Research Article

Pseudodifferential Operators on Weighted Hardy Spaces

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We study two sufficient conditions for the boundedness of a class of pseudodifferential operators $T$ with symbols in the H"olmander class $S^{m}_{p,\delta}(\mathbb{R}^n)$ on weighted Hardy spaces $H^1_\omega(\mathbb{R}^n)$, where $\omega$ belongs to Muckenhoupt class $A_\infty$. The first one is an estimate from $H^1_\omega(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. We get a better range of admissible $p$ and $m$. The second one is a weighted version bounded for the operators $T$ on $H^1_\omega(\mathbb{R}^n)$, and it is an addition to the literature.

1. Introduction

The purpose of this paper is to study some sufficient conditions for the boundedness of pseudodifferential operators $T$ on weighted Hardy space $H^1_\omega(\mathbb{R}^n)$, where the operators $T$ have symbols in the H"olmander class $S^{m}_{p,\delta}(\mathbb{R}^n)$. As in [1], for $m \in \mathbb{R}$ and $\rho, \delta \in [0, 1]$, a symbol $a(x, \xi) \in S^{m}_{p,\delta}(\mathbb{R}^n)$ is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-p|\beta|+\delta|\alpha|}$$

holds for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, where $C_{\alpha,\beta}$ is independent of $x$ and $\xi$. We now assume that the symbol $a(x, \xi)$ is smooth in both the spatial variable $x$ and the frequency variable $\xi$.

Given $f \in C_0^\infty(\mathbb{R}^n)$, the pseudodifferential operator $T \in \mathcal{L}_{p,\delta}$ associated with the symbol $a(x, \xi) \in S^{m}_{p,\delta}(\mathbb{R}^n)$ is given by

$$Tf(x) = \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i x \cdot \xi} \tilde{f}(\xi) d\xi,$$

where $\tilde{f}$ denotes the Fourier transform of $f$. Moreover, we can express $T$ by a kernel $K(x, y)$ as (see, e.g., [2])

$$Tf(x) = \int K(x, y) f(y) dy.$$

Pseudodifferential operators play an important role in the theory of partial differential equations. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ coincide with the Lebesgue spaces $L^p(\mathbb{R}^n)$ when $p > 1$. The $L^p$ and weighted $L^p$ boundedness of the operator $T \in \mathcal{L}_{p,\delta}$ have been extensively studied. We refer to [1, 2, 3, 4] for the $L^p$ bounds and [5, 6, 7, 8] for the weighted $L^p$ bounds.

For $p \in (0, 1]$, there is an estimate from $L^1(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ for the pseudodifferential operator $T \in \mathcal{L}_{p,\delta}$ (cf. [5]). As known, the Hardy space $H^p(\mathbb{R}^n)$ is an advantageous substitute for $L^p(\mathbb{R}^n)$. The behavior of the pseudodifferential operator $T$ on $H^p(\mathbb{R}^n)$ has attracted a lot of interest. For example, Alvarez and Hounie [5] have found that the pseudodifferential operator $T$ with symbol in $S^{m}_{p,\delta}(\mathbb{R}^n)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where $0 < \rho \leq 1$, $0 \leq \delta < 1$ and $m \leq -(n(1-\rho))/2$. Hounie and Kapp [9] have shown that the operator $T$ with $0 \leq \delta \leq \rho < 1$ and $m = -(n(1-\rho))/2$ is bounded from the local Hardy space $h^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Yabuta [10] has proved the operator $T$ involving a modulus of continuity $\omega(t)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

The bounds of the pseudodifferential operator $T$ from the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ into the weighted Lebesgue space $L^1_\omega(\mathbb{R}^n)$ have also been studied. Yabuta [11] has shown that the operator $T$ is bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$, where $T \in \mathcal{L}_{p,\delta}$ and $\omega \in A_1$. In view of this, it is natural to look for a wide range of operator $T$ in $\mathcal{L}_{p,\delta}$ to study the bounds on the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$. 
In this paper, we establish two estimates for the pseudodifferential operator $T$ with symbols in $S^m_{ρ,δ}(R^n)$. The first one is an estimate from $H^1_w(R^n)$ into $L^1_w(R^n)$. We extend the result in Yabuta [11] to $ω ∈ A_p$ with $1 ≤ p < 1 + (ε/n)$ and the operator $T$ with $0 < ρ ≤ 1$, $0 ≤ δ < 1$, and $−(n+1) < m ≤ −(n+1)(1−p)$. Our first main result is stated as follows.

**Theorem 1.** Let $ε = \min[1, (1 + m + n/p)]$, $p ∈ [1, 1 + (ε/n)]$, $ω ∈ A_p$, and $T ∈ S^m_{ρ,δ}(R^n)$ with $0 < ρ ≤ 1$, $0 ≤ δ < 1$. If $−(n+1) < m ≤ −(n+1)(1−p)$, then $T$ is bounded from $H^1_w(R^n)$ into $L^1_w(R^n)$, i.e., there exists a constant $C > 0$ such that

$$\|TF\|_{L^1_w(R^n)} ≤ C\|f\|_{H^1_w(R^n)}. \quad (4)$$

The second one is an estimate on weighted Hardy spaces $H^1_w(R^n)$ for the pseudodifferential operator $T$. It is well known that under certain conditions on $m, ρ, δ$, the operator $T$ is bounded on $h^1(R^n)$ (cf. [9, 12]). Alvarez and Hounie [5] have found that the pseudodifferential operator $T$ is bounded on $H^1(R^n)$, where $T ∈ S^m_{ρ,δ}(R^n)$ with $0 < ρ ≤ 1$, $0 ≤ δ < 1$, and $m ≤ −\frac{λ}{2}$. If $−(n+1) < m ≤ −(n+1)(1−p)$, then $T$ is bounded from $H^1_w(R^n)$ into $L^1_w(R^n)$. We now state our second main result.

**Theorem 2.** Let $μ = ((1 + m + n)/m) − n$, $p ∈ [1, 1 + m/n]$), $ω ∈ A_p$, and $T ∈ S^m_{ρ,δ}(R^n)$ with $0 < ρ ≤ 1$, $0 ≤ δ < 1$. Assume $pn − (n+1) < m ≤ −(n+1)(1−p)$ and $T^*1 = 0$. Then, $T$ is bounded on $H^1_w(R^n)$, i.e., there exists a constant $C > 0$ such that

$$\|TF\|_{H^1_w(R^n)} ≤ C\|f\|_{H^1_w(R^n)}. \quad (5)$$

The remainder of this paper is organized as follows. In Section 2, we present some definitions and well-known results we use later. The aim of Section 3 is to set up the estimate from $H^1_w(R^n)$ into $L^1_w(R^n)$ for pseudodifferential operators $T$ in $S^m_{ρ,δ}(R^n)$. We develop a method to handle $\|T(a)\|_{L^1_w(R^n)}$ (see Proposition 1). The aim of Section 4 is to establish the estimate on weighted Hardy spaces $H^1_w(R^n)$ for pseudodifferential operators $T$ in $S^m_{ρ,δ}(R^n)$.

Most of the notations we use are standard. $C$ denotes a constant that may change from line to line and we write $a ≤ b$ as shorthand for $a ≤Cb$. If $a ≤ b$ and $b ≤ a$, we mean $a = b$. For a measurable set $A$, $|A|$ denotes the Lebesgue measure of $A$ and $χ_A$ the characteristic function. $B$ will always denote a ball, and $tB(t > 0)$ denotes the ball $B$ dilated by $t$.

### 2. Notations and Auxiliary Lemma

In this section, we first present an auxiliary lemma about the pseudodifferential operator $T$ associated with the characteristic function of $A$ and $χ_A$ the characteristic function. $B$ will always denote a ball, and $tB(t > 0)$ denotes the ball $B$ dilated by $t$.
Let \( \omega \in A_p \) and \( p \geq 1 \). Then, there exists a constant \( C > 0 \) such that
\[
C \left( \frac{|E|}{|B|} \right)^p \leq \frac{\omega(E)}{\omega(B)}
\]  
for all balls \( B \) and measurable subsets \( E \subset B \).

Given a weight function \( \omega \) on \( \mathbb{R}^n \), we denote by \( L^p_\omega(\mathbb{R}^n) \) the weighted Lebesgue space of all functions \( f \) satisfying
\[
\|f\|_{L^p_\omega(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{1/p} < \infty.
\]  
When \( p = \infty \), \( L^\infty_\omega(\mathbb{R}^n) \) is \( L^\infty(\mathbb{R}^n) \). Analogous to the classical Hardy space, the weighted Hardy space \( H^1_\omega(\mathbb{R}^n) \) can be defined in terms of maximal functions.

**Definition 1.** Let \( \omega \in A_\infty \). The weighted Hardy space \( H^1_\omega(\mathbb{R}^n) \) is defined by
\[
H^1_\omega(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \phi^*(f)(x) = \sup_{r > 0} \phi_r * f(x) \in L^1_\omega(\mathbb{R}^n) \right\},
\]  
where \( \phi \in \mathcal{S}(\mathbb{R}^n) \) is a fixed function with \( \int \phi dx \neq 0 \) and \( \phi_r(y) = (1/r^n)\phi(y/t) \) for any \( t > 0 \). Moreover, we define
\[
\|f\|_{H^1_\omega(\mathbb{R}^n)} = \|\phi^*(f)\|_{L^1_\omega(\mathbb{R}^n)}.
\]

**Remark 5.** Definition 1 is independent of the choice of \( \phi \) (see [14]).

**Definition 2.** Let \( \omega \) be a weight with the critical index \( q_\omega \). An \((1,\infty)\)-atom with respect to \( \omega \) is a function \( a \) satisfying
\[
supp(a) \subset B \quad \|a\|_{L^\infty} \leq \omega(B)^{-1},
\]  
and \( \int a(x)x^\alpha dx = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq |q_\omega - 1| \).

The Hardy space \( H^1_\omega(\mathbb{R}^n) \) is a linear space spanned by all \((1,\infty)\)-atoms with respect to \( \omega \). Namely, \( f \in H^1_\omega(\mathbb{R}^n) \) if and only if \( f \) can be written as (see [13])
\[
f = \sum_{j=1}^\infty a_j \lambda_j,
\]  
in the sense of \( \mathcal{S}' \), where each \( a_j \) is an \((1,\infty)\)-atom with respect to \( \omega \) and \( \lambda_j \) satisfies
\[
\sum_j |\lambda_j| < \infty.
\]

Moreover, \( \|f\|_{H^1_\omega(\mathbb{R}^n)} = \inf \{ \sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty a_j \lambda_j \} \).

**Definition 3.** Let \( T \) be a pseudodifferential operator in \( \mathcal{L}^m_{\rho,\delta} \). We say \( T^*1 = 0 \) if \( \| \int_{\mathbb{R}^n} T(a(x))^* dx \| = 0 \) for all \( a \in L^\infty(\mathbb{R}^n) \) with compact support and \( \int_{\mathbb{R}^n} a(x) dx = 0 \).

### 3. The Proof of Theorem 1

In this section, we prove that the pseudodifferential operators \( T \) in \( \mathcal{L}^m_{\rho,\delta} \) are bounded from \( H^1_\omega(\mathbb{R}^n) \) into \( L^1_\omega(\mathbb{R}^n) \).

**Proposition 1.** Let \( \omega \in A_p \), \( p \in [1, \infty) \) and \( \varepsilon = \min \{1, (1 + m + n/p)\} \). Assume pseudodifferential operator \( T \in \mathcal{L}^m_{\rho,\delta} \) with \( 0 < \rho \leq 1 \), \( 0 < \delta < 1 \), and \( -(n+1) < m \leq -(n+1)(1-\rho) \). Then, there exists a constant \( C > 0 \) such that
\[
\|Ta\|_{L^1_\omega(2^{k+1}B,2^kB)} \leq C 2^{-k(\varepsilon+n(1-\rho))},
\]
holds for all \((1,\infty)\)-atoms \( a \) with respect to \( \omega \), where \( \text{supp}(a) \subset B = B(x_0,r) \).

**Proof.** Inspired by the proof of Lemma 3.2 in [15], we consider two cases about the radius \( r \).

**Case 1.** When \( r \geq 1 \). For every \( x \in 2^{k+1}B \setminus 2^kB \) and \( y \in B(x_0,r) \), we have
\[
|x-y| \geq |x-x_0| - |y-x_0| \geq 2^k r - r \geq 1.
\]

Hence, by (8) and properties of \((1,\infty)\)-atoms with respect to \( \omega \), we have
\[
|Ta(x)| = \left| \int_{\mathbb{R}^n} K(x,y) a(y) dy \right| \leq \int_{B} |K(x,y) - K(x,x_0)| |a(y)| dy
\]
\[
\leq C \int_{B} \frac{1}{|x-y|^\rho + 1} |a(y)| dy \leq C \frac{1}{|x-x_0|^\rho + 1} \int_{B} |a(y)| dy
\]
\[
\leq C 2^{-k} \frac{|B|}{2^kB} \omega(B)^{-1},
\]
for all \( x \in 2^{k+1}B \setminus 2^kB \). Thus, \( \|Ta\|_{L^1_\omega(2^{k+1}B,2^kB)} \leq C 2^{-k} \frac{|B|}{2^kB} \omega(B)^{-1} \leq C 2^{-k(\varepsilon+n(1-\rho))} \).

**Case 2.** When \( 0 < r < 1 \). For every \( x \in (2^{k+1}B) \setminus 2kB \) and \( y \in B(x_0,r) \), by moments condition, we have
\[
|Ta(x)| = \left| \int_{\mathbb{R}^n} K(x,y) a(y) dy \right| \leq \int_{B} |K(x,y) - K(x,x_0)| |a(y)| dy.
\]

By the mean value theorem, \( 1 + m + n > 0 \), and (7), we have
\[
|Ta(x)| \leq C \int_{B} \frac{|y-x_0|}{|x-x_0|^{1+m+n/p}} |a(y)| \leq C \frac{r}{(2^k)^{1+m+n/p}} \frac{|B|}{\omega(B)},
\]
where we take \( M = 1 \) and use the fact that \( |x - \xi| \sim |x - x_0| \) if \( \xi \in B(x_0,r) \). Let us now consider two subcases.

**Subcase 1.** If \((2^k-1)r \geq 1\), then, for any \( y \in B(x_0,r) \) and \( x \in 2^{k+1}B \setminus 2^kB \),
\[
|x-y| \geq |x-x_0| - |y-x_0| \geq (2^k - 1)r \geq 1.
\]

Similar to the case \( r \geq 1 \), we get
\[ |Ta(x)| \leq C \frac{|B|}{|2^k B|} \frac{r}{(2^k r)^{1+m+n+p}} \omega(B)^{-1}. \]  

(24)

Since \(0 < r < 1\) and \(2^k r > 1\), we have

\[
\frac{r}{(2^k r)^{1+m+n+p}} \leq \begin{cases} \frac{1}{2}, & \frac{1 + m + n}{p} \leq 1; \\
\frac{1}{2^k}, & \frac{1 + m + n}{p} \geq 1.
\end{cases}
\]

(25)

Noting \(\epsilon = \min\{1, (1 + m + n/p)\}\), it is easy to see \(|Ta(x)| \leq C 2^{-k} \frac{|B|}{|2^k B|} \omega(B)^{-1}\). This implies (17).

**Subcase 2.** If \((2^k - 1)r < 1\). Since \(m \leq -(n + 1)(1 - \rho)\), (22) yields

\[
\|Ta\|_{L^p_\omega(2^{k+1}B, 2^k B)} \leq \frac{C}{(2^k r)|B|^{1+m+n+p}} \omega(2^{k+1}B) \omega(B) \leq C 2^{-k} \frac{1}{(p-1)C}.
\]

(26)

In view of (20) and (26), we finish the proof of Proposition 1.

**Proof.** The proof of Theorem 1 is motivated by the atomic decomposition for \(H^1_\omega(\mathbb{R}^n)\). Let \(f \in H^1_\omega(\mathbb{R}^n)\). We obtain an atomic decomposition of \(f\) satisfying (15) and (16). So, to prove that the pseudodifferential operators \(T \) are bounded from \(H^1_\omega(\mathbb{R}^n)\) into \(L^1_\omega(\mathbb{R}^n)\), it suffices to show that for each \((1, \infty)\)-atom \(a\) with respect to \(\omega\), we have \(Ta \in L^1_\omega(\mathbb{R}^n)\). Recall that an \((1, \infty)\)-atom \(a\) with respect to \(\omega\) is a function satisfying

\[
\|a\|_{L^\infty} \leq \omega(B)^{-1},
\]

(27)

for some ball \(B = B(x_0, r)\).

Now, let \(a\) be such an atom and write

\[
\int |Ta| \omega = \int_{2B} |Ta| \omega + \int_{(2B)^c} |Ta| \omega = I_1 + I_2.
\]

(28)

It is easy to estimate the term \(I_1\). Using Hölder inequality and \(L^2_\omega\)-boundedness for the pseudodifferential operator \(T\) (see Remark 1), we get

\[
I_1 \leq \left( \int_{2B} |Ta|^2 \omega \right)^{1/2} \left( \int_{2B} \omega \right)^{1/2} \leq C \|Ta\|_{L^2_\omega(2^k B)} \omega(2B)^{1/2} \leq C \|a\|_{L^2_\omega(2^k B)} \omega(B)^{1/2} \leq C,
\]

(29)

where \(C\) is independent of \(a\).

For the term \(I_2\), we have

\[
I_2 = \int_{(2B)^c} |Ta| \omega \leq \int_{(2B)^c} \sum_{k=1}^{\infty} |Ta| \omega = \sum_{k=1}^{\infty} \|Ta\|_{L^1_\omega(2^{k+1}B, 2^k B)}.
\]

(30)

By Proposition 1, we get

\[
I_2 \leq \sum_{k=1}^{\infty} C 2^{-k} |B|^{1+m+n} \omega(B)^{-1} \leq C,
\]

(31)

since \(1 \leq p < 1 + (\epsilon/n)\). Combing (29) and (31), we finish the proof of Theorem 1.

\[
\square
\]

**4. The Proof of Theorem 2**

In this section, we establish the weighted norm inequality on weighted Hardy spaces \(H^1_\omega(\mathbb{R}^n)\) for pseudodifferential operators \(T\) in \(\mathcal{L}^m_{\rho, \delta}\).

**Proof.** Without loss of generality, we assume \(1 \leq p < 1 + (\mu/n)\), where \(\mu = (1 + m + n/p) - n\). Fix \(\phi \in \mathcal{S}(\mathbb{R}^n)\) and \(\int_{\mathbb{R}^n} \phi(x) \omega(x) dx \neq 0\). By (15), it is sufficient to show that for each \((1, \infty)\)-atom \(a\) with respect to \(\omega\), \(\| (Ta)^* \|_{L^1_\omega(\mathbb{R}^n)} \leq C\) with \(C\) independent of \(a\). In order to do this, one can suppose \(\text{supp}(a) \subset B = B(x_0, r)\) and write

\[
\| (Ta)^* \|_{L^1_\omega(\mathbb{R}^n)} = \int_{|x-x_0| < 4r} |(Ta)^*(x)| \omega(x) dx + \int_{|x-x_0| \geq 4r} |(Ta)^*(x)| \omega(x) dx
\]

(32)

\[
= I_1 + I_2.
\]

For the term \(I_1\), by Hölder inequality, \(L^2_\omega\)-boundedness of the maximal function \((Ta)^*\), \(L^2_\omega\)-boundedness of the pseudodifferential operator \(T\), and (14), we get

\[
I_1 \leq \left( \int_{|x-x_0| < 4r} |(Ta)^*(x)|^2 \omega(x) dx \right)^{1/2} \left( \int_{|x-x_0| < 4r} \omega(x) dx \right)^{1/2} \leq C \|a\|_{L^2_\omega(2^k B)} \omega(B(x_0, 4r))^{1/2} \leq C,
\]

(33)

where \(C\) is independent of \(a\).

To estimate \(I_2\), we first estimate \((Ta)^*(x)\) for \(|x-x_0| > 4r\). For any \(t > 0\), since \(T^*1 = 0\) (see Definition 3), we have

\[
|Ta * \phi_t(x)| = \left| \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \phi_t \left( \frac{x-y}{t} \right) dy \right| \leq \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) dy
\]

\[
\leq \frac{1}{t^n} \int_{|y-x| < 2r} |Ta(y)| \omega(y) dy \leq C \|a\|_{L^2_\omega(B(x_0, 2r))} \omega(B(x_0, 2r)) \leq C \|a\|_{L^2_\omega(2^k B)} \omega(B(x_0, 2r)) \leq C,
\]

(34)

For the term \(E_1\), by the mean value theorem and Hölder’s inequality, we have
\[ E_1 \leq \frac{1}{m^{r+1}} \|Ta\|_{L^2(\mathbb{R}^n)} \times \left( \int_{|y-x_0| < 2r} \|\nabla \phi \left( \frac{x-x_0 - y(y-x_0)}{t} \right)\|_2^2 \, dy \right)^{1/2} \]
\[ \leq C \frac{\rho^{r+1}}{|x-x_0|^{m+1} \omega(B)} \]  
(35)

where \( \rho \in (0, 1) \) depends on \( x, y, \) and \( x_0, \) and \( \nabla = ((\partial/\partial x_1), \ldots, (\partial/\partial x_n)) \). Here, we use the inequalities
\[ \left| x - x_0 - y(y-x_0) \right| \leq |x-x_0| - |y-y_0| \leq |x-x_0| - |y-y_0| \geq |x-x_0|/2, \]  
(36)

and \( |x-x_0 - y(y-x_0)| - |x-x_0| \geq y |y-y_0| \geq |x-x_0| - |y-y_0| \geq |x-x_0|/2, \) and \( L^2 \)-boundedness of the pseudodifferential operator \( T \) (see Lemma 3).

To estimate \( E_2 \) and \( E_3 \), we first estimate \( Ta(y) \) when \( |y-x_0| > 2r \) and consider two cases about \( r \).

Case 3. If \( r > 1 \), then for every \( y \in \mathbb{R}^n \setminus B(x_0, 2r) \) and \( z \in B \), we have \( |y-z| \geq |y-x_0| - |z-x_0| \geq r \). Hence, by (8), we have
\[ \|Ta(y)\| = \int_{\mathbb{R}^n} K(y, z) a(z) \, dz \leq \int_B |K(y, z)| |a(z)| \, dz \]
\[ \leq C \left( \frac{1}{|y-z|^{m+1}} \|a\|_{L^\infty} \right) \, dz \]
\[ \leq C \frac{1}{|y-x_0|^{m+1}} |a|_{L^\infty} \, |B| \]
\[ \leq C \frac{\rho^{r+1}}{|y-x_0|^{m+1} \omega(B)} \]  
(37)

Case 4. In the case of \( 0 < r < 1 \), we have \( \int_{|y-z| < 2} |a(z)| \, dz = 0 \). Thus, for every \( y \in \mathbb{R}^n \setminus B(x_0, 2r) \), from \( 1 + m + n > mp > 0 \), (7) yields
\[ \|Ta(y)\| = \left[ \int_{B} |K(y, z) - K(y, x_0)| |a(z)| \, dz \right] \]
\[ \leq \int_B \left| K(y, z) - K(y, x_0) \right| |a(z)| \, dz \]
\[ \leq C \left( \frac{1}{|y-x_0|^{1+mp+1}} \|a\|_{L^\infty} \right) \, dz \]
\[ \leq C \frac{r}{|y-x_0|^{1+mp+1} \omega(B)} \]  
(38)

where we use the fact that \( |y-x| \sim |y-x_0| \) if \( x \in B(x_0, r) \) and \( |y-x_0| > 2r \).

Let us now continue to estimate \( E_2 \). When \( r \geq 1 \), using the mean value theorem and (37), we have
\[ E_2 = \frac{1}{m^{r+1}} \int_{2r \leq |y-x_0| \leq |x-x_0|/2} \|Ta(y)\|_{L^2(\mathbb{R}^n)} \times \left( \int_{|y-x_0| < 2r} \|\nabla \phi \left( \frac{x-x_0 - y(y-x_0)}{t} \right)\|_2^2 \, dy \right)^{1/2} \]
\[ \leq C \frac{\rho^{r+1}}{|x-x_0|^{m+1} \omega(B)} \int_{2r \leq |y-x_0| \leq |x-x_0|/2} \frac{1}{\omega(B)} \, dy \]
\[ \leq C \frac{\rho^{r+1}}{|x-x_0|^{m+1} \omega(B)} \ln \left( \frac{|x-x_0|}{4r} \right) \]  
(39)

Here, we use the fact that \( |x-x_0 - y(y-x_0)| \sim |x-x_0| \) under the condition of \( 2r \leq |y-x_0| \leq (|x-x_0|/2) \).

Similarly, in the case of \( 0 < r < 1 \), by the moments condition for \( a \), the mean value condition, and (38), we get
\[ E_2 = \frac{1}{m^{r+1}} \int_{2r \leq |y-x_0| \leq |x-x_0|/2} \left[ \int_B (K(y, z) - k(y, x_0)) a(z) \, dz \right] \]
\[ \times \left[ \int_{|y-x_0| < 2r} \|\nabla \phi \left( \frac{x-x_0 - y(y-x_0)}{t} \right)\|_2^2 \, dy \right] \]
\[ \leq C \frac{\rho^{r+1}}{|x-x_0|^{m+1} \omega(B)} \int_{2r \leq |y-x_0| \leq |x-x_0|/2} \frac{1}{\omega(B)} \, dy \]
\[ \leq C \frac{r}{|x-x_0|^{1+mp+1} \omega(B)} \ln \left( \frac{|x-x_0|}{4r} \right), \quad \frac{1}{\omega(B)} \leq \frac{1 + m + n}{\rho} = n + 1; \]
\[ \leq C \frac{r}{|x-x_0|^{1+mp+1} \omega(B)} \ln \left( \frac{|x-x_0|}{4r} \right), \quad \frac{1 + m + n}{\rho} < n + 1. \]  
(40)

For the term \( E_3 \), we have
\[ E_3 \leq \frac{1}{m} \int_{|y-x_0| \geq |x-x_0|/2} \|Ta(y)\|_{L^\infty} \left[ \|\phi \left( \frac{x-x_0}{t} \right) \| + \|\phi \left( \frac{x-x_0}{t} \right) \| \right] \, dy. \]  
(41)

Since \( |y-x_0| \geq (|x-x_0|/2) \), we have \( |x-y| \geq (|x-x_0|/2) \). Thus,
\[ \frac{1}{m} \|\phi \left( \frac{x-x_0}{t} \right) \| \leq \frac{C}{|x-x_0|^m} \leq \frac{C}{|x-x_0|^m} \]  
(42)

Meanwhile,
\[ \frac{1}{m} \|\phi \left( \frac{x-x_0}{t} \right) \| \leq \frac{C}{|x-x_0|^m} \]  
(43)

So, in the case of \( r \geq 1 \), by (37), (42), and (43), we have
\[ E_3 \leq \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \left| \frac{1}{y-x_0} \right| dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \left| \frac{1}{y-x_0} \right| dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} dy \]

(44)

In the case of \(0 < r < 1\), by (38), (42), and (43), we have

\[ E_3 \leq \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \left| \frac{1}{y-x_0} \right| dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \left| \frac{1}{y-x_0} \right| dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} dy = \mathcal{C} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \]

(45)

since \(\rho n < 1 + n + m \leq \rho (n + 1)\).

Let \(x \notin B(x_0, 4r)\). In view of (35), (39), (40), (44), and (45), we shall unify these formulas. Firstly, \(\rho n - (n+1) = \rho - (n+1) \leq (\rho - 1)(n+1) \implies \mu \in (0, 1]\). Secondly, \(x \notin B(x_0, 4r)\) implies \((r/|x-x_0|) < 1\). Therefore,

\[ \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \leq \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \]

holds for any \(\mu \in (0, 1]\). Finally, since \(\ln x \leq (1/\alpha) x^\alpha\) holds for \(x \geq 1\) and \(\alpha \in (0, 1]\), we have

\[ \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \leq C \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \left( \frac{|x-x_0|}{r} \right)^{1-\mu} = C \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \]

(47)

Using these three facts, we have

\[ |(Ta)^\alpha(x)| \leq C \frac{r^{\nu+1}}{|x-x_0|^{\nu+1} \omega(B)} \]

(48)

Note that \(p \leq 1 + (\mu/n)\). Then, in all cases, we have \(n + \mu > np\) and

\[ I_2 \leq \mathcal{C} \int_{|x-x_0| > 4r} |x-x_0|^\nu + \mu \omega(x) dx = \mathcal{C} \int_{|x-x_0| > 4r} |x-x_0|^\nu + \mu \omega(x) dx \]

(49)

\[ \leq \mathcal{C} \sum_{l=0}^{\infty} \frac{1}{2^{|l(n+\mu) - np|}} \]

\(\leq \mathcal{C}\).

This concludes the proof of Theorem 2.

\[ \square \]

**Data Availability**

The authors confirm that no data were used to support this study. All references used were listed.

**Disclosure**

This study is a part of research work done by Yu-long Deng, a PhD student, under the supervision of the second author.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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