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Research Article

Monotone Iterative Method for Fractional Differential Equations with Integral Boundary Conditions

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In this paper, the existence of extremal solutions for fractional differential equations with integral boundary conditions is obtained by using the monotone iteration technique and the method of upper and lower solutions. The main equations studied are as follows: $\begin{cases} -D_{0+}^{\alpha}u(t) = f(t, u(t)),\ t \in [0,1], \\ u(0) = 0,\ u(1) = \int_0^1 u(t)\mathrm{d}A(t), \end{cases}$ where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order $\alpha \in (1,2)$ and $A(t)$ is a positive measure function. Moreover, an example is given to illustrate the main results.

1. Introduction

The purpose of this paper is to study the following differential equations with integral boundary conditions:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = 0, \\ u(1) = \int_0^1 u(t)\mathrm{d}A(t), \end{cases} \tag{1}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative and $1 < \alpha < 2$ and $A(t)$ is a positive measure function, and it satisfies $\int_0^1 G(t)\mathrm{d}A(t) < G(1)$.

Fractional differential equations have been widely used in physics, chemistry, aerodynamics, electrodynamics of complex media, and rheology of polymers [1–7]. As a result, various nonlinear functional analysis methods have been used to study the existence of solutions for differential equations [8–38]. However, it is difficult to obtain the exact solutions of fractional differential equations, so the monotone iteration method and the upper and lower solutions are generally used to obtain the approximate solutions of fractional differential problems. This method is also applicable to both initial value problems and boundary value problems for integer-order differential equations and differential systems [39–43]. In recent years, many scholars have used this method to study various fractional differential equation problems [44–55]. For example, in [47, 48, 52], the authors paid attention to the Riemann–Liouville fractional differential equations of order $\alpha \in (1,2)$, and in [44, 45, 49, 55], the authors considered the Riemann–Liouville fractional differential equation of order $\alpha \in (0,1)$, while in [46], the authors considered the boundary value problem with Riemann–Liouville fractional order $\alpha \in (2,3)$. Of course, some papers also use the monotone iterative method to deal with nonlinear Caputo fractional differential equations [56].

Based on the upper and lower solutions, this paper presents a method to prove the existence of solutions of Riemann–Liouville fractional differential equation (1). By using the monotone iteration technique coupled with the upper and lower solution method, a new comparison principle is established and the existence of the extremal solution of integral boundary value problems (1) is proved.

This paper is mainly divided into the following two parts: Section 2 mainly introduces the preparation of this article, and then in Section 3, the monotone sequence of solutions is constructed, and the main result of integral boundary value problems (1) is given.

2. Preliminaries

In this section, we will briefly introduce some of the necessary definitions and results that will be used in the main results.

Definition 1 (see [2, 5]). The fractional integral of order $\alpha > 0$ of a function u: $(0, \infty) \longrightarrow R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \mathrm{d}s, \tag{2}$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2 (see [2, 5]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function u: $(0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \mathrm{d}s, \tag{3}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Let $E = \{u \in C[0,1] \colon D_{0+}^{\alpha} u \in C[0,1]\}$ be endowed with the norm $\|u\| = \max\{\|u\|_{\infty}, \|D_{0+}^{\alpha} u\|_{\infty}\}$ in which $\|u\|_{\infty} = \max_{t \in [0,1]} |u(t)|$, and then $(E, \|\cdot\|)$ is a Banach space.

Definition 3. We say that $u \in E$ is an upper solution of (1) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha} u(t) \geq f(t, u(t)), & t \in [0,1], \\ u(0) = 0, \quad u(1) \geq \int_0^1 u(t) \mathrm{d}A(t). \end{cases} \tag{4}$$

Definition 4. We say that $v \in E$ is a lower solution of (1) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha} v(t) \leq f(t, v(t)), & t \in [0,1], \\ v(0) = 0, \quad v(1) \leq \int_0^1 v(t) \mathrm{d}A(t). \end{cases} \tag{5}$$

Denote

$$g(t) = \frac{\alpha - 2}{\Gamma(\alpha - 1)} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma((k+1)\alpha - 2)}. \tag{6}$$

It is easy to check that (see [57, 58])

$$\begin{gathered} g(0) = \frac{\alpha - 2}{\Gamma(\alpha - 1)} < 0, \\ g'(t) > 0, \quad t \in [0, +\infty), \\ \lim_{t \longrightarrow +\infty} g(t) = +\infty. \end{gathered} \tag{7}$$

Therefore, there exists a unique number $b^* > 0$ such that

$$g(b^*) = 0. \tag{8}$$

Set $G(t) = t^{\alpha-1} E_{\alpha,\alpha}(bt^{\alpha})$, where $E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty} (x^k / (\Gamma((k+1)\alpha)))$ is the Mittag-Leffler function (see [2, 5]).

In this article, we list the following assumption for convenience:

$(H1)$: the parameter b satisfies $b \in (0, b^*]$.

$(H2)$: $A(t)$ is a positive measure function and $0 < G(1) - \int_0^1 G(t) \mathrm{d}A(t)$.

$(H3)$: assume that $w_0, v_0 \in E$ are the upper and lower solutions of problem (1), respectively, and $v_0(t) \leq w_0(t)$, $t \in [0,1]$.

$(H4)$: $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and $f(t, w) - f(t, v) \geq -b(w - v)$ for $v_0(t) \leq v \leq w \leq w_0(t), t \in [0,1]$.

Next, we will consider the following auxiliary linear boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + bu(t) = y(t), & t \in [0,1], \\ u(0) = 0, \\ u(1) = \int_0^1 u(t) \mathrm{d}A(t) + c, & c \in \mathbb{R}. \end{cases} \tag{9}$$

Lemma 1. *Suppose that (H1) and (H2) hold and $y \in C[0,1]$. Then, fractional boundary value problem (9) has the following unique solution:*

$$u(t) = \int_0^1 K(t,s) y(s) \mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t) \mathrm{d}A(t)}, \tag{10}$$

where

$$\begin{aligned} K(t,s) &= K_0(t,s) + G(t)h(s), \\ h(s) &= \frac{\int_0^1 K_0(t,s) \mathrm{d}A(t)}{G(1) - \int_0^1 G(t) \mathrm{d}A(t)}, \\ K_0(t,s) &= \frac{1}{G(1)} \begin{cases} G(t)G(1-s), & 0 \leq t \leq s \leq 1, \\ G(t)G(1-s) - G(t-s)G(1), & 0 \leq s \leq t \leq 1. \end{cases} \end{aligned} \tag{11}$$

Proof. The main idea of Lemma 1 comes from [57]. By [2], we first find the solution of the fractional differential equations

$$-D_{0+}^{\alpha} u(t) + bu(t) = y(t), \quad t \in [0,1], \tag{12}$$

with two-point boundary condition

$$\begin{gathered} u(0) = 0, \\ u(1) = a \in \mathbb{R}, \end{gathered} \tag{13}$$

which can be expressed by

$$u(t) = -\int_0^t G(t-s) y(s) \mathrm{d}s + c_1 G(t) + c_2 G'(t). \tag{14}$$

Since $u(0)=0$, we have $c_2=0$. Then, by the condition $u(1)=a$, we calculated that

$$c_1 = \frac{\int_0^1 G(1-s)y(s)\mathrm{d}s + a}{G(1)}. \tag{15}$$

Therefore, the solution of (12) and (13) is

$$\begin{aligned} u(t) &= -\int_0^t G(t-s)y(s)\mathrm{d}s + \frac{\int_0^1 G(1-s)y(s)\mathrm{d}s + a}{G(1)} G(t) \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{u(1)}{G(1)} G(t). \end{aligned} \tag{16}$$

Next, we consider problem (9). Integrating equality (16) with respect to $A(t)$, we have

$$\begin{aligned} \int_0^1 u(t)\mathrm{d}A(t) &= \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) \\ &\quad + \frac{u(1)\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}. \end{aligned} \tag{17}$$

Making use of the condition $u(1)=\int_0^1 u(t)\mathrm{d}A(t)+c$ in the above equality yields

$$u(1) - c = \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + \frac{u(1)\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}, \tag{18}$$

and then we get

$$u(1)\left(1 - \frac{\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}\right) = \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + c. \tag{19}$$

Obviously,

$$u(1) = \frac{G(1)\int_0^1\int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + cG(1)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \tag{20}$$

Therefore,

$$\begin{aligned} u(t) &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{G(t)\left(G(1)\int_0^1\int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + cG(1)\right)}{G(1)\left(G(1) - \int_0^1 G(t)\mathrm{d}A(t)\right)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{G(t)\left(\int_0^1\int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + c\right)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{\int_0^1 G(t)\left(\int_0^1 K_0(t,s)\mathrm{d}A(t)\right)y(s)\mathrm{d}s}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \end{aligned} \tag{21}$$

The proof is completed. □

Lemma 2. *Suppose that (H1) and (H2) hold, and $u \in E$ satisfies*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) \geq 0, \\ u(0) = 0, \quad u(1) \geq \int_0^1 u(t)\mathrm{d}A(t). \end{cases} \tag{22}$$

Then, for $t \in [0,1]$, $u(t) \geq 0$.

Proof. Let $y(t) = -D_{0+}^{\alpha}u(t) + bu(t)$ and $c = u(1) - \int_0^1 u(t)\mathrm{d}A(t)$. From (12), we have $y(t) \geq 0$, $c \geq 0$, and

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = y(t), \\ u(0) = 0, \quad u(1) = \int_0^1 u(t)\mathrm{d}A(t) + c. \end{cases} \tag{23}$$

By Lemma 1, we obtain that problem (23) has unique solution u, which can be expressed as follows:

$$u(t) = \int_0^1 K(t,s)y(s)\mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \tag{24}$$

From [57], it follows that $K_0(t,s) \geq 0$ for $t,s \in [0,1]$. This together with $(H1)$ and $(H2)$ yields

$$K(t,s) = K_0(t,s) + G(t)\frac{\int_0^1 K_0(t,s)\mathrm{d}A(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \geq 0, \quad t,s \in [0,1]. \tag{25}$$

Hence, we conclude that

$$u(t) \geq \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \geq 0, \quad t \in [0,1], \tag{26}$$

which completes the proof. □

3. Main Results

For $v_0, w_0 \in E$ with $v_0(t) \leq w_0(t)$ for $t \in [0,1]$, we denote an ordered interval:

$$[v_0, w_0] = \{u \in E\colon v_0(t) \leq u(t) \leq w_0(t), \quad t \in [0,1]\}. \tag{27}$$

Theorem 1. *Suppose (H1)–(H4) hold, and then there exist monotone iterative sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ such that $v_n \longrightarrow v^*, w_n \longrightarrow w^*$ as $n \longrightarrow \infty$ uniformly in $[v_0, w_0]$, and v^*, w^* are a minimal and a maximal solution of (1) in $[v_0, w_0]$, respectively.*

Proof. For $v_{n-1}, w_{n-1} \in E$, $n \geq 1$, we define two sequences $\{v_n\}, \{w_n\} \subset E$ satisfying the following fractional differential equation:

$$\begin{cases} -D_{0+}^{\alpha} v_n(t) + bv_n(t) = f(t, v_{n-1}(t)) + bv_{n-1}(t), & t \in [0,1], \\ v_n(0) = 0, v_n(1) = \int_0^1 v_n(t)\mathrm{d}A(t), \end{cases} \tag{28}$$

$$\begin{cases} -D_{0+}^{\alpha} w_n(t) + bw_n(t) = f(t, w_{n-1}(t)) + bw_{n-1}(t), \\ \quad t \in [0,1], \\ w_n(0) = 0, \quad w_n(1) = \int_0^1 w_n(t)\mathrm{d}A(t). \end{cases} \tag{29}$$

By consideration of Lemma 1, for any $n \geq 1$, problems (28) and (29) have a unique solution $v_{n+1}(t), w_{n+1}(t)$, respectively, which are well defined.

Firstly, we need to show that, for any $t \in [0,1]$, $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Let $x(t) = v_1(t) - v_0(t)$, and the definition of $v_1(t)$ together with $(H3)$ yields

$$\begin{cases} -D_{0+}^{\alpha} x(t) + bx(t) = -D_{0+}^{\alpha}(v_1(t) - v_0(t)) + b(v_1(t) - v_0(t)) = -D_{0+}^{\alpha} v_1(t) + bv_1(t) - [-D_{0+}^{\alpha} v_0(t) + bv_0(t)] \\ \geq f(t, v_0(t)) + bv_0(t) - f(t, v_0(t)) - bv_0(t) = 0, \\ x(0) = v_1(0) - v_0(0) = 0, \\ x(1) = v_1(1) - v_0(1) \geq \int_0^1 v_1(t)\mathrm{d}A(t) - \int_0^1 v_0(t)\mathrm{d}A(t) = \int_0^1 x(t)\mathrm{d}A(t). \end{cases} \tag{30}$$

In the light of Lemma 2, we have $x(t) \geq 0, t \in [0,1]$, namely, $v_1(t) \geq v_0(t)$. Similarly, it can be shown that $w_0(t) \geq w_1(t)$, $t \in [0,1]$.

Secondly, we make $h(t) = w_1(t) - v_1(t)$. From $(H4)$, we get

$$\begin{aligned} -D_{0+}^{\alpha} h(t) &= f(t, w_0(t)) - b[w_1(t) - w_0(t)] - f(t, v_0(t)) \\ &\quad + b[v_1(t) - v_0(t)] \\ &\geq -b[w_0(t) - v_0(t)] - b[w_1(t) - w_0(t)] \\ &\quad + b[v_1(t) - v_0(t)] \\ &= -bh(t). \end{aligned} \tag{31}$$

Also, $h(0) = 0$ and $h(1) = \int_0^1 h(t)\mathrm{d}A(t)$. Thus, Lemma 2 implies that, for any $t \in [0,1]$, $w_1(t) \geq v_1(t)$.

Thirdly, we prove that w_1, v_1 are upper and lower solutions of problem (1), respectively. Note that

$$\begin{aligned} -D_{0+}^{\alpha} v_1(t) &= f(t, v_0(t)) + bv_0(t) - bv_1(t) - f(t, v_1(t)) \\ &\quad + f(t, v_1(t)) \\ &= f(t, v_0(t)) - f(t, v_1(t)) + f(t, v_1(t)) \\ &\quad - b[v_1(t) - v_0(t)] \\ &\leq b[v_1(t) - v_0(t)] + f(t, v_1(t)) - b[v_1(t) - v_0(t)] \\ &= f(t, v_1(t)). \end{aligned} \tag{32}$$

And by assumption $(H4)$, $v_1(0) = 0$ and $v_1(1) = \int_0^1 v_1(t)\mathrm{d}A(t)$. This shows that v_1 is a lower solution of problem (1). Similarly, we can infer that w_1 is an upper solution of (1).

Using mathematical induction, it is easy to verify that

$$\begin{aligned} &v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq w_n(t) \leq \cdots \\ &\quad \leq w_1(t) \leq w_0(t). \end{aligned} \tag{33}$$

Clearly, it is easy to conclude that $\{v_n\}$ and $\{w_n\}$ are uniformly bounded in $C[0,1]$. Moreover, by Lemma 1, problems (28) and (29) are equivalent to the following integral equation:

$$\begin{aligned} v_n(t) &= \int_0^1 K(t,s)[f(s,v_{n-1}(s)+bv_{n-1}(s)]\mathrm{d}s,\\ w_n(t) &= \int_0^1 K(t,s)[f(s,w_{n-1}(s)+bw_{n-1}(s)]\mathrm{d}s, \end{aligned} \tag{34}$$

respectively. Therefore, the continuity of the functions $K(t,s)$ allows us to conclude that $\{v_n\}$ and $\{w_n\}$ are equicontinuous in $C[0,1]$. Using (28) and (29) again, we know that $\{D^{\alpha}_{0+}v_n\}$ and $\{D^{\alpha}_{0+}w_n\}$ are uniformly bounded and equicontinuous in $C[0,1]$. So, $\{v_n\}$ and $\{w_n\}$ are uniformly bounded and equicontinuous in E. Using the standard arguments, we have $\{v_n\}$ and $\{w_n\}$ converging, say, to v^* and w^*, uniformly on $[0,1]$, respectively. That is,

$$\begin{gathered} \lim_{n\longrightarrow\infty} v_n(t) = v^*(t),\\ \lim_{n\longrightarrow\infty} w_n(t) = w^*(t),\\ t\in[0,1]. \end{gathered} \tag{35}$$

Furthermore, $v^*(t)$ and $w^*(t)$ are the solutions of problem (1), and $v_0 \le v^* \le w^* \le w_0$ on [0, 1].

Next, we need to prove that v^* and w^* are extremal solutions of (1) in $[v_0,w_0]$. Let $u\in[v_0,w_0]$ be any solution of problem (1). We assume that $v_m(t)\le u(t)\le w_m(t), t\in[0,1]$ for some m. Take $p(t)=u(t)-v_{m+1}(t)$, $q(t)=w_{m+1}(t)-u(t)$. Then, by assumption $(H4)$, we obtain

$$\begin{gathered} \begin{cases} -D^{\alpha}_{0+}p(t)\ge -bp(t), & t\in[0,1],\\ p(0)=0, \quad p(1)=\int_0^1 p(t)\mathrm{d}A(t), \end{cases}\\ \begin{cases} -D^{\alpha}_{0+}q(t)\ge -bq(t), & t\in[0,1],\\ q(0)=0, \quad q(1)=\int_0^1 q(t)\mathrm{d}A(t). \end{cases} \end{gathered} \tag{36}$$

By Lemma 2, we have

$$v_{m+1}(t)\le u(t)\le w_{m+1}(t), \quad t\in[0,1]. \tag{37}$$

Applying mathematical induction, one has $v_n(t)\le u(t)\le w_n(t)$ on $[0,1]$ for any n. Taking the limit, we conclude $v^*(t)\le u(t)\le w^*(t), t\in[0,1]$. The proof is complete. □

Example 1. Consider the following problem:

$$\begin{cases} -D^{4/3}_{0+}u(t) = -\dfrac{u^3(t)}{10}+\dfrac{\sin t}{10}, & t\in[0,1],\\ u(0)=0,\\ u(1)=\displaystyle\int_0^1 u(t)\mathrm{d}t, \end{cases} \tag{38}$$

where $\alpha=4/3$, $A(t)=t$, and $f(t,u)=-(u^3/10)+(\sin t/10)$.

Taking $v_0(t)=-t^{1/3}$ and $w_0(t)=t^{1/3}$, we have

$$\begin{gathered} \begin{cases} -D^{4/3}_{0+}v_0(t)=0\le\dfrac{t}{10}+\dfrac{\sin t}{10}=f(t,v_0(t)), & t\in[0,1],\\ v_0(0)=0, v_0(1)=-1<-\dfrac{3}{4}=\displaystyle\int_0^1 v_0(t)\mathrm{d}t, \end{cases}\\ \begin{cases} -D^{4/3}_{0+}w_0(t)=0\ge-\dfrac{t}{10}+\dfrac{\sin t}{10}=f(t,w_0(t)), & t\in[0,1],\\ w_0(0)=0,\ w_0(1)=1>\dfrac{3}{4}=\displaystyle\int_0^1 w_0(t)\mathrm{d}t, \end{cases} \end{gathered} \tag{39}$$

which shows that $v_0(t)$ and $w_0(t)$ are a lower and an upper solution of (38), respectively, and $v_0(t)\le w_0(t)$. So, $(H3)$ holds.

Using the strictly monotone increasing property of gamma function $\Gamma(\cdot)$ on $[2,+\infty)$, we have

$$\begin{aligned} g(t) &= -\frac{2}{3\Gamma(1/3)}+\sum_{k=1}^{\infty}\frac{t^k}{\Gamma((4k/3)-(2/3))} = -\frac{2}{3\Gamma(1/3)}\\ &\quad+\frac{t}{\Gamma(2/3)}+\sum_{k=2}^{\infty}\frac{t^k}{\Gamma(((4(k-2))/3)+2)}\\ &\le -\frac{2}{3\Gamma(1/3)}+\frac{t}{\Gamma(2/3)}+\sum_{k=2}^{\infty}\frac{t^k}{\Gamma(k)} = -\frac{2}{3\Gamma(1/3)}\\ &\quad+\frac{t}{\Gamma(2/3)}+t\left(e^t-1\right), \quad t\in[0,+\infty). \end{aligned} \tag{40}$$

By MATLAB, we obtain $g(3/10)<-0.0568$. Therefore, $b^*>b:=(3/10)$. In addition, we have

$$f(t,w)-f(t,v)=-\left[\frac{w^3}{10}-\frac{v^3}{10}\right]\ge-\frac{3}{10}(w-v), \tag{41}$$

for $v_0(t)\le v\le w\le w_0(t), t\in[0,1]$. Hence, $(H1)$ and $(H4)$ hold.

Note that $G(t)=\sum_{k=0}^{+\infty}(3/10)^k(t^{(4k+1)/3}/(\Gamma((4k+4)/3)))$. Then,

$$\begin{aligned} \int_0^1 G(t)\mathrm{d}t &= \sum_{k=0}^{+\infty}\left(\frac{3}{10}\right)^k\frac{\int_0^1 t^{((4k+1)/3)}\mathrm{d}t}{\Gamma((4k+4)/3)}\\ &= \sum_{k=0}^{+\infty}\left(\frac{3}{10}\right)^k\frac{1}{((4k+4)/3)\Gamma(((4k+4)/3))}\\ &< \sum_{k=0}^{+\infty}\left(\frac{3}{10}\right)^k\frac{1}{\Gamma((4k+4)/3)}=G(1). \end{aligned} \tag{42}$$

It shows that $(H2)$ holds. Thus, Theorem 1 ensures that problem (38) has extremal solutions in $[v_0,w_0]$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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