

Research Article

New Results for Some Generalizations of Starlike and Convex Functions

Ali Ebadian,¹ Nafya Hameed Mohammed,² Ebrahim Analouei Adegani ,¹ and Teodor Bulboacă³

¹Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran

²Department of Mathematics, College of Basic Education, University of Raparin, Kurdistan Region, Iraq

³Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

Correspondence should be addressed to Ebrahim Analouei Adegani; analoey.ebrahim@gmail.com

Received 17 July 2020; Accepted 31 August 2020; Published 25 September 2020

Academic Editor: Konstantin M. Dyakonov

Copyright © 2020 Ali Ebadian et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of the current paper is to investigate several various problems for the categories $\mathcal{ST}_L(s)$, \mathcal{S}_{Ne}^* , and other related categories such as various new outcomes for the coefficients of f , together with majorization issues, the Hankel determinant, and the logarithmic coefficients with sharp inequalities and differential subordination implications.

Dedicated to the memory of Professor Stephan Ruscheweyh (1944-2019)

1. Introduction and Preliminaries

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane \mathbb{C} . Let \mathcal{A} be the category of functions f analytic in U that has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \quad (1)$$

and denoted by \mathcal{S} the subclass of all functions of \mathcal{A} which are univalent in U . Then, the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined as the coefficients of the series expansion

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in U. \quad (2)$$

These coefficients play an important role for various estimates in the theory of univalent functions (see for example

[1–3] and [4], Chapter 2), and note that we will use the notation γ_n instead of $\gamma_n(f)$.

Utilizing the principle of subordination, Ma and Minda [5] introduced the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, where we make here the assumptions that the function φ is univalent in the unit disk U and satisfies $\varphi(0) = 1$, with the power series expansion of the form

$$\varphi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 \cdots, \quad z \in U. \quad (3)$$

They considered the abovementioned classes as follows:

$$\begin{aligned} \mathcal{S}^*(\varphi) &:= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \\ \mathcal{C}(\varphi) &:= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \end{aligned} \quad (4)$$

where the symbol “ \prec ” stands for the usual subordination. Some special subclasses of the class $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ play a significant role in the Geometric Function Theory because of their geometric properties.

For instance, the categories $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ reduce to the categories $\mathcal{S}^*[A, B]$ and $\mathcal{C}[A, B]$ of the popular *Janowski starlike* and *Janowski convex functions* for $\varphi(z) = (1 + Az)/(1 + Bz)$ with $-1 \leq B < A \leq 1$, respectively. By replacing $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in these function families, we get the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of the *starlike functions of order α* and *convex functions of order α* , respectively. Especially, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the family of starlike functions and of convex functions in the unit disk U , respectively.

Further, for $\varphi(z) = \sqrt{1+z}$, we get the family \mathcal{S}_L^* defined by Sok’ol and Stankiewicz [6], including functions f such that $w = zf'(z)/f(z)$ stands in the region bounded by the right-half branch of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Moreover, the properties of the classes $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ consisting of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1, z \in U$ were considered by Mendiratta et al. in [7]. Raina and Sokół [8] studied the family $\mathcal{S}_c^* := \mathcal{S}^*(h)$, where $h(z) = z + \sqrt{1+z^2}$ maps U onto the crescent-shaped region $\{w \in \mathbb{C} : |w^2 - 1| < 2|w|, \operatorname{Re} w > 0\}$.

In addition, for $\varphi(z) = (1-z)^{-s}$ ($0 < s \leq 1$), we obtain the categories $\mathcal{ST}_{hpl}(s)$ and $\mathcal{CV}_{hpl}(s)$ studied by Kanas et al. [9], with the property that $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ lies in a domain bounded by a right branch of a hyperbola $\rho(s) = (2 \cos(\phi/s))^{-s}$ where $|\phi| < \pi s/2$, respectively. Also, Goel and Kumar [10] introduced the classes \mathcal{S}_{SG}^* and \mathcal{C}_{SG} for $\varphi(z) = 2/(1+e^{-z})$ where this function maps U onto a domain $\Delta_{SG} := \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$.

Lately, Masih and Kanas [11] introduced and studied the categories $\mathcal{ST}_L(s)$ and $\mathcal{CV}_L(s)$ by

$$\begin{aligned} \mathcal{ST}_L(s) &:= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \mathbb{L}_s(z) \right. \\ &:= \left. (1 + sz)^2, 0 < s \leq \frac{1}{\sqrt{2}} \right\}, \\ \mathcal{CV}_L(s) &:= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \mathbb{L}_s(z) \right. \\ &:= \left. (1 + sz)^2, 0 < s \leq \frac{1}{\sqrt{2}} \right\}, \end{aligned} \tag{5}$$

and investigated some outcomes regarding the behavior of the functions of these classes. The function \mathbb{L}_s maps the unit disk onto a domain bounded by the limaçon $[(u-1)^2 + v^2 - s^4]^2 = 4s^2[(u-1+s)^2 + v^2]$. Further, $\mathbb{L}_s(U)$ is symmetric respecting the real axis, and \mathbb{L}_s is starlike with respect to $\mathbb{L}_s(0) = 1$. Also, $\mathbb{L}_s'(0) > 0$ and $\operatorname{Re} \mathbb{L}_s(z) > 0$ in U (see for more details [11]); hence, \mathbb{L}_s satisfies the conditions of the category of the Ma-Minda functions (see [5]). In addition, for $n = 1, 2, \dots$ the functions

$$\begin{aligned} \Phi_{s,n}(z) &:= z \exp \left(\int_0^z \frac{\mathbb{L}_s(t^n) - 1}{t} dt \right) \\ &= z + \frac{2s}{n} z^{n+1} + \frac{(n+4)s^2}{2n^2} z^{2n+1} + \dots, z \in U, \\ K_{s,n}(z) &:= \int_0^z \exp \left(\int_0^x \frac{\mathbb{L}_s(t^n) - 1}{t} dt \right) dx \\ &= z + \frac{2s}{n(n+1)} z^{n+1} + \frac{(n+4)s^2}{2n^2(2n+1)} z^{2n+1} + \dots, z \in U, \end{aligned} \tag{6}$$

play as extremal functions for some problems for the classes $\mathcal{ST}_L(s)$ and $\mathcal{CV}_L(s)$, respectively.

For instance, the quantity $zf'(z)/f(z)$ (or $1 + zf''(z)/f'(z)$) lies in a domain bounded by the limaçon $[(u-1)^2 + v^2 - s^4]^2 = 4s^2[(u-1+s)^2 + v^2]$ in the category $\mathcal{ST}_L(s)$ (or $\mathcal{CV}_L(s)$) while $zf'(z)/f(z)$ (or $1 + zf''(z)/f'(z)$) lies in a domain bounded by a right branch of a hyperbola $\rho(s) = e^{-s} |\phi| < (\pi s/2)(2 \cos(\phi/s))$ where in the category $\mathcal{ST}_{hpl}(s)$ (or $\mathcal{CV}_{hpl}(s)$). Therefore, it is observed that these categories have different structures and geometric properties (see for more details [9, 11]).

Recently, Wani and Swaminathan [12] investigated the new Ma-Minda-type function classes \mathcal{S}_{Ne}^* and \mathcal{C}_{Ne} and obtained some characteristic properties of these classes defined by

$$\begin{aligned} \mathcal{S}_{Ne}^* &:= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_{Ne}(z) := 1 + z - \frac{z^3}{3} \right\}, \\ \mathcal{C}_{Ne} &:= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_{Ne}(z) := 1 + z - \frac{z^3}{3} \right\}. \end{aligned} \tag{7}$$

The function φ_{Ne} maps U onto the interior of the nephroid, a 2-cusped kidney-shaped curve,

$$\left[(u-1)^2 + v^2 - \frac{4}{9} \right]^3 = \frac{4v^2}{3}. \tag{8}$$

Further, for $n = 2, 3, \dots$, the functions

$$\begin{aligned} \Omega_n(z) &:= z \exp \left(\int_0^z \frac{\varphi_{Ne}(t^{n-1}) - 1}{t} dt \right) \\ &= z + \frac{z^n}{n-1} + \frac{z^{2n-1}}{2(n-1)^2} + \dots, z \in U, \\ \Lambda_n(z) &:= \int_0^z \exp \left(\int_0^x \frac{\varphi_{Ne}(t^{n-1}) - 1}{t} dt \right) dx \\ &= z + \frac{z^n}{n(n-1)} + \frac{z^{2n-1}}{2(2n-1)(n-1)^2} + \dots, z \in U, \end{aligned} \tag{9}$$

play the role of extremal functions for several problems for the categories \mathcal{S}_{Ne}^* and \mathcal{C}_{Ne} , respectively.

Finding the upper bound for coefficients has been one of the central topics of research in the Geometric Function Theory as it gives several properties of functions. In particular, the bound for the second coefficient gives growth and distortion theorems for the functions of the class \mathcal{S} . In [13], Ebadian et al. studied some coefficient problems for the categories $\mathcal{ST}_{hpl}(s)$, $\mathcal{CV}_{hpl}(s)$, \mathcal{S}_{SG}^* and related categories like sharp bounds for initial coefficients, logarithmic coefficients, Hankel determinants, and Fekete-Szegő problems. Also, they investigated some geometric properties as applications of differential subordinations.

According to the abovementioned issues, motivated essentially by the recent work [13], this paper is aimed at investigating some various problems for the categories $\mathcal{ST}_L(s)$, \mathcal{S}_{Ne}^* , and other related categories like various new outcomes for the coefficients of the power series expansions of the functions that belong to these classes, together with majorization issue, the Hankel determinant, and the logarithmic coefficients with sharp inequalities, and differential subordination implications.

2. Logarithmic Coefficients, Coefficient Estimates, and Majorization Issue

We obtain the estimates for the logarithmic coefficients, the first three coefficients, and majorization issue (see [14]) for the functions belonging to $\mathcal{ST}_L(s)$, \mathcal{S}_{Ne}^* and similar classes. For this purpose, suppose Ω represents the set of all analytic functions ω in \mathbb{U} , with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$, i.e., Ω is the set of Schwarz functions. Also, we recall the following lemmas:

Lemma 1 (see [15], Theorem 1). *Let the function $f \in \mathcal{S}^*(\varphi)$. If φ is starlike with respect to 1, then the logarithmic coefficients of f satisfy the inequality:*

$$|\gamma_n| \leq \frac{|L_1|}{2}, n \in \mathbb{N} := \{1, 2, 3 \dots\}. \tag{10}$$

The above inequality is sharp for any $n \in \mathbb{N}$, for the function f_n given by $zf_n'(z)/f_n(z) = \varphi(z^n)$.

Lemma 2 (see [16], p. 172). *Assume that ω is a Schwarz function, such that $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, z \in \mathbb{U}$.*

Then,

$$|\omega_n| \leq 1, |\omega_n| \leq 1 - |\omega_1|^2, n = 2, 3, \dots \tag{11}$$

Lemma 3 (see [15], Theorem 2). *Let the function $f \in \mathcal{C}(\varphi)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|L_1|}{4}, \tag{12}$$

$$|\gamma_2| \leq \begin{cases} \frac{|L_1|}{12}, & \text{if } |4L_2 + L_1^2| \leq 4|L_1| \\ \frac{|4L_2 + L_1^2|}{48} & \text{if } |4L_2 + L_1^2| > 4|L_1| \end{cases}. \tag{13}$$

The bounds (12) and (13) are sharp. Also, if L_1, L_2 , and L_3 are real values, then

$$|\gamma_3| \leq \frac{|L_1|}{24} K(q_1; q_2), \tag{14}$$

where $q_1 = (L_1 + (4L_2/L_1))/2, q_2 = (L_2 + (2L_3/L_1))/2$ and $K(q_1; q_2)$ is given by (see [17, 18])

$$K(q_1; q_2) = \begin{cases} 1 & \text{if } (q_1, q_2) \in D_1 \cup D_2 \\ |q_2| & \text{if } (q_1, q_2) \in D_6 \\ \frac{2}{3}(|q_1| + 1) \left(\frac{|q_1| + 1}{3|q_1| + 1 + q_2} \right), & \text{if } (q_1, q_2) \in D_8 \cup D_9 \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)} \right), & \text{if } (q_1, q_2) \in D_{10} \end{cases} \tag{15}$$

and the sets $D_k, k = 1, 2, 6, 8, 9, 10$ are stated as given below:

$$\begin{aligned} D_1 &= \left\{ (q_1, q_2): |q_1| \leq \frac{1}{2}, |q_2| \leq 1 \right\}, \\ D_2 &= \left\{ (q_1, q_2): \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27}(|q_1| + 1)^2 \right. \\ &\quad \left. - (|q_1| + 1) \leq q_2 \leq 1 \right\}, \\ D_6 &= \left\{ (q_1, q_2): 2 \leq |q_1| \leq 4, q_2 \geq \frac{1}{12}(q_1^2 + 8) \right\}, \\ D_8 &= \left\{ (q_1, q_2): \frac{1}{2} \leq |q_1| \leq 2, -\frac{2}{3}(|q_1| + 1) \leq q_2 \right. \\ &\quad \left. \leq \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \right\}, \\ D_9 &= \left\{ (q_1, q_2): |q_1| \geq 2, -\frac{2}{3}(|q_1| + 1) \right. \\ &\quad \left. \leq q_2 \leq \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \right\}, \\ D_{10} &= \left\{ (q_1, q_2): 2 \leq |q_1| \leq 4, \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \right. \\ &\quad \left. \leq q_2 \leq \frac{1}{12}(q_1^2 + 8) \right\}. \end{aligned} \tag{16}$$

For two analytic functions f and g in U , the function g is said to be majorized to f , written as $g(z) \ll f(z)$, if there exists an analytic function ψ in U , with $|\psi(z)| \leq 1, z \in \mathbb{U}$, such that (see MacGregor [14]).

$$g(z) = \psi(z)f(z), z \in \mathbb{U}. \tag{17}$$

Cho et al. [19] studied the majorization issue for the category $\mathcal{S}^*(\varphi)$ of starlike functions as follows:

Lemma 4 (see [19], Theorem 2). *If $g \in \mathcal{A}, f \in \mathcal{S}^*(\varphi)$ with $g(z) \ll f(z)$, then $|g'(z)| \leq |f'(z)|$ for all z in the disk $|z| \leq r_1$, where r_1 is the smallest positive root of the equation*

$$\min_{|z|=r} |\varphi(z)| (1 - r^2) - 2r = 0, \quad r \in (0, 1). \quad (18)$$

Setting $\varphi =: \mathbb{L}_s$ and $\varphi =: \varphi_{N_e}$ in Lemma 1, since

$$\begin{aligned} \log \frac{\Phi_{s,1}(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n(\Phi_{s,1}) z^n = 2sz + \dots, z \in \mathbb{U}, \\ \log \frac{\Omega_2(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n(\Omega_2) z^n = z + \dots, z \in \mathbb{U}, \end{aligned} \quad (19)$$

we obtain the following two results:

Theorem 5. *If the function $f \in \mathcal{ST}_L(s)$, then*

$$|\gamma_n| \leq s, \quad n \in \mathbb{N}. \quad (20)$$

This inequality is sharp for $n = 1$ for the function $\Phi_{s,1}$.

Theorem 6. *If the function $f \in \mathcal{S}_{N_e}^*$, then*

$$|\gamma_n| \leq \frac{1}{2}, \quad n \in \mathbb{N}. \quad (21)$$

This inequality is sharp for $n = 1$ for the function Ω_2 .

Theorem 7. *If the function $f \in \mathcal{CV}_L(s)$, then*

$$|\gamma_1| \leq \frac{s}{2}, \quad |\gamma_2| \leq \frac{s}{6}, \quad |\gamma_3| \leq \frac{s}{12}. \quad (22)$$

These bounds are sharp for $f = K_{s,n}$ and $n = 1, 2, 3$, respectively.

Proof. Applying Lemma 3 with $\varphi = \mathbb{L}_s$, the first two estimates follows immediately, and the results are sharp. To prove the above inequality $|\gamma_3|$, by Lemma 3 we have

$$|\gamma_3| \leq \frac{|L_1|}{24} K(q_1; q_2) = \frac{s}{12} K(q_1; q_2), \quad (23)$$

where

$$q_1 = \frac{L_1 + (4L_2/L_1)}{2} = 2s, \quad \text{and} \quad q_2 = \frac{L_2 + (2L_3/L_1)}{2} = \frac{s^2}{2}. \quad (24)$$

First, regarding D_1 it is clear that $|q_1| \leq 1/2$ for $s \in (0, 1/4]$, and the inequality $|q_2| \leq 1$ holds for $s \in (0, 1/\sqrt{2}]$. Therefore, $(q_1, q_2) \in D_1$ for $s \in (0, 1/4]$.

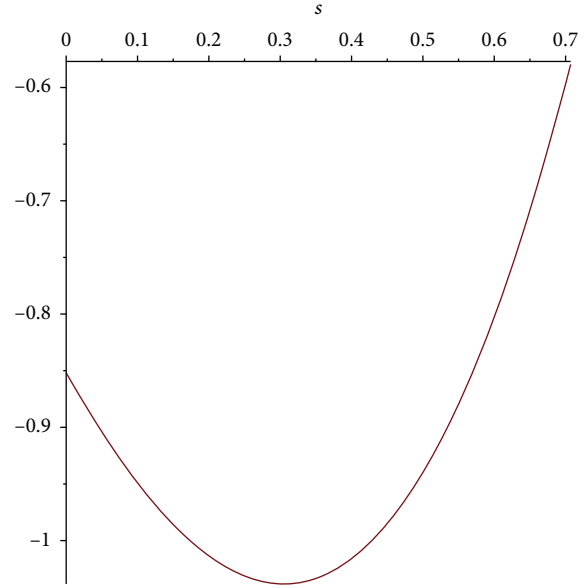


FIGURE 1: The plot of $G(s)$ for $s \in (0, 1/\sqrt{2}]$.

Next, if we select D_2 then $1/2 \leq |q_1| \leq 2$ holds for $s \in [1/4, 1/\sqrt{2}]$. Also, the inequality $(4/27)(|q_1| + 1)^3 - s(|q_1| + 1) \leq q_2$ is equivalent to

$$G(s) := \frac{32s^3}{27} + \frac{23s^2}{18} - \frac{10s}{9} - \frac{23}{27} \leq 0, \quad (25)$$

that holds for $s \in (0, 1/\sqrt{2}]$ (see Figure 1) and the fact that $G(1/\sqrt{2}) < 0$. Further, $q_2 \leq 1$ holds for $s \in (0, 1/\sqrt{2}]$. Therefore, regarding these results, we conclude $(q_1, q_2) \in D_2$ for $s \in [1/4, 1/\sqrt{2}]$.

From the abovementioned two results, using (23) and Lemma 3, we conclude that $|\gamma_3| \leq (s/24)K(q_1; q_2) = s/12$.

Since

$$\begin{aligned} \log \frac{K_{s,1}(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n(K_{s,1}) z^n = sz + \dots, z \in \mathbb{U}, \\ \log \frac{K_{s,2}(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n(K_{s,2}) z^n = \frac{2s}{6} z^2 + \dots, z \in \mathbb{U}, \\ \log \frac{K_{s,3}(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n(K_{s,3}) z^n = \frac{2s}{12} z^3 + \dots, z \in \mathbb{U}, \end{aligned} \quad (26)$$

it follows that the bounds are sharp for $f = Ks, n$ and $n = 1, 2, 3$, respectively.

Theorem 8. *If the function $f \in \mathcal{C}_{N_e}$, then*

$$|\gamma_1| \leq \frac{1}{4}, \quad |\gamma_2| \leq \frac{1}{12}, \quad |\gamma_3| \leq \frac{1}{24}. \quad (27)$$

These bounds are sharp for $f = \Lambda_{n+1}$ and $n = 1, 2, 3$, respectively.

Proof. Applying Lemma 3 for $\varphi = \varphi_{N_e}$, we obtain the first two bounds, and these results are sharp. To find the upper bound for $|\gamma_3|$, by Lemma 3, we have

$$|\gamma_3| \leq \frac{|L_1|}{24} K(q_1; q_2) = \frac{1}{24} K(q_1; q_2), \tag{28}$$

where

$$\begin{aligned} q_1 &= \frac{L_1 + (4L_2/L_1)}{2} = \frac{1}{2}, \\ q_2 &= \frac{L_2 + (2L_3/L_1)}{2} = -\frac{1}{3}. \end{aligned} \tag{29}$$

Therefore, $(q_1, q_2) \in D_1$ and from Lemma 3, we get

$$|\gamma_3| \leq \frac{1}{24} K(q_1; q_2) = \frac{1}{24}. \tag{30}$$

$$|a_4| \leq \frac{2s}{3} \times \begin{cases} 1, & \text{if } s \in (0, 0.48481 \dots], \\ \frac{2}{3} (|q_1| + 1) \left(\frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^{1/2}, & \text{if } s \in [0.48481 \dots, 0.55177 \dots], \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)} \right)^{1/2}, & \text{if } s \in \left[0.55177 \dots, 0.55470 = \frac{2}{\sqrt{13}} \right], \\ |q_2|, & \text{if } s \in \left[0.55470 \dots = \frac{2}{\sqrt{13}}, \frac{1}{\sqrt{2}} \right], \end{cases} \tag{33}$$

where $q_1 = 4s$ and $q_2 = 7s^2/2$. The first two bounds and the first and last bound for $|a_4|$ are sharp.

Proof. If $f \in \mathcal{ST}_L(s)$, by the concept of the subordination, there exists the function $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n \in \Omega$ such that

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \varphi(\omega(z)) = 1 + L_1 \omega_1 z + (L_1 \omega_2 + L_2 \omega_1^2) z^2 \\ &\quad + (L_1 \omega_3 + 2\omega_1 \omega_2 L_2 + L_3 \omega_1^3) z^3 + \dots, z \in \mathbb{U}, \end{aligned} \tag{34}$$

where $\varphi \in \mathbb{L}_s$. From the above equality, it follows that

$$\begin{cases} a_2 = L_1 \omega_1, \\ 2a_3 - a_2^2 = L_1 \omega_2 + L_2 \omega_1^2, \\ 3a_4 - 3a_2 a_3 + a_2^3 = L_1 \omega_3 + 2\omega_1 \omega_2 L_2 + L_3 \omega_1^3. \end{cases} \tag{35}$$

Since

$$\begin{aligned} \log \frac{\Lambda_2(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n (\Lambda_2) z^n = \frac{1}{2} z + \dots, z \in \mathbb{U}, \\ \log \frac{\Lambda_3(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n (\Lambda_3) z^n = \frac{1}{6} z^2 + \dots, z \in \mathbb{U}, \\ \log \frac{\Lambda_4(z)}{z} &= 2 \sum_{n=1}^{\infty} \gamma_n (\Lambda_4) z^n = \frac{1}{12} z^3 + \dots, z \in \mathbb{U}, \end{aligned} \tag{31}$$

we conclude that the bounds are sharp for $f = \Lambda_{n+1}$ and $n = 1, 2, 3$, respectively.

Theorem 9. If the function $f \in \mathcal{ST}_L(s)$ has the power expansion series given by (1), then

$$|a_2| \leq 2s, \quad |a_3| \leq \begin{cases} s, & \text{if } s \in \left(0, \frac{2}{5} \right], \\ \frac{5s^2}{2} & \text{if } s \in \left(\frac{2}{5}, \frac{1}{\sqrt{2}} \right], \end{cases} \tag{32}$$

and

$$\begin{aligned} &\text{if } s \in (0, 0.48481 \dots], \\ &\text{if } s \in [0.48481 \dots, 0.55177 \dots], \\ &\text{if } s \in \left[0.55177 \dots, 0.55470 = \frac{2}{\sqrt{13}} \right], \\ &\text{if } s \in \left[0.55470 \dots = \frac{2}{\sqrt{13}}, \frac{1}{\sqrt{2}} \right], \end{aligned} \tag{33}$$

For the estimate of $|a_2|$, applying Lemma 2 for the first of the above equalities, we get $|a_2| \leq 2s$. Next, using again Lemma 2, for $|a_3|$, it follows that

$$\begin{aligned} |a_3| &= \left| \frac{L_1 \omega_2 + (L_2 + L_1^2) \omega_1^2}{2} \right| \leq \frac{L_1(1 - |\omega_1|^2) + (L_2 + L_1^2) |\omega_1|^2}{2} \\ &= \frac{L_1 + (L_2 + L_1^2 - L_1) |\omega_1|^2}{2} \\ &= \begin{cases} \frac{L_1}{2}, & \text{if } L_2 + L_1^2 \leq L_1, \\ \frac{L_2 + L_1^2}{2}, & \text{if } L_2 + L_1^2 \leq L_1 \\ s, & \text{if } s \in \left(0, \frac{2}{5} \right], \\ \frac{5s^2}{2}, & \text{if } s \in \left(\frac{2}{5}, \frac{1}{\sqrt{2}} \right]. \end{cases} \end{aligned} \tag{36}$$

Now, for $|a_4|$, we obtain (see also [18])

$$\begin{aligned} |a_4| &\leq \frac{L_1}{3} \left| w_3 + \left(\frac{3}{2}L_1 + \frac{2L_2}{L_1} \right) w_1 w_2 + \left(\frac{3}{2}L_2 + \frac{1}{2}L_1^2 + \frac{L_3}{L_1} \right) w_1^3 \right| \\ &\leq \frac{L_1}{3} K(q_1, q_2), \end{aligned} \quad (37)$$

where

$$\begin{aligned} q_1 &= \frac{3}{2}L_1 + \frac{2L_2}{L_1} = 4s, \\ q_2 &= \frac{3}{2}L_2 + \frac{1}{2}L_1^2 + \frac{L_3}{L_1} = \frac{7s^2}{2}. \end{aligned} \quad (38)$$

First, it is clear that $|q_1| \leq 1/2$ for $s \in (0, 1/8]$, and the inequality $|q_2| \leq 1$ holds for $s \in (0, \sqrt{2/7}]$.

Therefore, according to the definition of D_1 , it follows that $(q_1, q_2) \in D_1$ for $s \in (0, 1/8]$.

Also, regarding the definition of D_2 , we conclude that the inequality $1/2 \leq |q_1| \leq 2$ holds for $s \in [1/8, 1/2]$. Also, the inequality $(4/27)(|q_1| + 1)^3 - (|q_1| + 1) \leq q_2$ is equivalent to

$$M_1(s) := \frac{256s^3}{27} + \frac{65s^2}{18} - \frac{20s}{9} - \frac{23}{27} \leq 0, \quad (39)$$

which holds for $s \in (0, s_0]$, where $s_0 = 0.48481 \dots$ (see Figure 2). Further, $q_2 \leq 1$ holds for $s \in (0, \sqrt{2/7}]$; therefore, considering these results, we conclude $(q_1, q_2) \in D_2$ for $s \in [1/8, 0.48481 \dots]$.

Next, regarding the definition of D_6 , $2 \leq |q_1| \leq 4$ holds for $s \in [1/2, 1]$, and the second inequality holds for $s \in [2/\sqrt{13} = 0.55470 \dots, 1/\sqrt{2}]$. Therefore, $(q_1, q_2) \in D_6$ for $s \in [2/\sqrt{13} = 0.55470 \dots, 1/\sqrt{2}]$.

According to the definition of D_8 , $1/2 \leq |q_1| \leq 2$ holds for $s \in [1/8, 1/2]$, and a simple computation shows that the first and second inequality from the right side hold for $s \in (0, 1/$

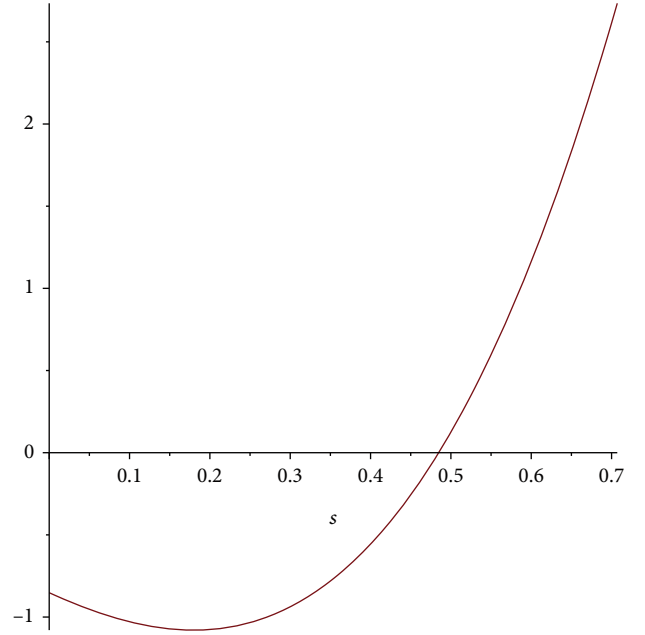


FIGURE 2: The plot of $M_1(s)$ for $s \in (0, 1/\sqrt{2}]$.

$\sqrt{2}]$ and $s \in [0.48481 \dots, 1/\sqrt{2}]$, respectively. Therefore, $(q_1, q_2) \in D_8$ for $s \in [0.48481 \dots, 1/2]$.

Further, for the definition of D_9 , it is clear that $|q_1| \geq 2$ for $s \in [1/2, 1/\sqrt{2}]$. On other the hand, the second inequality from the right side holds for $s \in (0, 1/\sqrt{2}]$, and the first inequality from the right side is equivalent to

$$M_2(s) := 56s^4 + 28s^3 - 18s^2 - 8s \leq 0, \quad (40)$$

holds for $s \in (0, 0.55177 \dots]$ (see Figure 3). Therefore, $(q_1, q_2) \in D_9$ for $s \in [1/2, 0.55177 \dots]$.

Finally, according to the definition of D_{10} , from the above explanations, it follows that $(q_1, q_2) \in D_{10}$ for $s \in [0.55177 \dots, 2/\sqrt{13} = 0.55470]$.

Hence, applying the mentioned outcomes from (37), we conclude that

$$|a_4| \leq \frac{L_1}{3} K(q_1, q_2) = \frac{2s}{3} \times \begin{cases} 1, & \text{if } s \in (0, 0.48481 \dots], \\ \frac{2}{3} (|q_1| + 1) \left(\frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^{1/2}, & \text{if } s \in [0, 0.48481 \dots, 0.55177 \dots], \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)} \right)^{1/2}, & \text{if } s \in \left[0.55177 \dots, 0.55470 \dots = \frac{2}{\sqrt{13}} \right], \\ |q_2|, & \text{if } s \in \left[0.55470 \dots = \frac{2}{\sqrt{13}}, \frac{1}{\sqrt{2}} \right]. \end{cases} \quad (41)$$

The extremal function for $|a_2|$ is $f = \Phi_{s,1}$, and for $|a_3|$ in the first and second inequalities is given by $f = \Phi_{s,2}$ and $f = \Phi_{s,1}$, respectively. Also, the extremal function for $|a_4|$ in the first and fourth above inequalities is given by $f = \Phi_{s,3}$ and $f = \Phi_{s,1}$, respectively.

For a function $f \in \mathcal{A}$, we have $f \in \mathcal{CV}_L(s)$ if and only if $zf'(z) \in \mathcal{ST}_L(s)$. Setting $g(z) := zf'(z)$, we have $g(z) = z +$

$$|a_2| \leq s, \quad |a_3| \leq \begin{cases} \frac{s}{3}, & \text{if } s \in \left(0, \frac{2}{5}\right], \\ \frac{5s^2}{6}, & \text{if } s \in \left(\frac{2}{5}, \frac{1}{\sqrt{2}}\right], \end{cases}$$

$$|a_4| \leq \frac{2s}{12} \times \begin{cases} 1, & \text{if } s \in (0, 0.48481 \dots], \\ \frac{2}{3}(|q_1| + 1) \left(\frac{|q_1| + 1}{3(|q_1| + 1 + q_2)}\right)^{1/2}, & \text{if } s \in [0.48481 \dots, 0.55177 \dots], \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2}\right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)}\right)^{1/2}, & \text{if } s \in \left[0.55177 \dots, 0.55470 \dots = \frac{2}{\sqrt{13}}\right], \\ |q_2|, & \text{if } s \in \left[0.55470 \dots = \frac{2}{\sqrt{13}}, \frac{1}{\sqrt{2}}\right], \end{cases} \quad (42)$$

where $q_1 = 4s$ and $q_2 = 7s^2/2$. The extremal function for $|a_2|$ is $f = K_{s,1}$, and for $|a_3|$ in the first and second inequalities is given by $f = K_{s,2}$ and $f = K_{s,1}$, respectively. Also, the extremal function for $|a_4|$ in the first and fourth above inequalities is given by $f = K_{s,3}$ and $f = K_{s,1}$, respectively.

Setting $\varphi = \varphi_{N_e}$ in the proof of Theorem 9, hence, $L_1 = 1$, $L_2 = 0$, and $L_3 = -1/3$, and we get similarly the following result:

Theorem 11. *If the function $f \in \mathcal{S}_{N_e}^*$ has the power expansion series given by (1), then*

$$|a_2| \leq 1, \quad |a_3| \leq \frac{1}{2}, \quad |a_4| \leq \frac{1}{18}. \quad (43)$$

The inequalities are sharp for $f = \Omega_2$.

For the above choice of φ , the analogue of Theorem 10 is the next one.

Theorem 12. *If the function $f \in \mathcal{C}_{N_e}$ has the power expansion series given by (1), then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{6}, \quad |a_4| \leq \frac{1}{72}. \quad (44)$$

The inequalities are sharp for $f = \Lambda_2$.

$\sum_{n=2}^{\infty} b_n z^n$ where $b_n = na_n$ for $n \geq 2$. Hence, we can get similarly the following theorem:

Theorem 10. *If the function $f \in \mathcal{CV}_L(s)$ has the power expansion series given by (1), then*

Our next result deals with a majorization problem for the functions of the class $\mathcal{ST}_L(s)$:

Theorem 13. *Let $g \in \mathcal{A}$ and $f \in \mathcal{ST}_L(s)$, with $g(z) \ll f(z)$. Then, for all z in the disk $|z| \leq r_2$, we have $|g'(z)| \leq |f'(z)|$, where r_2 is the smallest positive root of the equation*

$$(1-s)^2(1-r^2) - 2r = 0, \quad r \in (0, 1). \quad (45)$$

Proof. The result follows from Lemma 4 using the fact that $\min_{|z|=r} \min |\mathbb{L}_s(z)| = \mathbb{L}_s(-r)$ that can be found in [11], Lemma 2.

3. Second Hankel Determinant Problem

In this section, we investigate the problem of coefficients for the second Hankel determinant problem (see [20–24]) for the classes $\mathcal{ST}_L(s)$, $\mathcal{S}_{N_e}^*$ and similar classes. For this purpose, we need some parts of Theorems 1 and 2 of [25], as follows:

Lemma 14 (see [25], Theorem 1). *Let the function $f \in \mathcal{S}^*(\varphi)$. If L_1, L_2 , and L_3 satisfy the conditions*

$$|L_2| \leq L_1, \quad |4L_1L_3 - L_1^4 - 3L_2^2| - 3L_1^2 \leq 0, \quad (46)$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{L_1^2}{4}. \quad (47)$$

Lemma 15 (see [25], Theorem 2). Let the function $f \in \mathcal{C}(\varphi)$.

(i) If L_1, L_2 , and L_3 satisfy the conditions

$$\begin{aligned} L_1^2 + 4|L_2| - 2L_1 &\leq 0, \\ |6L_1L_3 + L_1^2L_2 - L_1^4 - 4L_2^2| - 4L_1^2 &\leq 0, \end{aligned} \quad (48)$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{L_1^2}{36}. \quad (49)$$

(ii) If L_1, L_2 , and L_3 satisfy the conditions

$$\begin{aligned} L_1^2 + 4|L_2| - 2L_1 &> 0, \\ 2|6L_1L_3 + L_1^2L_2 - L_1^4 - 4L_2^2| - L_1^3 - 4L_1|L_2| - 6L_1^2 &\leq 0, \end{aligned} \quad (50)$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{L_1^2}{576} \left(\frac{16|6L_1L_3 + L_1^2L_2 - L_1^4 - 4L_2^2| - 12L_1^3 - 48L_1|L_2| - 36L_1^2 - L_1^4 - 8L_1^2|L_2| - 16L_2^2}{|6L_1L_3 + L_1^2L_2 - L_1^4 - 4L_2^2| - L_1^3 - 4L_1|L_2| - 2L_1^2} \right). \quad (51)$$

By choosing $\varphi = \mathbb{L}_s$ and $\varphi = \varphi_{Ne}$ in Lemmas 14 and 15, then we obtain the following outcomes:

Theorem 16. If the function $f \in \mathcal{ST}_L(s)$, then the second Hankel determinant satisfies the inequality

$$|a_2a_4 - a_3^2| \leq s^2. \quad (52)$$

The inequality is sharp for $f = \Phi_{s,2}$.

Theorem 17. If the function $f \in \mathcal{S}_{Ne}^*$, then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \quad (53)$$

The inequality is sharp for $f = \Omega_3$.

Theorem 18. If the function $f \in \mathcal{CV}_L(s)$, then the second Hankel determinant satisfies the inequality

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{a^2}{9}, & \text{if } s \in \left(0, \frac{1}{2}\right], \\ \frac{s^2}{24} \left(\frac{4s^2 - 4s - 3}{2s^2 - 2s - 1} \right), & \text{if } s \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right]. \end{cases} \quad (54)$$

The first inequality is sharp for $f = K_{s,2}$.

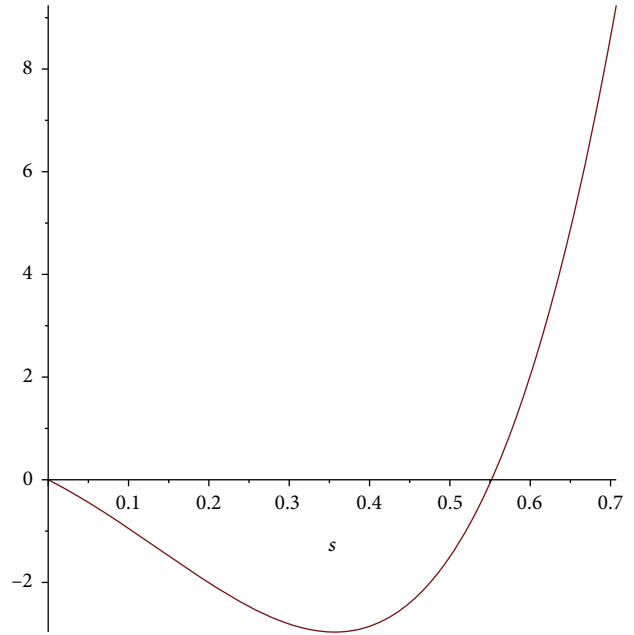


FIGURE 3: The plot of $M_2(s)$ for $s \in (0, 1/\sqrt{2}]$.

Proof. From Lemma 15 (i), a simple computation shows that the first and second inequalities of the assumption hold for $s \in (0, 1/2]$ and $s \in (0, 1/\sqrt{2}]$, respectively. Also, regarding Lemma 15 (ii), the first and second (which is equivalent to $8s^2(4s^2 - 2s - 3) \leq 0$) inequalities of the assumption hold for $s \in (1/2, 1/\sqrt{2}]$ and $s \in (0, 1/\sqrt{2}]$, respectively; hence, we obtain the required result.

Theorem 19. If the function $f \in \mathcal{C}_{Ne}$, then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{36}. \quad (55)$$

The inequality is sharp for $f = \Lambda_3$.

Proof. The result follows immediately from Lemma 15 (i).

4. Differential Subordinations

The principle of differential subordination has important usages in the theory of analytic functions (for details see [26]). The significant importance of the Briot-Bouquet differential subordination inspired many authors to study these types of subordinations, and recently, many generalizations and extensions of the Briot-Bouquet differential subordination have been obtained; for example, see

[27–30]. The following lemma will be a useful tool to get the main results:

Lemma 20 (see [26], Theorem 3.4h, p. 132). *Let q be analytic in \mathbb{U} and let ψ and θ be analytic in a domain D containing $q(\mathbb{U})$, with $\psi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$. Suppose that:*

- (i) *Either h is convex, or Q is starlike univalent in \mathbb{U}*
- (ii) *$\operatorname{Re}((zh'(z))/Q(z)) > 0$ for $z \in \mathbb{U}$*

If p is analytic in \mathbb{U} , with $p(0) = q(0), p(\mathbb{U}) \subset D$

$$\theta(p(z)) + zp'(z)\psi(p(z)) < \theta(q(z)) + zq'(z)\psi(q(z)), \quad (56)$$

then $p(z) \prec q(z)$, and q is the best dominant of (56).

We will obtain sufficient conditions for certain subordinations involving the different mentioned functions and the function \mathbb{L}_s .

Theorem 21. *Let p be an analytic function in \mathbb{U} , with $p(0) = 1$, such that*

$$1 + \beta zp'(z) \prec \mathbb{L}_s(z), \quad (57)$$

with $\beta > 0$, and let $j_s(z) := 2sz + 1/2s^2z^2$.

- (i) *For $-1 < A \leq 1$ if $\beta \geq -(2/(1+A))j_s(-1)$, then $p(z) \prec ((1+Az)/(1-z))$;*
- (ii) *If $\beta \geq \max\{(-j_{1/\sqrt{2}}(-1))/(2-\sqrt{2}); (-j_{1/\sqrt{2}}(1))/\sqrt{2}\} = 1.98743 \dots$, then $p(z) \prec z + \sqrt{1+z^2}$;*
- (iii) *If $\beta \geq \max\{-j_{1/\sqrt{2}}(-1); (-j_{1/\sqrt{2}}(1))/(\sqrt{2}-1)\} = 4.01776 \dots$, then $p(z) \prec \sqrt{1+z}$*
- (iv) *If $\beta \geq \max\{(e/(1-e)) - j_{1/\sqrt{2}}(-1); (j_{1/\sqrt{2}}(1))/(e-1)\} = 1.84175 \dots$, then $p(z) \prec e^z$*

Proof.

- (i) The differential equation

$$1 + \beta zq'(z) = \mathbb{L}_s(z), \quad (58)$$

has an analytic solution $q : \mathbb{U} \rightarrow \mathbb{C}$, with $q(0) = 1$, defined by

$$\begin{aligned} q(z) &= 1 + \frac{1}{\beta} \int_0^z \frac{\mathbb{L}_s(t) - 1}{t} dt \\ &= 1 + \frac{1}{\beta} \left(2sz + \frac{1}{2}s^2z^2 \right) = 1 + \frac{1}{\beta} j_s(z). \end{aligned} \quad (59)$$

In order to prove our result, we will use Lemma 20. Thus, let define the functions $\theta(w) = 1$ and $\psi(w) = \beta$, for $w \in \mathbb{C}$.

These functions are analytic in the domain $D := \mathbb{C}$ containing $q(\mathbb{U})$ and $\psi(w) \neq 0$ for all $w \in q(\mathbb{U})$. Let $Q, h : \mathbb{U} \rightarrow \mathbb{C}$ be defined by

$$\begin{aligned} Q(z) &:= zq'(z)\psi(q(z)) = \beta zq'(z) = 2sz + s^2z^2 = \mathbb{L}_s(z) - 1, \\ h(z) &:= \theta(q(z)) + Q(z) = 1 + \beta zq'(z). \end{aligned} \quad (60)$$

Since $\mathbb{L}_s(z) - 1$ is starlike with respect to 0, hence, Q is a starlike function in \mathbb{U} , and on the other hand

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, z \in \mathbb{U}. \quad (61)$$

Thus, according to Lemma 20

$$\begin{aligned} 1 + \beta zp'(z) &= \theta(p(z)) + zp'(z)\psi(p(z)) < \theta(q(z)) \\ &+ zq'(z)\psi(q(z)) = 1 + \beta zq'(z), \end{aligned} \quad (62)$$

implies $p(z) \prec q(z)$.

The required subordination $p(z) \prec (1+Az)/(1-z)$ holds if $q(z) \prec (1+Az)/(1-z)$. For the function $J(z) := (1+Az)/(1-z)$, with $-1 < A \leq 1$, we have

$$J(\mathbb{U}) = \left\{ w \in \mathbb{C} : \operatorname{Re} w > \frac{1-A}{2} \right\}. \quad (63)$$

Since J is univalent in \mathbb{U} and $q(0) = J(0)$, the subordination $q(z) \prec J(z)$ holds whenever $q(\mathbb{U}) \subset J(\mathbb{U})$. On the other hand, the domain $q(\mathbb{U})$ is symmetric with respect to the real axis because q has real coefficients. If $j_s(z) = 2sz + (1/2)s^2z^2$, then $zj_s'(z) = \mathbb{L}_s(z) - 1 = Q(z)$, $Q(0) = 0$, and Q is starlike with respect to $Q(0) = 0$. It follows that

$$\operatorname{Re} \left(1 + \frac{zj_s''(z)}{j_s'(z)} \right) = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, z \in \mathbb{U}, \quad (64)$$

that is j_s is convex in \mathbb{U} , and therefore, q is convex in \mathbb{U} . Since $q(\mathbb{U})$ is convex, $q(\mathbb{U})$ is symmetric with respect to the real axis, $q(-1) < q(1)$, so it follows that the inclusion $q(\mathbb{U}) \subset J(\mathbb{U})$ holds if

$$q(-1) = 1 + \frac{1}{\beta} j_s(-1) \geq \frac{1-A}{2}, \quad (65)$$

that is $\beta \geq -(2/(1+A))j_s(-1)$. Thus, the subordination $q(z) \prec (1+Az)/(1-z)$ holds whenever $\beta \geq -(2/(1+A))j_s(-1)$.

- (ii) Similarly to the proof of Theorem 21 (i), it is sufficient to establish that

$$q(z) = 1 + \frac{1}{\beta} j_s(z) \prec h(z) := z + \sqrt{1+z^2}, \quad (66)$$

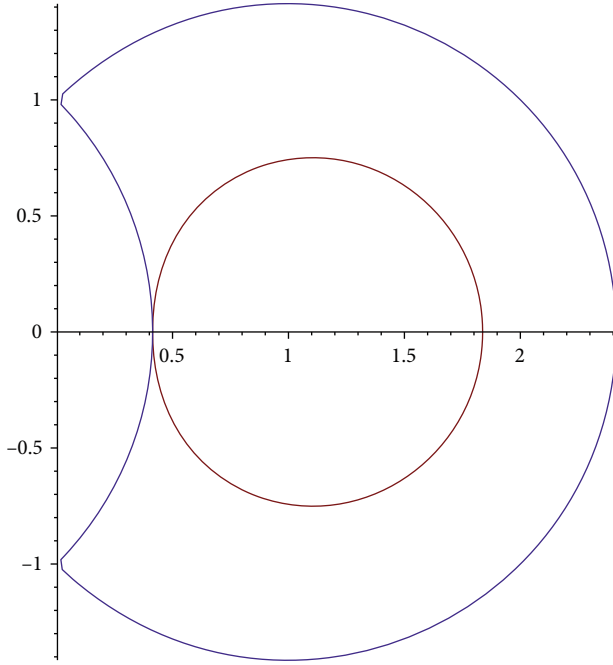


FIGURE 4: The image of $|z| = 1$ under $q(z)(s = 1/\sqrt{2}, \beta = 1.98743 \dots)$ and $h(z)$.

for some values of the parameters s and β . For this work, first, we will prove

$$j_{s_1}(z) \prec j_{s_2}(z), \quad \text{if } 0 < s_1 < s_2 \leq \frac{1}{\sqrt{2}}. \quad (67)$$

Let denote by $\phi(\zeta) := 2\zeta + (1/2)\zeta^2$, and $U_r := \{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq 1$. Since the function j_s is convex in U , it follows that it is univalent in \mathbb{U} for all $0 < s \leq 1/\sqrt{2}$, and $j_{s_1}(0) = j_{s_2}(0)$. Therefore, to prove the subordination (67), we need to show that $j_{s_1}(\mathbb{U}) \subset j_{s_2}(\mathbb{U})$.

It is easy to see that

$$\begin{aligned} j_{s_1}(\mathbb{U}) &= \{j_{s_1}(z) : z \in \mathbb{U}\} = \{\phi(s_1 z) : z \in \mathbb{U}\} \\ &= \{\phi(\zeta) : \zeta \in \mathbb{U}_{s_1}\} = \phi(\mathbb{U}_{s_1}) \subset \phi(\mathbb{U}_{s_2}) \\ &= \{\phi(\zeta) : \zeta \in \mathbb{U}_{s_2}\} = \{\phi(s_2 z) : z \in \mathbb{U}\} \\ &= \{j_{s_2}(z) : z \in \mathbb{U}\} = j_{s_2}(\mathbb{U}), \end{aligned} \quad (68)$$

therefore, the subordination (67) holds. Thus, from (67), it follows that

$$j_s(z) \prec j_{1/\sqrt{2}}(z), \quad 0 < s \leq \frac{1}{\sqrt{2}}, \quad (69)$$

which implies that (66) holds for all $0 < s \leq 1/\sqrt{2}$ if

$$\tilde{q}(z) := 1 + \frac{1}{\beta} j_{1/\sqrt{2}}(z) \prec h(z) = z + \sqrt{1+z^2}. \quad (70)$$

Since h is univalent in \mathbb{U} (see [8]) and $\tilde{q}(0) = h(0)$, it is enough to establish $\tilde{q}(\mathbb{U}) \subset h(\mathbb{U})$. Using the MAPLE™ software, from Figure 4, it follows that this subordination holds

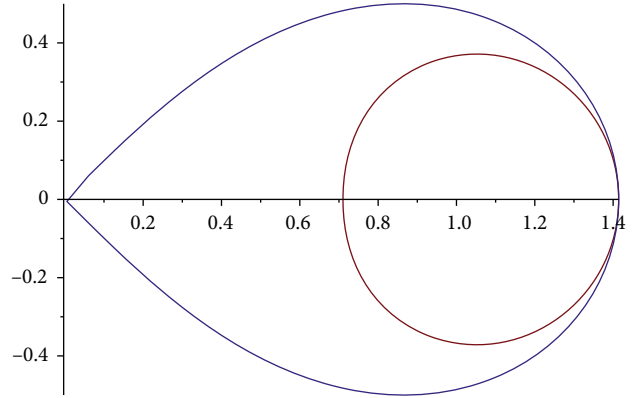


FIGURE 5: The image of $|z| = 1$ under $q(z)(s = 1/\sqrt{2}; \beta = 4.01776 \dots)$ and $\sqrt{1+z}$.

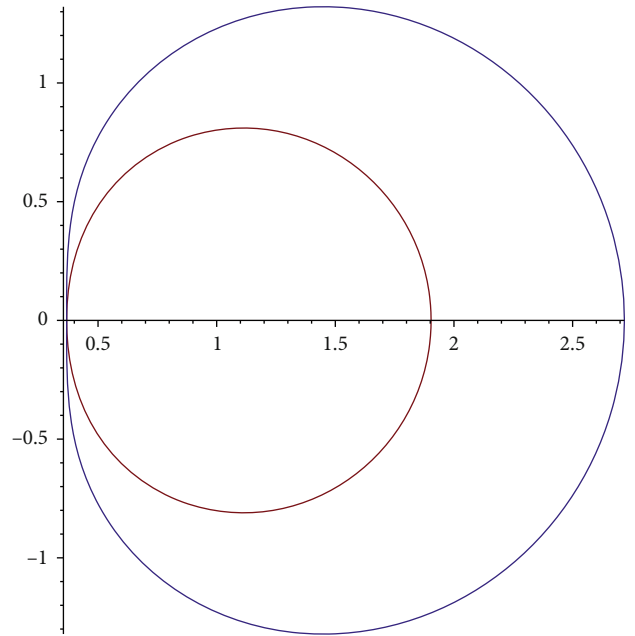


FIGURE 6: The image of $|z| = 1$ under $q(z)(s = 1/2; \beta = 1.84175 \dots)$ and e^z .

provided $\sqrt{2} - 1 \leq \tilde{q}(-1) < \tilde{q}(1) \leq \sqrt{2} + 1$, and these inequalities are equivalent to

$$\begin{aligned} \beta &\geq \beta_1 = \frac{-j_{1/\sqrt{2}}(-1)}{2 - \sqrt{2}}, \\ \beta &= \beta_2 \geq \frac{-j_{1/\sqrt{2}}(1)}{\sqrt{2}}. \end{aligned} \quad (71)$$

Thus, the subordination $q(z) \prec z + \sqrt{1+z^2}$ holds for all $0 < s \leq 1/\sqrt{2}$ whenever $\beta \geq \max\{\beta_1; \beta_2\} = 1.98743 \dots$.

(iii) Regarding the same proof of Theorem 21 (i), it is sufficient to establish that $\tilde{q}(z) := 1 + (1/\beta)j_{1/\sqrt{2}}(z) \prec \sqrt{1+z}$. Using again the MAPLE™ software, since the function $H(z) := \sqrt{1+z}$ is univalent in \mathbb{U} , from Figure 5, it follows that this subordination holds

provided $0 \leq \tilde{q}(-1) \leq \tilde{q}(1) \leq \sqrt{2}$. Hence, it is sufficient to assume that

$$\begin{aligned}\beta &\geq \beta_3 = -j_{1/\sqrt{2}}(-1), \\ \beta &= \beta_4 \geq \frac{j_{1/\sqrt{2}}(1)}{\sqrt{2}-1}.\end{aligned}\quad (72)$$

Thus, the subordination $q(z) \prec \sqrt{1+z}$ holds for all $0 < s \leq 1/\sqrt{2}$ whenever $\beta \geq \max\{\beta_3; \beta_4\} = 4.01776 \dots$.

(iv) Similarly to the proof of Theorem 21 (i) and the previous parts of this theorem, using the MAPLE™ software, since the function $T(z) := e^z$ is univalent in \mathbb{U} , from Figure 6, it follows that the subordination $q(z) \prec e^z$ holds provided

$$\begin{aligned}\beta &\geq \beta_5 = \frac{e}{1-e} j_{1/\sqrt{2}}(-1), \\ \beta &\geq \beta_6 = \frac{j_{1/\sqrt{2}}(1)}{e-1}.\end{aligned}\quad (73)$$

Thus, the subordination $q(z) \prec e^z$ holds for all $0 < s \leq 1/\sqrt{2}$ whenever $\beta \geq \max\{\beta_5; \beta_6\} = 1.84175 \dots$.

Setting $p(z) := (zf'(z))/(f(z))$ in the above theorem, we obtain the next special cases:

Corollary 22. *Suppose that the function $f \in A$ satisfies*

$$1 + \beta \frac{zf'(z)}{f(z)} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right) \prec \mathbb{L}_s(z). \quad (74)$$

- (i) For $-1 < A \leq 1$, if $\beta \geq -(2/(1+A))j_s(-1)$, then $f \in \mathcal{S}^*[A, -1]$
- (ii) If $\beta \geq \max\{(-j_{1/\sqrt{2}}(-1))/(2-\sqrt{2}); (j_{1/\sqrt{2}}(1))/\sqrt{2}\} = 1.98743 \dots$, then $f \in \mathcal{S}_c^*$
- (iii) If $\beta \geq \max\{-j_{1/\sqrt{2}}(-1); (j_{1/\sqrt{2}}(1))/(\sqrt{2}-1)\} = 4.01776 \dots$, then $f \in \mathcal{S}_L^*$
- (iv) If $\beta \geq \max\{(e/(1-e))j_{1/\sqrt{2}}(-1); (j_{1/\sqrt{2}}(1))/(e-1)\} = 1.84175 \dots$, then $f \in \mathcal{S}_e^*$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to and approved the final manuscript.

References

- [1] I. P. Kayumov, "On Brennan's conjecture for a special class of functions," *Mathematical Notes*, vol. 78, no. 3-4, pp. 498–502, 2005.
- [2] I. M. Milin, "On a property of the logarithmic coefficients of univalent functions," in *Metric Questions in the Theory of Functions*, pp. 86–90, Naukova Dumka, Kiev, Ukraine, 1980.
- [3] I. M. Milin, "On a conjecture for the logarithmic coefficients of univalent functions," *Zap. Nauch. Semin. Leningr. Otd. Mat. Inst. Steklova*, vol. 125, pp. 135–143, 1983.
- [4] I. M. Milin, *Univalent Functions and Orthonormal Systems*, Amer Mathematical Society, Providence, RI, USA, 1977.
- [5] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, pp. 157–169, Tianjin, China, 1992.
- [6] J. Sokół and J. Stankiewicz, "Radius of convexity of some subclasses of strongly starlike functions," *Zeszyty Nauk Politech Rzeszowskiej Mat*, vol. 19, pp. 101–105, 1996.
- [7] R. Mendiratta, S. Nagpal, and V. Ravichandran, "On a subclass of strongly starlike functions associated with exponential function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 365–386, 2015.
- [8] R. K. Raina and J. Sokół, "Some properties related to a certain class of starlike functions," *Comptes Rendus Mathematique*, vol. 353, no. 11, pp. 973–978, 2015.
- [9] S. Kanas, V. S. Masih, and A. Ebadian, "Relations of a planar domains bounded by hyperbolas with families of holomorphic functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.
- [10] P. Goel and S. S. Kumar, "Certain class of starlike functions associated with modified sigmoid function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 957–991, 2020.
- [11] V. S. Masih and S. Kanas, "Subclasses of starlike and convex functions associated with the limaçon domain," *Symmetry*, vol. 12, no. 6, p. 942, 2020.
- [12] L. A. Wani and A. Swaminathan, "Starlike and convex functions associated with a nephroid domain," *Bulletin of the Malaysian Mathematical Sciences Society*, 2020.
- [13] A. Ebadian, T. Bulboacă, N. E. Cho, and E. Analouei Adegani, "Coefficient bounds and differential subordinations for analytic functions associated with starlike functions," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, no. 3, 2020.
- [14] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, no. 1, pp. 95–102, 1967.
- [15] E. A. Adegani, N. E. Cho, and M. Jafari, "Logarithmic coefficients for univalent functions defined by subordination," *Mathematics*, vol. 7, no. 5, p. 408, 2019.
- [16] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, NY, USA, 1952.
- [17] D. Alimohammadi, N. E. Cho, E. A. Adegani, and A. Motamednezhad, "Argument and coefficient estimates for certain analytic functions," *Mathematics*, vol. 8, no. 1, p. 88, 2020.
- [18] D. V. Prokhorov and J. Szynal, "Inverse coefficients for $(\alpha; \beta)$ -convex functions," *Annales Universitatis Mariae Curie-Skłodowska, Section A*, vol. 35, pp. 125–143, 1981.

- [19] N. E. Cho, Z. Oroujy, E. Analouei Adegani, and A. Ebadian, "Majorization and coefficient problems for a general class of starlike functions," *Symmetry*, vol. 12, no. 3, p. 476, 2020.
- [20] N. E. Cho, E. Analouei Adegani, S. Bulut, and A. Motamednezhad, "The second Hankel determinant problem for a class of bi-close-to-convex functions," *Mathematics*, vol. 7, no. 10, p. 986, 2019.
- [21] S. Kanas, E. Analouei Adegani, and A. Zireh, "An unified approach to second Hankel determinant of bi-subordinate functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 6, 2017.
- [22] A. Motamednezhad, T. Bulboacă, E. Analouei Adegani, and N. Dibagar, "Second Hankel determinant for a subclass of analytic bi-univalent functions defined by subordination," *Turkish Journal of Mathematics*, vol. 42, no. 5, pp. 2798–2808, 2018.
- [23] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *Journal of the London Mathematical Society*, vol. s1-41, no. 1, pp. 111–122, 1966.
- [24] C. Pommerenke, "On the Hankel determinants of univalent functions," *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.
- [25] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Bounds for the second Hankel determinant of certain univalent functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.
- [26] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, NY, USA, 2000.
- [27] E. Analouei Adegani, T. Bulboacă, and A. Motamednezhad, "Simple sufficient subordination conditions for close-to-convexity," *Mathematics*, vol. 7, no. 3, p. 241, 2019.
- [28] T. Bulboacă, "On some classes of differential subordinations," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 31, pp. 45–50, 1986.
- [29] M. Nunokawa and J. Sokół, "On some differential subordinations," *Studia Scientiarum Mathematicarum Hungarica*, vol. 54, no. 4, pp. 436–445, 2017.
- [30] H. M. Srivastava, D. Răducanu, and P. Zaprawa, "A certain subclass of analytic functions defined by means of differential subordination," *Filomat*, vol. 30, pp. 3743–3757, 2016.