Research Article

Analysis on Existence of Positive Solutions for a Class Second Order Semipositone Differential Equations

Yunhai Wang\textsuperscript{1} and Xu Yang\textsuperscript{2,3}

\textsuperscript{1}School of Aeronautics, Shandong Jiao Tong University, Jinan 250357, China
\textsuperscript{2}Group of Mathematics, Xinhuai Senior High School, Huai’an 223001, China
\textsuperscript{3}Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China

Correspondence should be addressed to Yunhai Wang; wangyhnuma@126.com

Received 31 May 2020; Accepted 18 June 2020; Published 14 August 2020

Guest Editor: Zisen Mao

Copyright © 2020 Yunhai Wang and Xu Yang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we study the existence of positive solutions of the following second-order semipositone system (see equation 1).

By applying a well-known fixed-point theorem, we prove that the problem admits at least one positive solution, if \( f \) is bounded below.

1. Introduction

This paper is focused on the existence of positive solutions of a second-order semipositone system

\[
\begin{aligned}
-u'' + \rho u = \phi u + f(t, u, \phi), & \quad t \in (0, 1), \\
-\phi'' = \mu u, & \quad t \in (0, 1), \\
u(0) = u(1) = \phi(0) = \phi(1) = 0,
\end{aligned}
\]

(1)

where \( \mu \) is a positive constant and \( f \) satisfies the following assumption: \( (F_0)f : [0, 1] \times \mathbb{R}_+^2 \rightarrow \mathbb{R} \) is continuous, and

\[
f(t, u, \phi) \geq -e(t), \quad \text{for } (t, u, \phi) \in [0, 1] \times \mathbb{R}_+^2,
\]

(2)

where \( e : [0, 1] \rightarrow \mathbb{R}_+ \) is continuous and \( e(t) \neq 0 \) on \([0,1]\).

The second-order elliptic systems

\[
\begin{aligned}
-\Delta u + \rho u = \phi u + f(u), & \quad x \in \Omega, \\
-\Delta \phi = \mu u, & \quad x \in \Omega, \\
u = \phi = 0, & \quad x \in \partial\Omega,
\end{aligned}
\]

(3)

have a strong physical meaning in quantum mechanics models [1, 2], in semiconductor theory [3], or in a time- and space-dependent mathematical model of nuclear reactors in a closed container [4]. To the best of our knowledge, existence and multiplicity of nontrivial solutions of BVP(1) have been widely studied by using the variational method [5], bifurcation techniques [6, 7], or fixed-point theorems [8–11]. In general, in order to ensure the positivity of the solutions of Equation (1), one of the crucial assumptions is that the nonlinearity \( f \) is nonnegative. Of course, the natural question is whether Equation (1) has a positive solution or not if \( f \) satisfies the assumption \( (F_0) \).
On the other hand, many authors have been interested in finding the relations between the positivity of solutions and the changing sign of the nonlinearity in order to prove the existence of the positive solutions. We refer the readers to [12–16] and the references.

Inspired by these references, the purpose of this paper is to find some new conditions, which are used to study the existence and multiplicity of positive solutions of the semipositone Equation (1). The main tool is the following well-known fixed-point theorem.

**Lemma 1** [17]. Let E be a Banach space and K be a cone in E. Assume $\Omega_r$ and $\Omega_R$ are open bounded subsets of E with $\Omega_r \cap K \neq \phi$, $\Omega_r \cap K \subset \Omega_R \cap K$. Let $T : \Omega_R \cap K \rightarrow K$ be a completely continuous operator such that

(a) $\|Tu\| \leq \|u\|$, for $u \in \partial(\Omega_r \cap K)$, and

(b) there exists a $\eta(t) \in K \setminus \{0\}$ such that

$$u \neq Tu + \lambda \eta(t), \quad \text{for} \quad u \in \partial(\Omega_R \cap K), \lambda > 0. \quad (4)$$

Then, $T$ has a fixed point in $\Omega_R \cap K \cap \Omega_r \cap K$. The same conclusion remains valid if (a) holds on $\partial(\Omega_R \cap K)$ and (b) holds on $\partial(\Omega_r \cap K)$.

The paper is organized as follows: in Section 2, we give some preliminaries, which are about the properties of the Green functions, the notations of some sets, etc.; in Section 3, we give the main results and the corresponding proof. In Section 4, some examples are given to illustrate the main results.

2. Preliminary

Let $G(t,s)$ be the Green function of linear boundary value problem

$$-u'' + \rho u = 0, \quad u(0) = u(1) = 0, \quad (5)$$

where $\rho > -\pi^2$.

**Lemma 2** [18]. Let $\omega = \sqrt{|\rho|}$, then $G(t,s)$ can be expressed by

(i) $G(t,s) = \begin{cases} \sinh \omega t \sinh (1-s)/\omega \sinh \omega, & 0 \leq t \leq s \leq 1, \\ \sinh \omega s \sinh (1-t)/\omega \sinh \omega, & 0 \leq s \leq t \leq 1, \text{ if } \rho > 0 \end{cases}$

(ii) $G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \text{ if } \rho = 0 \end{cases}$

(iii) $G(t,s) = \begin{cases} \sin \omega t \sin (1-s)/\omega \sin \omega, & 0 \leq t \leq s \leq 1, \\ \sin \omega s \sin (1-t)/\omega \sin \omega, & 0 \leq s \leq t \leq 1, \text{ if } -\pi^2 < \rho > 0 \end{cases}$

**Lemma 3** [18]. The function $G(t,s)$ has the following properties:

(i) $G(t,s) > 0, \forall t, s \in (0, 1)$

(ii) $G(t,s) \leq CG(s,s), \forall t, s \in [0, 1]$

(iii) $G(t,s) \geq CG(t,t)G(s,s), \forall t, s \in [0, 1]$

where $C = 1, \delta = \omega/\sinh \omega$, if $\rho > 0$; $C = 1, \delta = 1, \text{ if } \rho = 0$; and $C = 1/\sin \omega, \delta = \omega \sin \omega, \text{ if } -\pi^2 < \rho < 0$.

**Lemma 4.** For the function $G(t,s)$, there exists a $\xi > 0$ such that

$$G(t,s) \geq \xi \int_0^s G(t,s)ds. \quad (6)$$

**Proof.**

(i) For $\rho > 0$, we have

$$\int_0^s G(t,s)ds = \frac{1}{\omega^2 \sin \omega} \left( \sin \omega (\cosh (1-t) - 1) + \sinh \omega (1-t) \cosh \omega t - 1 \right). \quad (7)$$

Let

$$J_1(t) = \begin{cases} \frac{\cosh \omega (1-t) - 1}{\sinh \omega (1-t)}, & 0 \leq t < 1, \\ 0, & t = 1. \end{cases} \quad (8)$$

$$J_2(t) = \begin{cases} \frac{\cosh \omega t - 1}{\sinh \omega t}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

Since $J_1(t)$ is positive and continuous on $[0,1]$ and $J_1(1) = 0$, we have

$$J'_1 = \max_{t \in [0,1]} J_1(t) > 0. \quad (9)$$

In the similar way, we also have

$$J'_2 = \max_{t \in [0,1]} J_2(t) > 0. \quad (10)$$

Choosing $\xi < \omega/(J'_1 + J'_2)$. Then, for any $t \in (0, 1)$, we have

$$\xi \int_0^s G(t,s)ds = \frac{1}{\omega^2 \sin \omega} \left[ \sin \omega (\cosh (1-t) - 1) + \sinh \omega (1-t) \cosh \omega t - 1 \right]$$

$$= \frac{1}{\omega^2 \sin \omega} \left[ \sin \omega t \sinh (1-t) \cosh (1-t) - 1 \right]$$

$$= \frac{1}{\omega^2 \sin \omega} \left[ \sin \omega (1-t) \cosh (1-t) - 1 \right]$$

$$= \frac{1}{\omega^2 \sin \omega} \left[ \sin \omega (1-t) \cosh (1-t) - 1 \right]$$

$$= \frac{1}{\omega^2 \sin \omega} \left[ \sinh \omega (1-t) J'_1 + \sin Nh \omega (1-t) J'_2 \right]$$

$$= \frac{\xi (J'_1 + J'_2)}{\omega} G(t,t) \leq G(t,t). \quad (11)$$
Since \( G(0, 0) = \int_0^1 G(0, s)ds = G(1, 1) = \int_0^1 G(1, s)ds = 0 \),
then for any \( \xi > 0 \), we have
\[
G(t, t) = \xi \int_0^1 G(t, s)ds, \quad \text{for } t = 0, 1. \tag{12}
\]

Therefore, there exists a \( \xi > 0 \) such that
\[
G(t, t) \geq \xi \int_0^1 G(t, s)ds. \tag{13}
\]

(ii) For \( \rho = 0 \), it is obvious that
\[
\int_0^1 G(t, s)ds = \frac{1}{2} t^2 (1 - t) + \frac{1}{2} t(1 - t)^2 \leq t(1 - t)^2 = G(t, t). \tag{14}
\]

(iii) For \( -\pi^2 < \rho < 0 \), we have
\[
\int_0^1 G(t, s)ds = \frac{1}{\omega^2} \sin \omega [1 - \cos \omega(1 - t)]
+ \sinh \omega(1 - t) [1 - \cos \omega t]. \tag{15}
\]

Using the similar discussion of Case (i), it follows that there exists a \( \xi > 0 \) such that
\[
G(t, t) \geq \xi \int_0^1 G(t, s)ds. \tag{16}
\]

For convenience, let \( K(t, s) \) denote the Green function for \( \rho = 0 \). Then, Equation (1) can be rewritten as
\[
\begin{aligned}
-u'' + \rho u &= \mu u \int_0^1 K(t, s)u(s)ds + f(t, u \int_0^1 K(t, s)u(s)ds), \\
0 &= u(1) = 0. 
\end{aligned} \tag{17}
\]

Furthermore, let \( x = u + \omega \), where \( \omega(t) = \int_0^1 G(t, s)e(s)ds \)
is the unique solution of the linear boundary value problem
\[
\begin{aligned}
-u'' + \rho u &= e(t), \\
0 &= u(1) = 0. 
\end{aligned} \tag{18}
\]

Then, we rewrite (17) as
\[
\begin{aligned}
-x'' + \rho x &= F(t, x - \omega), \\
x(0) &= x(1) = 0, 
\end{aligned} \tag{19}
\]

where
\[
F(t, x - \lambda \omega) = \mu(x - \omega) \int_0^1 K(t, s)[x(s) - \omega(s)]ds
+ \left\{ f\left( t, x - \omega, \int_0^1 K(t, s)[x(s) - \omega(s)]ds \right) + e(t) \right\}. \tag{20}
\]

From the above discussion, then we have the following lemma.

**Lemma 5.** Assume that \( (F_0) \) holds. Then, \( u(t) \) is a positive solution of (1) if only if \( x(t) \) is a positive solution of the following problem:
\[
-x'' + \rho x = F(t, H(x - \omega)(x - \omega)), \tag{21}
\]

with \( x(t) \geq \omega(t) \). Here, \( H(t) \) denotes the Heaviside function of a single real variable:
\[
H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t \leq 0. \end{cases} \tag{22}
\]

Let \( E \) denote the Banach space \( C[0, 1] \) with the norm \( ||x|| = \max_{t \in [0, 1]} |x(t)| \).
Define a cone \( K \subset E \) by
\[
K = \left\{ x(t) \in C[0, 1] : \min_{t \in [0, 1]} x(t) \geq \sigma ||x|| \right\}, \tag{23}
\]

where \( \theta \in (0, 1/2) \), \( \sigma = \min_{t \in [0, 1]} \theta (\sigma / ||C|| G(t, t) \in (0, 1) \). Define an operator \( T \) by
\[
T(x)(t) = \int_0^1 G(t, s)F(s, H(x - \omega)(x - \omega))ds \tag{24}
\]

**Lemma 6.** Assume that \( (F_\rho) \) holds. Then, \( T(K) \subset K \), and \( T : K \rightarrow K \) is completely continuous.

**Proof.** For any \( x(t) \in K \), from Lemma 3, it follows that
\[
T(x)(t) = \int_0^1 G(t, s)F(s, H(x - \omega)(x - \omega))ds
\geq \delta \int_0^1 G(t, s)G(s, s)F(s, H(x - \omega)(x - \omega))ds
= \frac{\delta}{C} \int_0^1 CG(s, s)F(s, H(x - \omega)(x - \omega))ds
\geq \frac{\delta}{C} \int_0^1 G(t, s)F(s, H(x - \omega)(x - \omega))ds, \tag{25}
\]

which implies that \( T(K) \subset K \).

Now, we show that \( T : K \rightarrow K \) is completely continuous.
First, we show that $T$ maps the bounded set into itself. Since $e$ and $f$ are continuous, for any given $c > 0$, let

$$L = \max \left\{ F(t, H(x - \omega)(x - \omega)) : 0 \leq t \leq 1, 0 \leq x \leq c \right\}. \quad (26)$$

Then, for $x \in \mathcal{K}_e$, we have

$$|T(x)(t)|_\infty = \left| \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds \right|_\infty \leq L \max G(t, s),$$

which implies that $T(\mathcal{K}_e)$ is uniformly bounded.

Second, for $t_1, t_2 \in [0, 1]$, we have

$$|Tx(t_2) - Tx(t_1)| = \left| \int_0^1 [G(t_2, s) - G(t_1, s)] F(s, H(x - \omega)(x - \omega)) ds \right| \leq L \max \left| \frac{\partial G(t, s)}{\partial t} \right| |t_2 - t_1|,$$

which implies that the operator $T$ is equicontinuous.

Thus, by applying the Arzela-Ascoli theorem [17], we obtain that $T(\mathcal{K}_e)$ is relatively compact, namely, the operator $T$ is compact.

Finally, we claim that $T : \mathcal{K}_e \rightarrow K$ is continuous. Assume that $(x_n)_{n=1}^{\infty} \subset \mathcal{K}_e$ which converges to $x(t)$ uniformly on $[0, 1]$. By Lebesgue’s dominated convergence theorem and letting $n \rightarrow \infty$, we have

$$\|Tx_n(t) - Tx(t)\| = \left| \int_0^1 G(t, s) F(s, H(x_n(x - \omega))(x_n - \omega)) \right. - F(s, H(x_n(x - \omega))(x_n - \omega)) ds \left| \leq C \int_0^1 G(s, s) H(x_n(x - \omega))(x_n - \omega) ds \rightarrow 0, \right.$$  

as $n \rightarrow +\infty$. \quad (29)

So, $T$ is continuous on $\mathcal{K}_e$. The proof is completed. At the end of this section, let

$$e^* = \max_{t \in [0, 1]} e(t) > 0, \omega^* = \max_{t \in [0, 1]} \omega(t). \quad (30)$$

Define the height functions

$$\Phi_e(r) = \min \left\{ f(t, u, \phi) : (t, u, \phi) \in [0, 1] \times [0, r] \times \left[ \frac{0, r}{\beta} \right] \right\}, \quad \Phi_e(t, r) = \max \left\{ f(t, u, \phi) : (u, \phi) \in [0, r] \times \left[ \frac{0, r}{\beta} \right] \right\}. \quad (31)$$

In addition, we need to select some suitable open bounded sets. For any $\gamma > 0$, let

$$\Omega' = \left\{ x \in E : \min_{t \in [0, 1]} x(t) < \sigma \gamma \right\}, B' = \{ x \in E : ||x|| < \gamma \},$$

$$\Omega' = \Omega' \cap K, \partial \Omega' = \partial \Omega' \cap K,$$

$$B' = B' \cap K, \partial B' = \partial B' \cap K. \quad (32)$$

From [19, 20], we can conclude the lemma below.

**Lemma 7.**

(i) $\Omega', B'$ are open relative to $K$

(ii) $B' \subset \Omega' \subset B'$

(iii) $x \in \partial \Omega'$ if and only if $\min_{t \in [0, 1]} x(t) = \sigma \gamma$

(iv) If $x \in \partial \Omega'$, then $\sigma \gamma \leq x(t) \leq \gamma$, for $t \in [0, 1 - \theta]$

3. Main Results

**Theorem 8.** Assume that $(F_0)$ holds. In addition, the function $f$ satisfies the following assumption:

$$(F_0)$$

Then, we have

(i) If $\sigma \alpha > \omega^*$, then (1) has at least one partly positive solution $(u, \phi)$, namely,

$$u(t) > 0, \quad \text{for } t \in [\theta, 1 - \theta] \quad (34)$$

(ii) If $\alpha \delta > C \alpha^*$, then (1) has at least one positive solution $(u, \phi)$, which satisfies

$$u(t) > 0, \quad \text{for } t \in [0, 1] \quad (35)$$

Proof. For any $x \in \partial \mathcal{B}'$, it is obvious that

$$H(x - \omega)(x - \omega) \leq ||x|| = \alpha,$$

$$\int_0^1 K(t, s) H(x - \omega)(x(s) - \omega(s)) ds \leq \frac{1}{6} \alpha. \quad (36)$$
Then, from \((F_1)\) it follows that

\[
T(x)(t) = \int_0^1 G(t, s) F(s, H(x - \omega)(x - \omega)) ds \\
= \mu \int_0^1 G(t, s) \left[ H(x - \omega)(x - \omega) \int_0^1 K(s, \tau) H(x - \omega) \right. \\
\left. \cdot |x(\tau) - \omega(\tau)| d\tau \right] ds + \int_0^1 G(t, s) [f(s, H(x - \omega) \\
\cdot (x - \omega) \int_0^1 K(s, \tau) H(x - \omega) |x(\tau) - \omega(\tau)| d\tau] ds \\
+ e(s) ds \leq \mu \frac{\alpha^2}{6} C \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds \\
+ C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds < \alpha = \|x\|,
\]

(37)

which implies that (a) of Lemma 1 holds.

Let

\[
\Psi(\rho) = \max_{\|\rho\|\leq \rho} \left\{ H(x - \omega)(x - \omega) \cdot \int_0^{\theta} K(\tau, \tau) H(x - \omega) |x(\tau) - \omega(\tau)| d\tau \right\}.
\]

(38)

From [21], we have that

\[
\lim_{\rho \to +\infty} \frac{\Psi(\rho)}{\rho} = +\infty.
\]

(39)

Then, there exists a \(\beta > 0\) with \(\sigma \beta > \alpha\) such that

\[
H(\sigma \beta - \omega)(\sigma \beta - \omega) \cdot \int_0^{\theta} K(\tau, \tau) H(\sigma \beta - \omega) |\sigma \beta - \omega(\tau)| d\tau > \Lambda \beta,
\]

(40)

where \(\Lambda\) satisfies

\[
\Lambda \rho C \int_0^{\theta} G(s, s) K(s, s) ds > 1.
\]

(41)

Let \(\eta(t) = 1\); now we prove that \(x \neq Tx + \lambda\), for \(x \in \partial \Omega_\beta^K\) and \(\lambda > 0\). On the contrary, if there exists a pair of \(x_0 \in \partial \Omega_\beta^K\) and \(\lambda_0 > 0\) such that \(x_0(t) = T(x_0)(t) + \lambda_0\), then from (iv) of Lemma 7, it follows that

\[
\sigma \beta = \sigma \|x_0\| \leq x_0(t) \leq \beta, \quad \text{for } t \in [\theta, 1 - \theta].
\]

(42)

Furthermore, for \(t \in [\theta, 1 - \theta]\), we have

\[
\|x_0\| \geq \min_{\theta \leq \tau \leq 1 - \theta} x_0(t) = \min_{\theta \leq \tau \leq 1 - \theta} T x_0(t) + \lambda_0 \\
= \min_{\theta \leq \tau \leq 1 - \theta} \mu \int_0^1 G(t, s) H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x_0 - \omega) \cdot |x(\tau) - \omega(\tau)| d\tau ds + \min_{\theta \leq \tau \leq 1 - \theta} G(t, s) \\
\cdot f(s, H(x_0 - \omega)(x_0 - \omega)), \\
\cdot \left[ f(s, H_0 - \omega)(x_0 - \omega), \right] \int_0^1 K(s, \tau) H(x_0 - \omega) - \omega(\tau) d\tau ds + \lambda_0 \\
\geq \min_{\theta \leq \tau \leq 1 - \theta} \mu \int_0^1 G(t, s) H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x_0 - \omega) \\
\cdot (x - \omega) |x(\tau) - \omega(\tau)| d\tau ds \geq \min_{\theta \leq \tau \leq 1 - \theta} \mu \delta H(t, t) \\
\cdot \int_0^1 G(t, s) H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x_0 - \omega) \\
\cdot \left[ f(s, H(x_0 - \omega)(x_0 - \omega)), \right] \int_0^1 K(s, \tau) H(x_0 - \omega) - \omega(\tau) d\tau ds + \lambda_0 \\
\geq \min_{\theta \leq \tau \leq 1 - \theta} \mu \int_0^1 G(t, s) H(x_0 - \omega)(x_0 - \omega) \int_0^1 K(s, \tau) H(x_0 - \omega) \\
\cdot |x(\tau) - \omega(\tau)| d\tau ds = \mu \sigma C \int_0^1 G(s, s) K(s, s) \\
\cdot \left[ f(s, H(x_0 - \omega)(x_0 - \omega)), \right] \int_0^1 K(s, \tau) H(x_0 - \omega) - \omega(\tau) d\tau ds \\
\cdot \cdot \cdot \Lambda \beta > \beta = \|x_0\|,
\]

(43)

which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Since \(\alpha < \sigma \beta\), from Lemma 7, we have \(\overline{B}_\beta^K \subset B_{\sigma \beta}^\rho \subset \Omega_\beta^K\). Therefore, by Lemma 7, we can get that \(T\) has at least one positive fixed-point \(x(t) \in \Omega_\beta^K \setminus B_\beta^K\). Hence, the inequalities hold,

\[
\|x\| \geq \sigma \alpha \leq \min_{\theta \leq \tau \leq 1 - \theta} x(t) \leq \sigma \beta.
\]

(44)

On the other hand, since \(\sigma \|x\| \leq \min_{\theta \leq \tau \leq 1 - \theta} x(t) \leq \sigma \beta\), we have \(\|x\| \leq \beta\).

(i) Since

\[
\min_{\theta \leq \tau \leq 1 - \theta} x(t) \geq \sigma \alpha > \omega^* > \omega(t) = \int_0^1 G(t, s) e(s) ds,
\]

(45)
we have
\[ u(t) = x(t) - \omega(t) > 0, \quad t \in [\theta, 1 - \theta] \] (46)

(ii) From Lemmas 3 and 4, we have
\[ x(t) \geq \frac{\delta}{C} G(t, t) \|t\| = \frac{\delta}{C} G(t, t)x(t) \geq \frac{\delta}{C} \frac{\xi}{\xi^*} \int_0^t G(t, s)ds \]
\[ \geq \frac{\delta}{C} \frac{\xi}{\xi^*} \int_0^t G(t, s)ds = \frac{\delta}{C} \frac{\xi}{\xi^*} \cdot \omega(t) > \omega(t), \] (47)
which implies that \( u(t) = x(t) - \omega(t) > 0 \)

Therefore, (1) has one positive solution \((u, \phi) = (u, \int_0^K (t, s)u(s)ds)\).

**Theorem 9.** Assume that \((F_3)\) holds. In addition, the function \(f\) satisfies the following assumptions:

- \((F_3)\) There exists \(a > \max \{\omega^*/\sigma, Ce^*/\delta \xi\}\) such that \(\Phi_+(a) \leq 0\) and
\[
\frac{\sigma^2}{6} C \int_0^t G(s, ds) + C \int_0^t G(s, s)e(s)ds + C \int_0^t G(s, s)\Phi^+(s, a, \alpha)ds < a. \] (48)

- \((F_4)\) There exists a \(r^* \in (0, \sigma - \omega^*)\) such that
\[
\frac{\mu^{r^2}}{6} C \int_0^t G(s, s)ds + C \int_0^t G(s, s)e(s)ds + C \int_0^t G(s, s)\Phi^+(s, r^*)ds < r^*. \] (49)

- \((F_5)\) \(\lim_{u \to 0^+} f(t, u, \phi)/(u + \phi) = +\infty, \) uniformly for \(t \in [0, 1]\).
Then, (1) has at least two positive solutions \((u_i, \phi_i)\) \((i = 1, 2)\), which satisfies
\[ 0 < u_1(t) < r^*, \min_{t \in [0, 1]} u_2(t) > \sigma - \omega^*. \] (50)

**Proof.** From \((F_2)\) and Theorem 8, it follows that there exists a solution \(u_2(t) \geq 0\) and
\[
\min_{t \in [0, 1]} u_2(t) > \sigma - \omega^*. \] (51)

Now, we apply Lemma 1 to prove the existence of another solution \(u_1(t)\).
Since \(r^* < \sigma - \omega^* < \alpha\) and \(\Phi_+(\alpha)\), then we can define the operator
\[
\tilde{T}(u)(t) = \int_0^t G(t, s)F(s, u(s))ds. \] (52)

For any \(u \in \partial B_{r^*}^C\), it is obvious that
\[
\int_0^1 K(t, s)u(s)ds \leq \int_0^1 K(t, s)u(s)ds \leq \frac{1}{6} r^*. \] (53)

Then, we have
\[
\tilde{T}(u)(t) = \int_0^t G(t, s)F(s, u(s))ds \]
\[
\leq \mu \frac{r^2}{6} C \int_0^t G(s, s)ds + C \int_0^t G(s, s)e(s)ds + C \int_0^t G(s, s)\Phi^+(s, r^*)ds \]
\[
+ C \int_0^t G(s, s)\Phi^+(s, r^*)ds < r^* = \|u\|. \] (54)

which implies that (a) of Lemma 1 holds.
Since \(\lim_{u \to 0^+} f(t, u, \phi)/(u + \phi) = +\infty, \) uniformly for \(t \in [0, 1]\), there exists a \(r^* < r^*\) such that
\[ f(t, u, \phi) > M(u + \phi), \quad \text{for} \ 0 < u + \phi \leq r^*, \] (55)
where \(M\) satisfies
\[
M\delta\sigma \cdot \max_{0 \leq t \leq 1} G(t, t) \cdot \int_0^1 G(s, s) \left[ 1 + \int_0^t K(s, \tau)d\tau \right] ds > 1. \] (56)

Let \(\eta(t) = 1, \) now we prove that \(u \not\in \tilde{T}u + \lambda, \) for \(u \in \partial \Omega_{C/2}^C\) and \(\lambda > 0.\) On the contrary, if there exists a pair of \(u_0 \in \partial \Omega_{C/2}^C\) and \(\lambda_0 > 0\) such that \(u_0(t) = \tilde{T}(u_0)(t) + \lambda_0, \) then from (iv) of Lemma 7, it follows that
\[ u_0(t) \leq \frac{r^*}{2}, \phi_0(t) = \int_0^t K(t, s)u_0(s)ds \leq \frac{r^*}{12}. \] (57)

Furthermore, we have
\[
\|u_0\| = \|\tilde{T}(u_0)(t)\| + \lambda_0 \geq \left\| \int_0^1 G(t, s)f(s, u(s), \int_0^t K(s, \tau)u(s)d\tau)ds \right\| \]
\[
\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^t G(s, s)f(s, u(s), \int_0^t K(s, \tau)u(s)d\tau)ds \]
\[
\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^t G(s, s)f(s, u(s), \int_0^t \int_0^s K(s, \tau)u(\tau)d\tau)d\tau)ds \]
\[
\geq \delta \max_{0 \leq t \leq 1} G(t, t) \int_0^t \int_0^s K(s, t)u(t)d\tau)ds \]
\[
\geq M\delta\sigma \cdot \max_{0 \leq t \leq 1} G(t, t) \cdot \int_0^1 G(s, s) \left[ 1 + \int_0^t K(s, \tau)d\tau \right] ds \]
\[
\cdot \|u_0\| > \|u_0\|. \] (58)
which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Therefore, from Lemma 1, we can get that $\tilde{T}$ has at least one positive fixed-point $u_1(t) \in \overline{\Omega^\varepsilon K \setminus \partial B^\varepsilon K}$. Hence, the inequalities hold

$$
\|u_1\| \geq \frac{r_1}{2}, \quad \sigma \frac{r_1}{2} \leq \min_{t \in [0,1]} u_1(t) \leq \sigma r^*.
$$

(59)

On the other hand, since $\sigma \|u_1\| \leq \min_{t \in [0,1]} u_1(t) \leq \sigma r^*$, we have $\|u_1\| \leq r^*$.

Finally, since

$$
\min_{t \in [0,1]} u_2(t) \leq \sigma \alpha - \omega^* > r^*, \quad \|u_1\| \leq r^*,
$$

(60)

(1) has at least two positive solutions.

4. Examples

Example 10. Let us consider the following system:

$$
\begin{cases}
-u'' + u = \phi u + \frac{1}{4} e^u + \frac{1}{4} \cos \pi \phi - \frac{1}{4} t^2 & 0 < t < 1, \\
-\phi'' = \frac{1}{4} u, \\
\phi(0) = \phi(1) = 0, \quad \phi''(0) = \phi''(1) = 0,
\end{cases}
$$

(61)

where $\mu = 1, \rho = 1, f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and

$$
f = \frac{1}{4} (e^u + \cos \pi \phi - t^2) \geq -\frac{1}{4} t^2 = -e(t) \quad \text{for} \ (t, u, \phi) \in [0,1] \times \mathbb{R}^2.
$$

(62)

It is obvious that $(F_0)$ holds. Via some computations, we have

$$
e^* = \frac{1}{4}, \quad C = 1, \quad \delta = \frac{1}{\sinh 1}, \quad J^*_1 = J^*_2 = \frac{e^{1/2} - e^{-(1/2)}}{e^{1/2} + e^{-(1/2)}} < 1.
$$

Choosing $\theta = 1/4 \in (0,1/2)$, $\xi = 1/2$. Then, we have

$$
\sigma = \min_{t \in [1/4,3/4]} \frac{\delta G(t, t)}{C} = \frac{\sinh (3/4) \sinh (1/4)}{(\sinh 1)^2},
$$

$$
\omega^* = \frac{\sinh 1 - 2 \sinh (1/2)}{\sinh 1}.
$$

(64)

Furthermore, we have

$$
\frac{\omega^*}{\sigma} = \frac{\sinh 1 - 2 \sinh (1/2)}{\sinh (3/4) \sinh (1/4)} \sinh 1 < 3,
$$

$$
Ce^* < \sin 1 (J^*_1 + J^*_2) < 3.
$$

Choosing $\alpha = 3 > \max \{\omega^*/\sigma, Ce^* / \delta \xi\}$, $r^* = 3/10 \in (0, \sigma \alpha - \omega^*)$. Then,

$$
\Phi_* (3, t) \geq \frac{1}{4} (e^t + 1),
$$

$$
\Phi_*(t, 3) \geq \frac{1}{4} (e^{3/10} + 1).
$$

(66)

It is easy to get

$$
\frac{\mu^2}{6} (1 + C) \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, \alpha) ds
$$

$$
\leq \frac{3}{2} \frac{e^{-1}}{\sinh 1} + \frac{1}{2} \frac{e^{-1}}{\sinh 1} \left[ \frac{1}{4} (e^3 + 1) \right] \leq 3,
$$

(67)

$$
\frac{\mu^2}{6} (1 + C) \int_0^1 G(s, s) ds + C \int_0^1 G(s, s) e(s) ds + C \int_0^1 G(s, s) \Phi^*(s, \alpha^*) ds
$$

$$
\leq \frac{3}{200} \frac{e^{-1}}{\sinh 1} + \frac{1}{2} \frac{e^{-1}}{\sinh 1} \left[ \frac{1}{4} (e^{3/10} + 1) \right] \leq \frac{3}{10},
$$

(68)

which implies that $(F_2)$ and $(F_3)$ hold.

Finally, it is obvious that

$$
\lim_{u+\phi\to0+} \frac{f(t, u, \phi)}{u + \phi} = +\infty, \quad \text{uniformly for} \ t \in [0,1].
$$

(69)

So $(F_4)$ holds.

Therefore, by Theorem 9, (61) has two positive solutions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

Acknowledgments

The authors were supported by the Fundamental Research Funds for the Central Universities (B200202003), the Guizhou Provincial Science and Technology Fund (QKH-JICHU[2017], Grant no. 1408) and the Scientific Research Starting Foundation for Doctorate Research from Shandong Jiaotong University.
References


