Research Article

A Classification of Fourth-Order Dissipative Differential Operators

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This paper is devoted to the classification of the fourth-order dissipative differential operators by the boundary conditions. Subject to certain conditions, we determine some nonself-adjoint boundary conditions that generate the fourth-order differential operators to be dissipative. And under certain conditions, we prove that these dissipative operators have no real eigenvalues.

1. Introduction

It is well known, the nonself-adjoint spectral problems have wide applications. For example, interesting nonclassical wavelets can be obtained from eigenfunctions and associated functions for nonself-adjoint spectral problems. Thus, such problems have received more and more attention by the scholars [1–6].

The nonself-adjointness of spectral problems can be caused by one or more of the following factors: the nonlinear dependence of the problems on the spectral parameter, the nonsymmetry of the differential expressions used, and the nonself-adjointness of the boundary conditions (BCs) involved.

Nonself-adjoint differential operators generated by symmetric differential expressions together with nonself-adjoint BCs have been investigated in [6–14]. The determinant of perturbation connected with the dissipative operator L generated in $L^2[a, b]$ by the Sturm–Liouville differential expression in Weyl’s limit-circle case has been studied by Bairamov and Ugurlu in [11]. By using the Livšic theorem, they showed the completeness of the system of eigenfunctions and associated functions of the operator.

In recent years, there are several results for fourth-order dissipative operators [6, 12–14]. However, these studies only restricted into some special boundary conditions. For second order Sturm–Liouville differential expression, Wang and Wu in 2012 gave all the nonself-adjoint boundary

conditions which generate the operators dissipative [8]. As far as we know, there are no such results on higher order cases.

The dissipative operators have very important application backgrounds such as the scattering theory and telegraphist’s equation. As is mentioned above, the boundary conditions can make the operators to be dissipative. A general question is how to justify the operator as a dissipative operator from the boundary conditions? Motivated by this reason and [8], in this paper, we will consider the fourth-order case and show a classification of the fourth-order dissipative differential operators. The results here are more general than previous known results, and the process is much complicated.

2. Notation and Main Results

In this paper, we consider the following fourth-order differential expression:

$$l(y) = y^{(4)} + q(x)y, \quad x \in I = (a, b), \quad (1)$$

where $-\infty \leq a < b \leq +\infty$ and $q(x)$ is a real-valued function on I and $q(x) \in L^1_{\text{loc}}(I)$. Suppose that the endpoints a and b are quasi regular (regular or limit circle); hence, without loss of generality, we assume that Weyl’s limit-circle case holds for the differential expression $l(y)$ throughout this paper.

For any $m, n \in \mathbb{N}$, we use $M_{m,n}(\mathbb{C})$ to denote the vector space of m by n complex matrices.

Let

$$\Omega = \{y \in L^2(I): y, y', y'', y''' \in AC_{\text{loc}}(I), \quad l(y) \in L^2(I)\}. \quad (2)$$

For $y, z \in \Omega$, the so-called Lagrange bracket is given by

$$[y, z]_x = -y\bar{z}''' + y'\bar{z}'' - y''\bar{z}' + y'''\bar{z} = -R_z(x)QC_y(x), \quad x \in I, \quad (3)$$

where the bar over a function denotes complex conjugate, and

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

$$R_z(x) = (z(x), z'(x), z''(x), z'''(x)),$$

$$C_z(x) = R_z^*(x),$$

$R_z^*(x)$ is the complex conjugate transpose of $R_z(x)$.

Let $\phi_i(x, \lambda)$, $i = 1, 2, 3, 4$, represent a set of linearly independent solutions of the equation $l(y) = \lambda y$, where λ is a complex parameter. From the well-known Naimark's Patching Lemma, the solutions can be chosen to satisfied with any initial conditions; hence, we set $v_i(x) = \phi_i(x, 0)$, $\lambda = 0$, $i = 1, 2, 3, 4$, and satisfy the condition

$$[[v_i, v_j]_a] = J, \quad i, j = 1, 2, 3, 4, \quad (5)$$

where

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (6)$$

From [15], the solutions $v_i(x)$, $i = 1, 2, 3, 4$, as described above exist and are linearly independent. Since Weyl's limit-circle case holds for the differential expression $l(y)$ on I , the solutions $v_i(x)$, $i = 1, 2, 3, 4$, belong to $L^2(I)$. Since $v_i(x)$, $i = 1, 2, 3, 4$, are solutions of equation $l(y) = 0$, according to Green's formula, we can get $[v_i, v_j]_x = \text{const}$, so for any $x \in I$, $[[v_i, v_j]_x] = J$.

Now let $l(y) = \lambda y$, and we consider the boundary value problem consisting of the differential equation

$$y^{(4)} + q(x)y = \lambda y, \quad x \in I, \quad (7)$$

and the boundary condition (BC):

$$AY(a) + BY(b) = 0, \quad (8)$$

where

$$\begin{aligned} Y(a) &= \begin{pmatrix} [y, v_1]_a \\ [y, v_2]_a \\ [y, v_3]_a \\ [y, v_4]_a \end{pmatrix}, \\ Y(b) &= \begin{pmatrix} [y, v_1]_b \\ [y, v_2]_b \\ [y, v_3]_b \\ [y, v_4]_b \end{pmatrix}, \\ A &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \\ B &= \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \end{aligned} \quad (9)$$

and $A, B \in M_{4,4}(\mathbb{C})$.

For the convenience to the next discussion we combine the matrices A and B as a whole and write it as the following block matrix form:

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix}, \quad (10)$$

where A_i, B_i , $i = 1, 2, 3, 4 \in M_{2,2}(\mathbb{C})$. $(A : B)$ is called as the boundary condition matrix throughout this paper.

Remark 1. When the endpoints a and b are both regular, the BCs are the same as given in (8), with $Y(a)$ and $Y(b)$ replaced by

$$\begin{aligned} Y(a) &= \begin{pmatrix} y(a) \\ y'(a) \\ y''(a) \\ y'''(a) \end{pmatrix}, \\ Y(b) &= \begin{pmatrix} y(b) \\ y'(b) \\ y''(b) \\ y'''(b) \end{pmatrix}. \end{aligned} \quad (11)$$

At this time the matrices A and B (or the boundary condition matrix $(A : B)$) are the same. There is a similar statement about one endpoint is regular and the other one is in limit-circle case. Hence, without loss of generality we will discuss the results in this paper under the BC (8), which is in limit-circle case.

Now in $L^2(I)$, we define the operator L as $Ly = l(y)$ on $D(L)$, where

$$D(L) = \{y \in \Omega \mid AY(a) + BY(b) = 0\}. \quad (12)$$

Definition 1 (see [1]). A linear operator L , acting in the Hilbert space $L^2(I)$ and having domain $D(L)$, is said to be dissipative if $\Im(Lf, f) \geq 0$, $\forall f \in D(L)$, where $\Im$ denotes the imaginary part of the value.

Then, we can state our main results.

Theorem 1. The fourth-order differential operator L generated by the differential expression (1) and a BC (8) is dissipative if up to a $GL(4, \mathbb{C})$ factor on the left, where $GL(4, \mathbb{C})$ is the set of invertible complex matrices in dimension 4, the boundary condition matrix $(A: B)$ has one of the following forms:

$$\begin{pmatrix} a_{11} & a_{12} & 1 & 0 & -\overline{a_{31}} & \overline{a_{41}} & 0 & 0 \\ a_{21} & a_{22} & 0 & -1 & \overline{a_{32}} & -\overline{a_{42}} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & b_{31} & b_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 0 & b_{41} & b_{42} & 0 & -1 \end{pmatrix}, \quad (13)$$

where $a_{i1}, a_{i2}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 2 \in \mathbb{C}$ and satisfy $\Im(a_{22} - a_{11}) \geq 0, -4\Im a_{22}\Im a_{11} \geq |a_{21} + \overline{a_{12}}|^2$ and $\Im(b_{31} - b_{42}) \geq 0, -4\Im b_{31}\Im b_{42} \geq |b_{32} + \overline{b_{41}}|^2$.

$$\begin{pmatrix} a_{11} & a_{12} & 1 & 0 & 0 & 0 & \overline{a_{31}} & -\overline{a_{41}} \\ a_{21} & a_{22} & 0 & -1 & 0 & 0 & -\overline{a_{32}} & \overline{a_{42}} \\ a_{31} & a_{32} & 0 & 0 & 1 & 0 & b_{33} & b_{34} \\ a_{41} & a_{42} & 0 & 0 & 0 & -1 & b_{43} & b_{44} \end{pmatrix}, \quad (14)$$

where $a_{i1}, a_{i2}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 3, 4 \in \mathbb{C}$ and satisfy $\Im(a_{22} - a_{11}) \geq 0, -4\Im a_{22}\Im a_{11} \geq |a_{21} + \overline{a_{12}}|^2$ and $\Im(b_{44} - b_{33}) \geq 0, -4\Im b_{33}\Im b_{44} \geq |b_{43} + \overline{b_{34}}|^2$.

$$\begin{pmatrix} a_{11} & a_{12} & 1 & 0 & \overline{a_{41}} & 0 & 0 & \overline{a_{31}} \\ a_{21} & a_{22} & 0 & -1 & -\overline{a_{42}} & 0 & 0 & -\overline{a_{32}} \\ a_{31} & a_{32} & 0 & 0 & b_{31} & 1 & 0 & b_{34} \\ a_{41} & a_{42} & 0 & 0 & b_{41} & 0 & -1 & b_{44} \end{pmatrix}, \quad (15)$$

where $a_{i1}, a_{i2}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 4 \in \mathbb{C}$ and satisfy $\Im(a_{22} - a_{11}) \geq 0, -4\Im a_{22}\Im a_{11} \geq |a_{21} + \overline{a_{12}}|^2$ and $\Im(b_{41} + b_{34}) \leq 0, 4\Im b_{41}\Im b_{34} \geq |b_{44} - b_{31}|^2$.

$$\begin{pmatrix} a_{11} & a_{12} & 1 & 0 & 0 & \overline{a_{41}} & \overline{a_{31}} & 0 \\ a_{21} & a_{22} & 0 & -1 & 0 & -\overline{a_{42}} & -\overline{a_{32}} & 0 \\ a_{31} & a_{32} & 0 & 0 & 1 & b_{32} & b_{33} & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & b_{42} & b_{43} & -1 \end{pmatrix}, \quad (16)$$

where $a_{i1}, a_{i2}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 2, 3 \in \mathbb{C}$ and satisfy $\Im(a_{22} - a_{11}) \geq 0, -4\Im a_{22}\Im a_{11} \geq |a_{21} + \overline{a_{12}}|^2$ and $\Im(b_{42} + b_{33}) \leq 0, 4\Im b_{42}\Im b_{33} \geq |b_{32} - b_{43}|^2$.

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} & \overline{a_{33}} & -\overline{a_{43}} & 0 & 0 \\ 0 & -1 & a_{23} & a_{24} & -\overline{a_{34}} & \overline{a_{44}} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & b_{31} & b_{32} & 1 & 0 \\ 0 & 0 & a_{43} & a_{44} & b_{41} & b_{42} & 0 & -1 \end{pmatrix}, \quad (17)$$

where $a_{i3}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 2 \in \mathbb{C}$ and satisfy $\Im(a_{13} - a_{24}) \geq 0, -4\Im a_{13}\Im a_{24} \geq |a_{23} + \overline{a_{14}}|^2$ and $\Im(b_{31} - b_{42}) \geq 0, -4\Im b_{31}\Im b_{42} \geq |b_{32} + \overline{b_{41}}|^2$.

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} & 0 & 0 & -\overline{a_{33}} & \overline{a_{43}} \\ 0 & -1 & a_{23} & a_{24} & 0 & 0 & \overline{a_{34}} & -\overline{a_{44}} \\ 0 & 0 & a_{33} & a_{34} & 1 & 0 & b_{33} & b_{34} \\ 0 & 0 & a_{43} & a_{44} & 0 & -1 & b_{43} & b_{44} \end{pmatrix}, \quad (18)$$

where $a_{i3}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 3, 4 \in \mathbb{C}$ and satisfy $\Im(a_{13} - a_{24}) \geq 0, -4\Im a_{13}\Im a_{24} \geq |a_{14} + \overline{a_{23}}|^2$ and $\Im(b_{44} - b_{33}) \geq 0, -4\Im b_{33}\Im b_{44} \geq |b_{43} + \overline{b_{34}}|^2$.

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} & -\overline{a_{43}} & 0 & 0 & -\overline{a_{33}} \\ 0 & -1 & a_{23} & a_{24} & \overline{a_{44}} & 0 & 0 & \overline{a_{34}} \\ 0 & 0 & a_{33} & a_{34} & b_{31} & 1 & 0 & b_{34} \\ 0 & 0 & a_{43} & a_{44} & b_{41} & 0 & -1 & b_{44} \end{pmatrix}, \quad (19)$$

where $a_{i3}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 4 \in \mathbb{C}$ and satisfy $\Im(a_{13} - a_{24}) \geq 0, -4\Im a_{13}\Im a_{24} \geq |a_{14} + \overline{a_{23}}|^2$ and $\Im(b_{41} + b_{34}) \leq 0, 4\Im b_{41}\Im b_{34} \geq |b_{44} - b_{31}|^2$.

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} & 0 & -\overline{a_{43}} & -\overline{a_{33}} & 0 \\ 0 & -1 & a_{23} & a_{24} & 0 & \overline{a_{44}} & \overline{a_{34}} & 0 \\ 0 & 0 & a_{33} & a_{34} & 1 & b_{32} & b_{33} & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & b_{42} & b_{43} & -1 \end{pmatrix}, \quad (20)$$

where $a_{i3}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 2, 3 \in \mathbb{C}$ and satisfy $\Im(a_{13} - a_{24}) \geq 0, -4\Im a_{13}\Im a_{24} \geq |a_{14} + \overline{a_{23}}|^2$ and $\Im(b_{42} + b_{33}) \leq 0, 4\Im b_{42}\Im b_{33} \geq |b_{32} - b_{43}|^2$.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 & -\overline{a_{33}} & \overline{a_{43}} \\ 0 & a_{22} & a_{23} & -1 & 0 & 0 & -\overline{a_{32}} & \overline{a_{42}} \\ 0 & a_{32} & a_{33} & 0 & 1 & 0 & b_{33} & b_{34} \\ 0 & a_{42} & a_{43} & 0 & 0 & -1 & b_{43} & b_{44} \end{pmatrix}, \quad (21)$$

where $a_{i2}, a_{i3}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 3, 4 \in \mathbb{C}$ and satisfy $\Im(a_{22} + a_{13}) \geq 0, 4\Im a_{22}\Im a_{13} \geq |a_{12} - \overline{a_{23}}|^2$ and $\Im(b_{44} - b_{33}) \geq 0, -4\Im b_{33}\Im b_{44} \geq |b_{43} + \overline{b_{34}}|^2$.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & \overline{a_{33}} & -\overline{a_{43}} & 0 & 0 \\ 0 & a_{22} & a_{23} & -1 & \overline{a_{32}} & -\overline{a_{42}} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & b_{31} & b_{32} & 1 & 0 \\ 0 & a_{42} & a_{43} & 0 & b_{41} & b_{42} & 0 & -1 \end{pmatrix}, \quad (22)$$

where $a_{i2}, a_{i3}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 2 \in \mathbb{C}$ and satisfy $\Im(a_{22} + a_{13}) \geq 0, 4\Im a_{22}\Im a_{13} \geq |a_{12} - \overline{a_{23}}|^2$ and $\Im(b_{31} - b_{42}) \geq 0, -4\Im b_{33}\Im b_{41} \geq |b_{32} + \overline{b_{41}}|^2$.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & -\overline{a_{43}} & -\overline{a_{33}} & 0 \\ 0 & a_{22} & a_{23} & -1 & 0 & -\overline{a_{42}} & -\overline{a_{32}} & 0 \\ 0 & a_{32} & a_{33} & 0 & 1 & b_{32} & b_{33} & 0 \\ 0 & a_{42} & a_{43} & 0 & 0 & b_{42} & b_{43} & -1 \end{pmatrix}, \quad (23)$$

where $a_{i2}, a_{i3}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 2, 3 \in \mathbb{C}$ and satisfy $\Im(a_{22} + a_{13}) \geq 0, 4\Im a_{22}\Im a_{13} \geq |a_{12} - \overline{a_{23}}|^2$ and $\Im(b_{42} + b_{33}) \leq 0, 4\Im b_{42}\Im b_{33} \geq |b_{43} - \overline{b_{32}}|^2$.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & -\overline{a_{43}} & 0 & 0 & -\overline{a_{33}} \\ 0 & a_{22} & a_{23} & -1 & -\overline{a_{42}} & 0 & 0 & -\overline{a_{32}} \\ 0 & a_{32} & a_{33} & 0 & b_{31} & 1 & 0 & b_{34} \\ 0 & a_{42} & a_{43} & 0 & b_{41} & 0 & -1 & b_{44} \end{pmatrix}, \quad (24)$$

where $a_{i2}, a_{i3}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 4 \in \mathbb{C}$ and satisfy $\Im(a_{22} + a_{13}) \geq 0, 4\Im a_{22}\Im a_{13} \geq |a_{12} - \overline{a_{23}}|^2$ and $\Im(b_{41} + b_{34}) \leq 0, 4\Im b_{41}\Im b_{34} \geq |b_{31} - \overline{b_{44}}|^2$.

$$\begin{pmatrix} a_{11} & 1 & 0 & a_{14} & \overline{a_{31}} & -\overline{a_{41}} & 0 & 0 \\ a_{21} & 0 & -1 & a_{24} & \overline{a_{34}} & -\overline{a_{44}} & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} & b_{31} & b_{32} & 1 & 0 \\ a_{41} & 0 & 0 & a_{44} & b_{41} & b_{42} & 0 & -1 \end{pmatrix}, \quad (25)$$

where $a_{i1}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 2 \in \mathbb{C}$ and satisfy $\Im(a_{21} + a_{14}) \geq 0, 4\Im a_{21}\Im a_{14} \geq |a_{24} - \overline{a_{11}}|^2$ and $\Im(b_{31} - b_{42}) \geq 0, -4\Im b_{31}\Im b_{42} \geq |b_{32} + \overline{b_{41}}|^2$.

$$\begin{pmatrix} a_{11} & 1 & 0 & a_{14} & 0 & 0 & -\overline{a_{34}} & \overline{a_{44}} \\ a_{21} & 0 & -1 & a_{24} & 0 & 0 & -\overline{a_{31}} & \overline{a_{41}} \\ a_{31} & 0 & 0 & a_{34} & 1 & 0 & b_{33} & b_{34} \\ a_{41} & 0 & 0 & a_{44} & 0 & -1 & b_{43} & b_{44} \end{pmatrix}, \quad (26)$$

where $a_{i1}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 3, 4 \in \mathbb{C}$ and satisfy $\Im(a_{21} + a_{14}) \geq 0, 4\Im a_{21}\Im a_{14} \geq |a_{24} - \overline{a_{11}}|^2$, and $\Im(b_{44} - b_{33}) \geq 0, -4\Im b_{33}\Im b_{44} \geq |b_{43} + \overline{b_{34}}|^2$;

$$\begin{pmatrix} a_{11} & 1 & 0 & a_{14} & -\overline{a_{44}} & 0 & 0 & -\overline{a_{34}} \\ a_{21} & 0 & -1 & a_{24} & -\overline{a_{41}} & 0 & 0 & -\overline{a_{31}} \\ a_{31} & 0 & 0 & a_{34} & b_{31} & 1 & 0 & b_{34} \\ a_{41} & 0 & 0 & a_{44} & b_{41} & 0 & -1 & b_{44} \end{pmatrix}, \quad (27)$$

where $a_{i1}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 1, 4 \in \mathbb{C}$ and satisfy $\Im(a_{21} + a_{14}) \geq 0, 4\Im a_{21}\Im a_{14} \geq |a_{24} - \overline{a_{11}}|^2$ and $\Im(b_{41} + b_{34}) \leq 0, 4\Im b_{41}\Im b_{34} \geq |b_{31} - \overline{b_{44}}|^2$.

$$\begin{pmatrix} a_{11} & 1 & 0 & a_{14} & 0 & -\overline{a_{44}} & -\overline{a_{34}} & 0 \\ a_{21} & 0 & -1 & a_{24} & 0 & -\overline{a_{41}} & -\overline{a_{31}} & 0 \\ a_{31} & 0 & 0 & a_{34} & 1 & b_{32} & b_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} & 0 & b_{42} & b_{43} & -1 \end{pmatrix}, \quad (28)$$

where $a_{i1}, a_{i4}, i = 1, 2, 3, 4, b_{3j}, b_{4j}, j = 2, 3 \in \mathbb{C}$ and satisfy $\Im(a_{21} + a_{14}) \geq 0, 4\Im a_{21}\Im a_{14} \geq |a_{24} - \overline{a_{11}}|^2$ and $\Im(b_{42} + b_{33}) \leq 0, 4\Im b_{42}\Im b_{33} \geq |b_{43} - \overline{b_{32}}|^2$.

Proof. It is given in Section 3. $\square$

Remark 2. The BCs of the fourth-order dissipative differential operators given in Theorem 1 can be classified as separated, mixed, and coupled. Let $\text{rank}(A_3 : A_4) = \text{rank}(B_1 : B_2) = r$, then it can be concluded that when $r = 0$, the BCs are separated, when $r = 1$, the BCs are mixed, and when $r = 2$, the BCs are coupled.

Remark 3. From Theorem 1, it can also be concluded that the fourth-order differential operators consist of the symmetric differential expression and certain types of BCs are dissipative when the BC matrix takes one of the forms (13)–(28). The BCs can be separated, mixed, or coupled. The classification includes all the separated cases; however, for the mixed BCs and the coupled BCs we still cannot say that the results include all the BCs which generate the fourth-order operators (7) and (8) to be dissipative. Furthermore, for the convenience to justify the dissipation of the fourth-order dissipative differential operators, the separated BCs can also be classified as follows, which is a supplementary explanation to the separated BCs given in Theorem 1.

Theorem 2. Suppose that the fourth-order differential operator L is generated by the differential expression (1) and a BC (8) and suppose that the BC (8) is separated, then L is dissipative if up to a $GL(4, \mathbb{C})$ factor on the left the boundary condition matrix $(A : B)$ takes one of the following forms:

$$\begin{pmatrix} 1 & a_{12} & 0 & a_{14} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & -1 & a_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{32} & 0 & b_{34} \\ 0 & 0 & 0 & 0 & 0 & b_{42} & -1 & b_{44} \end{pmatrix}, \quad (29)$$

where $a_{1i}, a_{2i}, i = 2, 4, b_{3j}, b_{4j}, j = 2, 4 \in \mathbb{C}$ and satisfy $\Im(a_{12}\overline{a_{22}} + a_{14}\overline{a_{24}}) \geq 0, 4\Im(a_{12}\overline{a_{22}})\Im(a_{14}\overline{a_{24}}) \geq |1 - \overline{a_{12}}a_{24} + a_{14}\overline{a_{22}}|^2$ and $\Im(b_{32}b_{42} + \overline{b_{34}}b_{44}) \geq 0, 4\Im(b_{32}b_{42})\Im(b_{34}b_{44}) \geq |1 - b_{32}\overline{b_{44}} + \overline{b_{34}}b_{42}|^2$.

$$\begin{pmatrix} 1 & a_{12} & 0 & a_{14} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & -1 & a_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{31} & 1 & b_{33} & 0 \\ 0 & 0 & 0 & 0 & b_{41} & 0 & b_{43} & -1 \end{pmatrix}, \quad (30)$$

where $a_{1i}, a_{2i}, i = 2, 4, b_{3j}, b_{4j}, j = 1, 3 \in \mathbb{C}$ and satisfy $\Im(a_{12}\overline{a_{22}} + a_{14}\overline{a_{24}}) \geq 0, 4\Im(a_{12}\overline{a_{22}})\Im(a_{14}\overline{a_{24}}) \geq |1 - \overline{a_{12}}a_{24} + a_{14}\overline{a_{22}}|^2$ and $\Im(b_{31}b_{41} + \overline{b_{33}}b_{43}) \geq 0, 4\Im(b_{31}b_{41})\Im(b_{33}b_{43}) \geq |1 - b_{31}\overline{b_{43}} + \overline{b_{33}}b_{41}|^2$.

$$\begin{pmatrix} a_{11} & 1 & a_{13} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{32} & 0 & b_{34} \\ 0 & 0 & 0 & 0 & 0 & b_{42} & -1 & b_{44} \end{pmatrix}, \quad (31)$$

where $a_{1i}, a_{2i}, i = 1, 3, b_{3j}, b_{4j}, j = 2, 4 \in \mathbb{C}$ and satisfy $\Im(a_{11}\overline{a_{21}} + a_{13}\overline{a_{23}}) \geq 0, 4\Im(a_{11}\overline{a_{21}})\Im(a_{13}\overline{a_{23}}) \geq |1 - \overline{a_{11}}a_{23} + a_{13}\overline{a_{21}}|^2$ and $\Im(\overline{b_{32}}b_{42} + \overline{b_{34}}b_{44}) \geq 0, 4\Im(\overline{b_{32}}b_{42})\Im(\overline{b_{34}}b_{44}) \geq |1 - b_{32}\overline{b_{44}} + \overline{b_{34}}b_{42}|^2$.

$$\begin{pmatrix} a_{11} & 1 & a_{13} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{31} & 1 & b_{33} & 0 \\ 0 & 0 & 0 & 0 & b_{41} & 0 & b_{43} & -1 \end{pmatrix}, \quad (32)$$

where $a_{1i}, a_{2i}, i = 1, 3, b_{3j}, b_{4j}, j = 1, 3 \in \mathbb{C}$ and satisfy $\Im(a_{11}\overline{a_{21}} + a_{13}\overline{a_{23}}) \geq 0, 4\Im(a_{11}\overline{a_{21}})\Im(a_{13}\overline{a_{23}}) \geq |1 - \overline{a_{11}}a_{23} + a_{13}\overline{a_{21}}|^2$ and $\Im(\overline{b_{31}}b_{41} + \overline{b_{33}}b_{43}) \geq 0, 4\Im(\overline{b_{31}}b_{41})\Im(\overline{b_{33}}b_{43}) \geq |1 - b_{31}\overline{b_{43}} + \overline{b_{33}}b_{41}|^2$.

Proof. It is given in Section 3. $\square$

Remark 4. The separated BCs given in Theorem 2 can be transformed into the separated BCs given in Theorem 1 by elementary row transformation of boundary condition matrix $(A : B)$, but the conditions satisfied by the coefficients are more complex than those in Theorem 1. Therefore, it is necessary to list them in brief additionally.

Remark 5. In the regular case, there are similar statements as Theorems 1 and 2.

3. Proofs of the Main Results

In this section, we prove Theorems 1 and 2 and present additional results about dissipative operators. Before

proving these theorems, we need to introduce the following results. Since these results and their proofs are regular, we will state them without proofs. The readers may find them in [16].

Let $\Phi(x)$ be the Wronskian matrix of the solutions $v_i(x), i = 1, 2, 3, 4$, in I , then we have

$$\Phi(x) = (C_{v_1}(x), C_{v_2}(x), C_{v_3}(x), C_{v_4}(x)). \quad (33)$$

Lemma 1

$$Q = -(\Phi^*(x))^{-1}J\Phi^{-1}(x), \quad x \in I. \quad (34)$$

Proof. (See [16]) $\square$

Lemma 2. For arbitrary $y \in D(L)$,

$$\left([y, v_1]_x, [y, v_2]_x, [y, v_3]_x, [y, v_4]_x \right)^T = J\Phi^{-1}(x)C_y(x), \quad x \in I. \quad (35)$$

Proof. (See [16]) $\square$

Corollary 1. For arbitrary $y_1, y_2, y_3, y_4 \in D(L)$, let

$$\Psi(x) = (C_{y_1}(x), C_{y_2}(x), C_{y_3}(x), C_{y_4}(x)), \quad (36)$$

be the Wronskian matrix of y_1, y_2, y_3, y_4 , then

$$J\Phi^{-1}(x)\Psi(x) = \begin{pmatrix} [y_1, v_1]_x & [y_2, v_1]_x & [y_3, v_1]_x & [y_4, v_1]_x \\ [y_1, v_2]_x & [y_2, v_2]_x & [y_3, v_2]_x & [y_4, v_2]_x \\ [y_1, v_3]_x & [y_2, v_3]_x & [y_3, v_3]_x & [y_4, v_3]_x \\ [y_1, v_4]_x & [y_2, v_4]_x & [y_3, v_4]_x & [y_4, v_4]_x \end{pmatrix}, \quad x \in I. \quad (37)$$

Lemma 3. For arbitrary $y, z \in D(L)$, we have

$$[y, z]_x = ([y, v_1]_x \overline{[z, v_3]_x} + [y, v_2]_x \overline{[z, v_4]_x} - [y, v_3]_x \overline{[z, v_1]_x} - [y, v_4]_x \overline{[z, v_2]_x}), \quad x \in I. \quad (38)$$

Proof. By Lemma 1 and Lemma 2, it follows

$$\begin{aligned}
 [y, z]_x &= -R_{\bar{z}}(x)QC_y(x) \\
 &= R_{\bar{z}}(x)(\Phi^*(x))^{-1}J\Phi^{-1}(x)C_y(x) \\
 &= (J\Phi^{-1}(x)C_z(x))^*J(J\Phi^{-1}(x)C_y(x)) \\
 &= (\overline{[z, v_1]_x}, \overline{[z, v_2]_x}, \overline{[z, v_3]_x}, \overline{[z, v_4]_x})J([y, v_1]_x, [y, v_2]_x, [y, v_3]_x, [y, v_4]_x)^T \\
 &= ([y, v_1]_x\overline{[z, v_3]_x} + [y, v_2]_x\overline{[z, v_4]_x} - [y, v_3]_x\overline{[z, v_1]_x} - [y, v_4]_x\overline{[z, v_2]_x}).
 \end{aligned} \tag{39}$$

This completes the proof.

Now let us prove Theorems 1 and 2. $\square$

$$2i\mathfrak{I}(Ly, y) = (Ly, y) - (y, Ly) = [y, y]_b - [y, y]_a. \tag{40}$$

Then, applying (38), one obtains

Proof of Theorem 1. For arbitrary $y \in D(L)$, we have

$$\begin{aligned}
 2i\mathfrak{I}(Ly, y) &= ([y, v_1]_b\overline{[y, v_3]_b} + [y, v_2]_b\overline{[y, v_4]_b} - [y, v_3]_b\overline{[y, v_1]_b} - [y, v_4]_b\overline{[y, v_2]_b}) \\
 &\quad - ([y, v_1]_a\overline{[y, v_3]_a} + [y, v_2]_a\overline{[y, v_4]_a} - [y, v_3]_a\overline{[y, v_1]_a} - [y, v_4]_a\overline{[y, v_2]_a}).
 \end{aligned} \tag{41}$$

If the boundary condition matrix $(A: B)$ is equivalent to the matrix in (13), i.e., the BC is

$$\begin{cases} [y, v_3]_a = -(a_{11}[y, v_1]_a + a_{12}[y, v_2]_a - \overline{a_{31}}[y, v_1]_b + \overline{a_{41}}[y, v_2]_b), \\ [y, v_4]_a = a_{21}[y, v_1]_a + a_{22}[y, v_2]_a + \overline{a_{32}}[y, v_1]_b - \overline{a_{42}}[y, v_2]_b, \\ [y, v_3]_b = -(a_{31}[y, v_1]_a + a_{32}[y, v_2]_a + b_{31}[y, v_1]_b + b_{32}[y, v_2]_b), \\ [y, v_4]_b = a_{41}[y, v_1]_a + a_{42}[y, v_2]_a + b_{41}[y, v_1]_b + b_{42}[y, v_2]_b. \end{cases} \tag{42}$$

Substituting (42) into (41), one obtains

$$\begin{aligned}
 2i\mathfrak{I}(Ly, y) &= (Ly, y) - (y, Ly) = (\overline{[y, v_1]_a}, \overline{[y, v_2]_a}, \overline{[y, v_1]_b}, \overline{[y, v_2]_b}) \\
 &\quad \cdot \begin{pmatrix} \overline{a_{11}} - a_{11} & -a_{12} - \overline{a_{21}} & 0 & 0 \\ a_{21} + \overline{a_{12}} & a_{22} - \overline{a_{22}} & 0 & 0 \\ 0 & 0 & b_{31} - \overline{b_{31}} & b_{32} + \overline{b_{41}} \\ 0 & 0 & -b_{41} - \overline{b_{32}} & \overline{b_{42}} - b_{42} \end{pmatrix} \begin{pmatrix} [y, v_1]_a \\ [y, v_2]_a \\ [y, v_1]_b \\ [y, v_2]_b \end{pmatrix},
 \end{aligned} \tag{43}$$

and hence

$$2\mathfrak{I}(Ly, y) = (\overline{[y, v_1]_a}, \overline{[y, v_2]_a}, \overline{[y, v_1]_b}, \overline{[y, v_2]_b}) \begin{pmatrix} d & c & 0 & 0 \\ \bar{c} & s & 0 & 0 \\ 0 & 0 & p & g \\ 0 & 0 & \bar{g} & f \end{pmatrix} \begin{pmatrix} [y, v_1]_a \\ [y, v_2]_a \\ [y, v_1]_b \\ [y, v_2]_b \end{pmatrix}, \tag{44}$$

where

$$\begin{aligned} d &= -2\Im a_{11}, \\ s &= 2\Im a_{22}, \\ c &= i(a_{12} + \overline{a_{21}}), \\ p &= 2\Im b_{31}, \\ f &= -2\Im b_{42}, \\ g &= -i(b_{32} + \overline{b_{41}}). \end{aligned} \quad (45)$$

Note that the 4 by 4 matrix in (44) is Hermitian. The eigenvalues of the Hermitian matrix are

$$\begin{aligned} \frac{d+s \pm \sqrt{(d-s)^2 + 4|c|^2}}{2}, \\ \frac{p+f \pm \sqrt{(p-f)^2 + 4|g|^2}}{2}, \end{aligned} \quad (46)$$

and they are all nonnegative if and only if

$$\begin{aligned} d+s &\geq 0, \quad ds \geq |c|^2, \\ p+f &\geq 0, \quad pf \geq |g|^2. \end{aligned} \quad (47)$$

From condition (13), we have

$$\Im(Ly, y) \geq 0, \quad (48)$$

i.e., L is dissipative in $L^2(I)$.

The other cases can be proved in the same way. $\square$

Proof of Theorem 2. If the boundary condition matrix $(A: B)$ is equivalent to the matrix in (30), then

$$\begin{cases} [y, v_1]_a = -a_{12}[y, v_2]_a - a_{14}[y, v_4]_a, \\ [y, v_3]_a = a_{22}[y, v_2]_a + a_{24}[y, v_4]_a, \\ [y, v_2]_b = -b_{31}[y, v_1]_b - b_{33}[y, v_3]_b, \\ [y, v_4]_b = b_{41}[y, v_1]_b + b_{43}[y, v_3]_b. \end{cases} \quad (49)$$

Substituting (49) into (41), one obtains

$$\begin{aligned} 2i\Im(Ly, y) &= (Ly, y) - (y, Ly) = (\overline{[y, v_2]_a}, \overline{[y, v_4]_a}, \overline{[y, v_1]_b}, \overline{[y, v_3]_b}) \\ &\cdot \begin{pmatrix} a_{12}\overline{a_{22}} - a_{22}\overline{a_{12}} & a_{14}\overline{a_{22}} - \overline{a_{12}}a_{24} + 1 & 0 & 0 \\ a_{12}\overline{a_{24}} - \overline{a_{14}}a_{22} - 1 & a_{14}\overline{a_{24}} - a_{24}\overline{a_{14}} & 0 & 0 \\ 0 & 0 & -b_{31}\overline{b_{41}} + \overline{b_{31}}b_{41} & -b_{33}\overline{b_{41}} + \overline{b_{31}}b_{43} - 1 \\ 0 & 0 & -b_{31}\overline{b_{43}} + \overline{b_{33}}b_{41} + 1 & -b_{33}\overline{b_{43}} + \overline{b_{33}}b_{43} \end{pmatrix} \begin{pmatrix} [y, v_2]_a \\ [y, v_4]_a \\ [y, v_1]_b \\ [y, v_3]_b \end{pmatrix}, \end{aligned} \quad (50)$$

and hence

$$\begin{aligned} 2\Im(Ly, y) &= (\overline{[y, v_2]_a}, \overline{[y, v_4]_a}, \overline{[y, v_1]_b}, \overline{[y, v_3]_b}) \\ &\cdot \begin{pmatrix} d' & c' & 0 & 0 \\ \overline{c'} & s' & 0 & 0 \\ 0 & 0 & p' & g' \\ 0 & 0 & \overline{g'} & f' \end{pmatrix} \begin{pmatrix} [y, v_2]_a \\ [y, v_4]_a \\ [y, v_1]_b \\ [y, v_3]_b \end{pmatrix}, \end{aligned} \quad (51)$$

where

$$\begin{aligned} d' &= 2\Im(a_{12}\overline{a_{22}}), \\ s' &= 2\Im(a_{14}\overline{a_{24}}), \\ c' &= -i(a_{14}\overline{a_{22}} - \overline{a_{12}}a_{24} + 1), \\ p' &= 2\Im(\overline{b_{31}}b_{41}), \\ f' &= 2\Im(\overline{b_{33}}b_{43}), \\ g' &= -i(-b_{33}\overline{b_{41}} + \overline{b_{31}}b_{43} - 1). \end{aligned} \quad (52)$$

Note that the 4 by 4 matrix in (51) is Hermitian. The eigenvalues of the Hermitian matrix are

$$\begin{aligned} \frac{d'+s' \pm \sqrt{(d'-s')^2 + 4|c'|^2}}{2}, \\ \frac{p'+f' \pm \sqrt{(p'-f')^2 + 4|g'|^2}}{2}, \end{aligned} \quad (53)$$

and they are all nonnegative if and only if

$$\begin{aligned} d'+s' &\geq 0, \quad d's' \geq |c'|^2, \\ p'+f' &\geq 0, \quad p'f' \geq |g'|^2. \end{aligned} \quad (54)$$

Since condition (30) holds, we have

$$\Im(Ly, y) \geq 0, \quad (55)$$

i.e., L is dissipative in $L^2(I)$.

The other cases can be proved similarly. $\square$

Finally, we will state a spectral property of the dissipative operator L .

Theorem 3. Let the notations of (45) hold. And if $d > 0, s > 0$, and $ds > |c|^2$, $p > 0, f > 0$, and $pf > |g|^2$, then the operator L has no real eigenvalue.

Proof. Let λ_0 be a real eigenvalue of L and let $\phi_0(x) = \phi(x, \lambda_0) \neq 0$ be the corresponding eigenfunction, since

$$\Im(L\phi_0, \phi_0) = \Im(\lambda_0 \|\phi_0\|^2) = 0, \quad (56)$$

from (44), it follows

$$\Im(L\phi_0, \phi_0) = \frac{1}{2} \left([\overline{\phi_0}, v_1]_a, [\overline{\phi_0}, v_2]_a, [\overline{\phi_0}, v_1]_b, [\overline{\phi_0}, v_2]_b \right) \cdot \begin{pmatrix} d & c & 0 & 0 \\ \bar{c} & s & 0 & 0 \\ 0 & 0 & p & g \\ 0 & 0 & \bar{g} & f \end{pmatrix} \begin{pmatrix} [\phi_0, v_1]_a \\ [\phi_0, v_2]_a \\ [\phi_0, v_1]_b \\ [\phi_0, v_2]_b \end{pmatrix} = 0, \quad (57)$$

since $d > 0, s > 0$, and $ds > |c|^2$ and $p > 0, f > 0$, and $pf > |g|^2$,

the matrix $\begin{pmatrix} d & c & 0 & 0 \\ \bar{c} & s & 0 & 0 \\ 0 & 0 & p & g \\ 0 & 0 & \bar{g} & f \end{pmatrix}$ is positive definite. So, it can be

funded that $[\phi_0, v_1]_a = 0, [\phi_0, v_2]_a = 0, [\phi_0, v_1]_b = 0$, and $[\phi_0, v_2]_b = 0$. By the boundary condition matrix (42), one has that $[\phi_0, v_3]_a = 0, [\phi_0, v_4]_a = 0, [\phi_0, v_3]_b = 0$, and $[\phi_0, v_4]_b = 0$. Let $\tau_0(x) = \tau(x, \lambda_0), \eta_0(x) = \eta(x, \lambda_0), \delta_0(x) = \delta(x, \lambda_0)$, and $\phi_0(x) = \phi(x, \lambda_0)$ be the independent solutions of $l(y) = \lambda_0 y$. Then,

$$\begin{pmatrix} [\phi_0, v_1]_b & [\tau_0, v_1]_b & [\eta_0, v_1]_b & [\delta_0, v_1]_b \\ [\phi_0, v_2]_b & [\tau_0, v_2]_b & [\eta_0, v_2]_b & [\delta_0, v_2]_b \\ [\phi_0, v_3]_b & [\tau_0, v_3]_b & [\eta_0, v_3]_b & [\delta_0, v_3]_b \\ [\phi_0, v_4]_b & [\tau_0, v_4]_b & [\eta_0, v_4]_b & [\delta_0, v_4]_b \end{pmatrix} = J\Phi^{-1}(b) \begin{pmatrix} C_{\phi_0}(b) \\ C_{\tau_0}(b) \\ C_{\eta_0}(b) \\ C_{\delta_0}(b) \end{pmatrix}. \quad (58)$$

It is evident that the determinant of the left-hand side is equal to zero, and the value of the Wronskian of the solutions $\tau_0(x), \eta_0(x), \delta_0(x)$, and $\phi_0(x)$ is not equal to zero, so the determinant of the right hand side is not equal to zero. This is a contradiction, thus the theorem is proven. $\square$

4. Conclusion Remarks

In this paper, a classification of nonself-adjoint fourth-order dissipative differential operators generated by a fixed symmetric differential expression together with general two points BCs is investigated. This classification is based on the boundary conditions which generate the fourth-order differential operators to be dissipative. It seems that such a classification has not been introduced previously.

We want to show all the classifications of the fourth-order dissipative differential operators; however, due to the complexity of the data of the boundary condition matrix, in this paper, we showed all the separated BCs and some of the mixed and coupled BCs which generate the fourth-order operators to be dissipative, and we still cannot show all the

mixed and coupled BCs which generate the fourth-order operators to be dissipative. It seems an interesting problem; hence, we plan to continue the studies on this problem in our future works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] I. C. Gohberg and M. G. Krein, "Introduction to the theory of linear non-self-adjoint operator," in *Translations of Mathematical Monographs*, Vol. 18, American Mathematical Society, Providence, RI, USA, 1969.
- [2] M. V. Keldysh, "On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators," *Russian Mathematical Surveys*, vol. 26, no. 4, pp. 15–44, 1971.
- [3] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, Israel, 1965.
- [4] A. Sims, "Secondary conditions for linear differential operators of the second order," *Indiana University Mathematics Journal*, vol. 6, no. 2, pp. 247–285, 1957.
- [5] V. A. Marchenko, "Expansion in eigenfunctions of non-self-adjoint singular second order differential operators," *Matematicheskii Sbornik*, vol. 52, no. 94, pp. 739–788, 1960.
- [6] X.-y. Zhang and J. Sun, "The determinants of fourth order dissipative operators with transmission conditions," *Journal of Mathematical Analysis and Applications*, vol. 410, no. 1, pp. 55–69, 2014.
- [7] M. A. Naimark, *Linear Differential Operators*, Ungar, New York, NY, USA, 1968.
- [8] Z. Wang and H. Wu, "Dissipative non-self-adjoint Sturm-Liouville operators and completeness of their eigenfunctions," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 1, pp. 1–12, 2012.
- [9] G. S. Guseinov and H. Tuncay, "The determinants of perturbation connected with a dissipative Sturm-Liouville operator," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 39–49, 1995.
- [10] E. Bairamov and A. M. Krall, "Dissipative operators generated by the Sturm-Liouville differential expression in the Weyl limit circle case," *Journal of Mathematical Analysis and Applications*, vol. 254, no. 1, pp. 178–190, 2001.
- [11] E. Bairamov and E. Ugurlu, "The determinants of dissipative Sturm-Liouville operators with transmission conditions," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 805–813, 2011.
- [12] E. Ugurlu and E. Bairamov, "On singular dissipative fourth-order differential operator in lim-4 case," *ISRN Mathematical Analysis*, vol. 2013, Article ID 549876, 5 pages, 2013.

- [13] E. Uğurlu and E. Bairamov, “On the rate of the convergence of the characteristic values of an integral operator associated with a dissipative fourth order differential operator in lim-4 case with finite transmission conditions,” *Journal of Mathematical Chemistry*, vol. 52, no. 10, pp. 2627–2644, 2014.
- [14] H. Tuna, “On spectral properties of dissipative fourth order boundary-value problem with a spectral parameter in the boundary condition,” *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 9377–9387, 2013.
- [15] W. N. Everitt, “Integrable-square solutions of ordinary differential equations,” *The Quarterly Journal of Mathematics*, vol. 10, no. 1, pp. 145–155, 1959.
- [16] Z. Cao, *Ordinary Differential Operators*, Science Press, Beijing, China, 2016, in Chinese.

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