac{1}{|y-z|^{N+n+1}}\|a\|_{L^\infty}\mathrm{d}z \\ &\le C\frac{1}{|y-x_0|^{n+1}}\|a\|_{L^\infty}|B| \\ &\le C\frac{r^{n+1}}{|y-x_0|^{n+1}\omega(B)}.\end{aligned} \tag{37}$$

Case 4. In the case of $0 < r < 1$, we have $\int_B a(z)\mathrm{d}z = 0$. Thus, for every $y \in \mathbb{R}^n \backslash B(x_0, 2r)$, from $1+n+m > n\rho > 0$, (7) yields

$$\begin{aligned}|Ta(y)| &= \left|\int_{\mathbb{R}^n} K(y,z)a(z)\mathrm{d}z\right| \\ &\le \int_B |K(y,z) - K(y,x_0)||a(z)|\mathrm{d}z \\ &\le C\int_B \frac{|z-x_0|}{|y-x_0|^{(1+n+m/\rho)}}\|a\|_{L^\infty}\mathrm{d}z \\ &\le C\frac{r}{|y-x_0|^{(1+n+m/\rho)}}\|a\|_{L^\infty}|B| \\ &\le C\frac{r^{n+1}}{|y-x_0|^{(1+n+m/\rho)}\omega(B)},\end{aligned} \tag{38}$$

where we use the fact that $|y-\xi| \sim |y-x_0|$ if $\xi \in B(x_0, r)$ and $|y-x_0| > 2r$.

Let us now continue to estimate E_2. When $r \ge 1$, using the mean value theorem and (37), we have

$$\begin{aligned}E_2 &= \frac{1}{t^{n+1}}\int_{2r\le|y-x_0|<|x-x_0|/2}\int_B 2r \le |y-x_0| < \frac{|x-x_0|}{2} \times \left|\nabla\phi\left(\frac{x-x_0-\gamma(y-x_0)}{t}\right)\right||y-x_0|\mathrm{d}y \\ &\le C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\int_{2r\le|y-x_0|<|x-x_0|/2}\frac{1}{|y-x_0|^n}\mathrm{d}y \\ &\le C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\ln\left(\frac{|x-x_0|}{4r}\right).\end{aligned} \tag{39}$$

Here, we use the fact that $|x-x_0-\gamma(y-x_0)| \sim |x-x_0|$ under the condition of $2r \le |y-x_0| < (|x-x_0|/2)$.

Similarly, in the case of $0 < r < 1$, by the moments condition for a, the mean value condition, and (38), we get

$$\begin{aligned}E_2 &= \frac{1}{t^{n+1}}\int_{2r\le|y-x_0|<|x-x_0|/2}\left|\int_B (K(y,z)-k(y,x_0))a(z)\mathrm{d}z\right| \times \left|\nabla\phi\left(\frac{x-x_0-\gamma(y-x_0)}{t}\right)\right||y-x_0|\mathrm{d}y \\ &\le C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\int_{2r\le|y-x_0|<|x-x_0|/2}\frac{1}{|y-x_0|^{(1+n+m/\rho)-1}}\mathrm{d}y \\ &\le \begin{cases} C\dfrac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\ln\left(\dfrac{|x-x_0|}{4r}\right), & \dfrac{1+m+n}{\rho} = n+1; \\ C\dfrac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)}, & \dfrac{1+m+n}{\rho} < n+1.\end{cases}\end{aligned} \tag{40}$$

For the term E_3, we have

$$E_3 \le \frac{1}{t^n}\int_{|y-x_0|\ge|x-x_0|/2}|Ta(y)|\left(\left|\phi\left(\frac{x-y}{t}\right)\right| + \left|\phi\left(\frac{x-x_0}{t}\right)\right|\right)\mathrm{d}y. \tag{41}$$

Since $|y-x_0| \ge (|x-x_0|/2)$, we have $|x-y| \ge (|x-x_0|/2)$. Thus,

$$\frac{1}{t^n}\left|\phi\left(\frac{x-y}{t}\right)\right| \le \frac{C}{|x-y|^n} \le \frac{C}{|x-x_0|^n}. \tag{42}$$

Meanwhile,

$$\frac{1}{t^n}\left|\phi\left(\frac{x-x_0}{t}\right)\right| \le \frac{C}{|x-x_0|^n}. \tag{43}$$

So, in the case of $r \ge 1$, by (37), (42), and (43), we have

$$E_3 \le C\int_{|y-x_0|\ge(|x-x_0|/2)} \frac{r^{n+1}}{|y-x_0|^{n+1}\omega(B)} \frac{1}{|x-x_0|^n}\mathrm{d}y$$
$$= C\frac{r^{n+1}}{|x-x_0|^n\omega(B)}\int_{|y-x_0|\ge(|x-x_0|/2)} \frac{1}{|y-x_0|^{n+1}}\mathrm{d}y$$
$$= C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}. \tag{44}$$

In the case of $0<r<1$, by (38), (42), and (43), we have

$$E_3 \le C\int_{|y-x_0|\ge(|x-x_0|/2)} \frac{r^{n+1}}{|y-x_0|^{(1+n+m/\rho)}\omega(B)} \frac{1}{|x-x_0|^n}\mathrm{d}y$$
$$= C\frac{r^{n+1}}{|x-x_0|^n\omega(B)}\int_{|y-x_0|\ge(|x-x_0|/2)} \frac{1}{|y-x_0|^{(1+n+m/\rho)}}\mathrm{d}y$$
$$= C\frac{r^{n+1}}{|x-x_0|^{n+\mu}\omega(B)}$$
$$= C\frac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)}, \tag{45}$$

since $\rho n < 1+n+m \le \rho(n+1)$.

Let $x \notin B(x_0, 4r)$. In view of (35), (39), (40), (44), and (45), we shall unify these formulas. Firstly, $\rho n-(n+1)<m\le(\rho-1)(n+1)$ implies $\mu\in(0,1]$. Secondly, $x\notin B(x_0,4r)$ implies $(r/|x-x_0|)<1$. Therefore,

$$\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)} \le \frac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)} \tag{46}$$

holds for any $\mu\in(0,1)$. Finally, since $\ln x \le (1/\alpha)x^\alpha$ holds for $x\ge 1$ and $\alpha\in(0,1)$, we have

$$\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\ln\left(\frac{|x-x_0|}{4r}\right) \le C\frac{r^{n+1}}{|x-x_0|^{n+1}\omega(B)}\left(\frac{|x-x_0|}{r}\right)^{1-\mu}$$
$$= C\frac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)}. \tag{47}$$

Using these three facts, we have

$$|(Ta)^*(x)| \le C\frac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)}. \tag{48}$$

Note that $p\le 1+(\mu/n)$. Then, in all cases, we have $n+\mu>np$ and

$$I_2 \le C\int_{|x-x_0|>4r} \frac{r^{n+\mu}}{|x-x_0|^{n+\mu}\omega(B)}\omega(x)\mathrm{d}x$$
$$= C\frac{r^{n+\mu}}{\omega(B)}\sum_{l=0}^{\infty}\int_{2^{l+2}r\le|x-x_0|<2^{l+3}r} \frac{\omega(x)}{|x-x_0|^{n+\mu}}$$
$$\le C\sum_{l=0}^{\infty}\frac{1}{2^{l[(n+\mu)-np]}}$$
$$\le C. \tag{49}$$

This concludes the proof of Theorem 2. □

Data Availability

The author confirm that no data were used to support this study. All References used were listed.

Disclosure

This study is a part of research work done by Yu-long Deng, a PhD student, under the supervision of the second author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. Hörmander, "Pseudo-differential operators and hypoelliptic equations," *Singular Integrals*, vol. 10, pp. 138–183, 1967.

[2] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, USA, 1993.

[3] A. P. Calderon and R. Vaillancourt, "A class of bounded pseudo-differential operators," *Proceedings of the National Academy of Sciences*, vol. 69, no. 5, pp. 1185–1187, 1972.

[4] C. Feffermn, ".L^p bounds for pseudo-differential operators," *Israel Journal of Mathematics*, vol. 14, no. 4, pp. 413–417, 1973.

[5] J. Alvarez and J. Hounie, "Estimates for the kernel and continuity properties of pseudo-differential operators," *Arkiv För Matematik*, vol. 28, no. 1-2, pp. 1–22, 1990.

[6] S. Chanillo and A. Torchinsky, "Sharp function and weighted L^p estimates for a class of pseudodifferential operators," *Arkiv för Matematik*, vol. 24, no. 1-2, pp. 1–25, 1985.

[7] N. Michalowski, D. J. Rule, and W. Staubach, "Weighted lp boundedness of pseudodifferential operators and applications," *Canadian Mathematical Bulletin*, vol. 55, no. 3, pp. 555–570, 2012.
[8] N. Miller, "Weighted Sobolev spaces and pseudodifferential operators with smooth symbols," *Transactions of the American Mathematical Society*, vol. 269, no. 1, p. 91, 1982.
[9] J. Hounie and R. A. d. S. Kapp, "Pseudodifferential operators on local Hardy spaces," *Journal of Fourier Analysis and Applications*, vol. 15, no. 2, pp. 153–178, 2009.
[10] K. Yabuta, "Generalizations of calderón-zygmund operators," *Studia Mathematica*, vol. 82, no. 1, pp. 17–31, 1985.
[11] K. Yabuta, "A note on weighted norm inequalities for pseudodifferential operators," *Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics*, vol. 18, pp. 7–12, 1986.
[12] D. Goldberg, "A local version of real Hardy spaces," *Duke Mathematical Journal*, vol. 46, no. 1, pp. 27–42, 1979.
[13] J. García-Cuerva and J. L. Rubio De Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Publishing, Amsterdam, Netherlands, 1985.
[14] J. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Springer-Verlag, Berlin, Germany, 1989.
[15] H. D. Hung and L. D. Ky, "An Hardy estimate for commutators of pseudo-differential operators," *Taiwanese Journal of Mathematics*, vol. 19, no. 4, pp. 1097–1109, 2015.