

## Research Article

# Fixed Point Problems in Cone Rectangular Metric Spaces with Applications

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In this paper, we introduce an ordered implicit relation and investigate some new fixed point theorems in a cone rectangular metric space subject to this relation. Some examples are presented as illustrations. We obtain a homotopy result as an application. Our results generalize and extend several fixed point results in literature.

## 1. Introduction

Many authors generalized the classical concept of metric space, by changing the metric conditions partially. Branciari [1] introduced rectangular metric space (RMS), where the triangular inequality condition of metric space was replaced by rectangular inequality. He also proved an analog of the Banach contraction principle in rectangular metric spaces. Azam and Arshad [2] mentioned some necessary conditions to get a unique fixed point for Kannan-type mappings in this context. Later, Karapinar et al. [3] investigated some fixed points for  $(\psi, \phi)$  contractions on rectangular metric spaces. On the other hand, Di Bari and Vetro [4] used  $(\psi, \phi)$ -weakly contractive condition to give an extension of the results in [3]. Subsequently, a number of authors were engrossed in rectangular metric spaces and proved the existence and uniqueness of fixed point theorems for certain types of mappings [2–6].

The significance of the Banach contraction principle lies in the fact that it is a very essential tool to check the existence of solutions for differential equations, integral equations, matrix equations, and functional equations made by mathematical models of real-world problems. There has been a ten-

dency for consistent theorists to improve both the underlying space and the contractive condition (explicit type) used by Banach [7] under the effect of one of the structures like order metric structure [8, 9], graphic metric structure [10, 11], multivalued mapping structure [12–14],  $\alpha$ -admissible mapping structure [15], comparison functions, and auxiliary functions. The process of developing new fixed point theorems in the complete metric spaces is in progress under various new restrictions. In this regard, we can find very nice results by Debnath et al. that appeared in [10, 12, 14].

Later on, Popa [16] introduced self-mappings satisfying implicit relation and obtained fixed points, under the effect of these functions. Popa [16–18] obtained some fixed point theorems in metric spaces. However, scrutiny into the fixed points of self-mappings satisfying implicit relations in order metric structure was made by Beg and Butt [19, 20], and some common fixed point theorems were established by Berinde and Vetro [21, 22] and Sedghi et al. [23]. Huang and Zhang [24] introduced cone metric by replacing real numbers with ordering Banach spaces and established a convergence criterion for sequences in cone metric space to generalize Banach fixed point theorem. Huang and Zhang [24] considered the concept of normal cone for their drawn

outcomes; however, Rezapour and Hambarani [25] left the normality condition in some results by Huang. Many authors have investigated fixed point theorems and common fixed point theorems of self-mappings for normal and nonnormal cones in cone metric spaces (see [26–29]).

Azam et al. [5] introduced the notion of a cone rectangular metric space and proved the Banach contraction principle in this context. In 2012, Rashwan [6] extended this idea as a continuation, which improved the results in [5]. The appealing nature of these spaces has enticed scrutiny into fixed point theorems for various contractions on cone rectangular metric spaces (see [5, 6]).

In this paper, we continue the study initiated by Azam et al. [5] subject to an ordered implicit relation. Since every cone metric space is a cone rectangular metric space but not conversely, therefore we prefer to establish results in cone rectangular metric spaces. These results are supported by some examples and an application in homotopy theory.

## 2. Preliminaries

*Definition 1.* A binary relation  $\mathcal{R}$  over a set  $\mathbb{Y} \neq \emptyset$  defines a partial order if  $\mathcal{R}$  has the following axioms:

- (1) reflexive
- (2) antisymmetric
- (3) transitive

A set having partial order  $\mathcal{R}$  is known as a partially ordered set denoted by  $(\mathbb{Y}, \mathcal{R})$ .

In the present article,  $\mathcal{E}$  stands for a real Banach space. Now, we present some definitions and relevant results, which will be required in the sequel.

*Definition 2* (see [24]). A subset  $\mathcal{P}$  of  $\mathcal{E}$  is called a cone if and only if the following conditions are satisfied:

- (1)  $\mathcal{P}$  is closed, nonempty, and  $\mathcal{P} \neq \{0\}$
- (2)  $\alpha\sigma + \beta\varsigma \in \mathcal{P}$ , for all  $\sigma, \varsigma \in \mathcal{P}$  and  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha, \beta \geq 0$
- (3)  $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$

Given  $\mathcal{P} \subset \mathcal{E}$ , define the partial order  $\leq$  with respect to  $\mathcal{P}$  as follows:

$\sigma \leq \varsigma$  if and only if  $\varsigma - \sigma \in \mathcal{P}$  for all  $\sigma, \varsigma \in \mathcal{E}$ . (5)

$\sigma < \varsigma$  represents that  $\sigma \leq \varsigma$  but  $\sigma \neq \varsigma$ , while  $\sigma \ll \varsigma$  stands for  $\varsigma - \sigma \in \mathcal{P}^\circ$  (interior of  $\mathcal{P}$ ).

*Definition 3* (see [24]). The cone  $\mathcal{P} \subseteq \mathcal{E}$  is called normal if, for all  $\sigma, \varsigma \in \mathcal{P}$ , there exists  $\mathcal{K} > 0$  such that

$$0 \leq \sigma \leq \varsigma \Rightarrow \|\sigma\| \leq \mathcal{K}\|\varsigma\|. \quad (1)$$

Throughout this paper, we assume  $\mathbb{Y} = (\mathbb{Y}, \mathcal{R})$  and  $\leq$  a partial order with respect to the cone  $\mathcal{P}$  defined in  $\mathcal{E}$ . If  $\mathbb{Y} \subseteq \mathcal{E}$ , then  $\mathcal{R}$  and  $\leq$  are identical; otherwise, they are different.

*Definition 4* (see [24]). Let  $\mathbb{Y}$  be a nonempty set, and  $d_c : \mathbb{Y} \times \mathbb{Y} \mapsto \mathcal{E}$  satisfies the following:

(d1)  $d_c(\sigma, \varsigma) \geq 0$ ,  $\forall \sigma, \varsigma \in \mathbb{Y}$  and  $d_c(\sigma, \varsigma) = 0$  if and only if  $\sigma = \varsigma$

(d2)  $d_c(\sigma, \varsigma) = d_c(\varsigma, \sigma)$

(d3)  $d_c(\sigma, \xi) \leq d_c(\sigma, \varsigma) + d_c(\varsigma, \xi)$ ,  $\forall \sigma, \varsigma, \xi \in \mathbb{Y}$

Then,  $d_c$  is called a cone metric on  $\mathbb{Y}$ , and  $(\mathbb{Y}, d_c)$  is then known as a cone metric space.

*Example 1* (see [5]). Let  $\mathbb{Y} = \mathbb{R}$ ,  $\mathcal{E} = \mathbb{R}^2$ , and  $\mathcal{P} = \{(\sigma, \varsigma) \in \mathcal{E} : \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$ . Define  $d_c : \mathbb{Y} \times \mathbb{Y} \rightarrow \mathcal{E}$  by

$$d_c(\sigma, \varsigma) = (|\sigma - \varsigma|, \alpha|\sigma - \varsigma|), \quad (2)$$

where  $\alpha \geq 0$  is a constant. Then,  $d_c$  defines a cone metric on  $\mathbb{Y}$ .

*Proposition 5* (see [5]). Consider a cone metric space  $(\mathbb{Y}, d_c)$ , with cone  $\mathcal{P}$ . Then, for  $u, v, w \in \mathcal{E}$ , we have

- (1) If  $u \leq \alpha u$  and  $\alpha \in [0, 1)$ , then  $u = 0$
- (2) If  $0 \leq u \ll v$  for each  $0 \ll v$ , then  $u = 0$
- (3) If  $u \leq v$  and  $v \ll w$ , then  $u \ll w$

Surely, cone metric space “being space” generalizes metric space, because in cone metric space, the range of a metric function is an ordered vector space instead of real numbers. Although the set of real numbers is an ordered vector space, we can find many significant ordered vector spaces in the literature (see [25, 27, 28, 30]). In Theorem 1.4 and Lemma 2.1 that appeared in [31, 32], respectively, the authors developed a metric depending on a given cone metric and proved that a complete cone metric space is always a complete metric space, and then, this relationship between metric and cone metric led them to say that every contraction mapping in a cone metric space is essentially contraction mapping in a metric space.

This paper addresses the fixed point results in the cone rectangular metric spaces. We know that every metric is a rectangular metric but rectangular metric needs not to be a metric (see [1–3, 33]). Also, we know that every cone metric is a cone rectangular metric but conversely does not hold in general (see [5, 6, 29]). In view of observations given in [5, 29], we infer that results in this paper are independent of what authors investigated in [31, 32]. The implicit relation and hence the contractive condition employed are even new and original in the rectangular metric space. The theorems in this paper are new in rectangular metric space, but we choose the cone rectangular metric space for the sake of the generality of our results.

*Definition 6* (see [24]). Let A mapping  $d_{cr} : \mathbb{Y} \times \mathbb{Y} \mapsto \mathcal{E}$  is said to be a cone rectangular metric if for all  $\sigma, \varsigma, \xi, v \in \mathbb{Y}$  the following conditions are satisfied:

(dR1)  $0 \leq d_{cr}(\sigma, \varsigma)$  and  $d_{cr}(\sigma, \varsigma) = 0$  if and only if  $\sigma = \varsigma$

(dR2)  $d_{cr}(\sigma, \varsigma) = d_{cr}(\varsigma, \sigma)$

(dR3)  $d_{cr}(\sigma, v) \leq d_{cr}(\sigma, \varsigma) + d_{cr}(\varsigma, \xi) + d_{cr}(\xi, v)$  for all distinct  $\varsigma, \xi \in \mathbb{Y} \setminus \{\sigma, v\}$

The cone rectangular metric space is denoted by  $(\mathbb{Y}, d_{cr})$ .

*Example 2* (see [5]). Let  $\mathcal{E} = \mathbb{R}^2$ ,  $\mathcal{P} = \{(\sigma, \varsigma) \in \mathcal{E} : \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$ ,  $\mathbb{Y} = \mathbb{R}$ . Take  $\beta > 0$ , and define

$$d_{cr}(\sigma, \varsigma) = \begin{cases} (0, 0) & \text{if } \sigma = \varsigma, \\ (3\beta, 3) & \text{if } \sigma, \varsigma \in \{1, 2\}, \sigma \neq \varsigma, \\ (\beta, 1) & \text{if } \sigma \text{ and } \varsigma \text{ cannot be both at a time in } \{1, 2\}, \sigma \neq \varsigma. \end{cases} \quad (3)$$

One can easily check that  $(\mathbb{Y}, d_{cr})$  is a cone rectangular metric space, but not a cone metric space, since  $d_{cr}(1, 2) = (3\beta, 3) \succ d_{cr}(1, 3) + d_{cr}(3, 2) = (2\beta, 2)$ .

*Definition 7* (see [1]). Let  $\mathcal{E}$  be a real Banach space,  $(\mathbb{Y}, d_{cr})$  be a cone rectangular metric space and  $c \in \mathcal{E}$  with  $0 \ll c$ .

- (1) A sequence  $\{\sigma_n\}$  in  $(\mathbb{Y}, d_{cr})$  is called a Cauchy sequence, if there exists a natural number  $N \in \mathbb{N}$  such that  $d_{cr}(\sigma_n, \sigma_m) \ll c$  for all  $n, m > N$
- (2) The sequence  $\{\sigma_n\}$  is said to be convergent if there exists an  $N \in \mathbb{N}$  such that  $d_{cr}(\sigma_n, \sigma) \ll c$  for all  $n \geq N$  and  $\sigma \in \mathbb{Y}$
- (3) The  $(\mathbb{Y}, d_{cr})$  is called complete if every Cauchy sequence converges in  $\mathbb{Y}$

### 3. Ordered Implicit Relations

Many authors have used implicit relations to establish fixed point results and have applied these results to solve nonlinear functional equations (see [19–22, 34, 35]).

In this section, we define a new ordered implicit relation and explain it with an example. In the next section, we use it along with some other assumptions to develop some new fixed point theorems in the cone rectangular metric space.

Let  $(\mathcal{E}, \|\cdot\|)$  be a real Banach space and  $B(\mathcal{E}, \mathcal{E})$  be the space of all bounded linear operators  $T : \mathcal{E} \rightarrow \mathcal{E}$  with the usual norm  $\|\cdot\|_1$  defined in  $B(\mathcal{E}, \mathcal{E})$  that is  $\|T\|_1 = \sup_{v \in \mathcal{E}} (\|T(v)\| / \|v\|) \neq 0$ .

In this section, generalizing the idea of [16], we define the following notion:

Let  $\mathcal{L} : \mathcal{E}^6 \rightarrow \mathcal{E}$  be a continuous operator which satisfies the conditions given below:

( $\mathcal{L}_1$ )  $v_5 \leq v_5$  and  $v_6 \leq v_6 \Rightarrow \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) \leq \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6)$

( $\mathcal{L}_2$ ) If  $\mathcal{L}(v_1, v_2, v_2, v_1, v_1 + v_2 + v_3, v_1) \leq 0$ , then there exists an order preserving operator  $S \in B(\mathcal{E}, \mathcal{E})$  with  $\|S\|_1 < 1$  such that  $v_1 \leq S(v_2)$  and  $v_3 \leq S(v_1)$  for all  $v_1, v_2, v_3 \in \mathcal{E}$  or if  $\mathcal{L}(v_1, v_2, v_1, v_2, v_1, v_1 + v_2 + v_3) \leq 0$ , then  $v_2 \leq S(v_3)$  and  $v_1 \leq S(v_2)$  for all  $v_1, v_2, v_3 \in \mathcal{E}$

( $\mathcal{L}_3$ )  $\mathcal{L}(v, 0, 0, v, v, 0) > 0$  whenever  $\|v\| > 0$

Let  $\mathcal{L} = \{\mathcal{L} : \mathcal{E}^6 \rightarrow \mathcal{E} \mid \mathcal{L} \text{ satisfies conditions } \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ .

*Example 3.* Let  $\leq$  be the partial order with respect to cone  $\mathcal{P}$  as defined in Section 2 and let  $(\mathcal{E}, \|\cdot\|)$  be a real Banach space. For all  $v_i \in \mathcal{E}$  ( $i = 1$  to 6),  $\alpha \in (0, 1/3)$  and  $(1 + \alpha)/2 \leq \beta \leq 1 + \alpha$ , define  $\mathcal{L} : \mathcal{E}^6 \rightarrow \mathcal{E}$ , by

$$\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = \alpha\{v_2 + v_3 + v_4\} - \beta(v_5 + v_6) + v_1. \quad (4)$$

Then, the operator  $\mathcal{L} \in \mathcal{L}$ :

( $\mathcal{L}_1$ ) Let  $v_5 \leq v_5$  and  $v_6 \leq v_6$ , then  $v_5 - v_5 \in \mathcal{P}$  and  $v_6 - v_6 \in \mathcal{P}$ . Now, we show that  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) \in \mathcal{P}$ . Given that  $v_5 - v_5 \in \mathcal{P}$  and  $v_6 - v_6 \in \mathcal{P}$  and by Definition 2 (2), we have

$$\begin{aligned} & \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) \\ &= \alpha\{v_2 + v_3 + v_4\} - \beta(v_5 + v_6) + v_1 - (\alpha\{v_2 + v_3 + v_4\} \\ & \quad - \beta(v_5 + v_6) + v_1) = \beta(v_5 - v_5 + v_6 - v_6) \in \mathcal{P}. \end{aligned} \quad (5)$$

Thus,  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) \leq \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6)$ .

( $\mathcal{L}_2$ ) Let  $v_1, v_2, v_3 \in \mathcal{E}$  be such that  $0 \leq v_1, 0 \leq v_2, 0 \leq v_3$ . If  $\mathcal{L}(v_1, v_2, v_2, v_1, v_1 + v_2 + v_3, v_1) \leq 0$  then, we have  $-v_1 - \alpha\{v_2 + v_2 + v_1\} + \beta(v_1 + v_2 + v_3 + v_1) \in \mathcal{P}$ .

So  $(2\beta - \alpha - 1)v_1 + (\beta - 2\alpha)v_2 + \beta v_3 \in \mathcal{P}$ .

By Definition 2 (2), we have for  $2\beta \geq \alpha + 1$  either

$$(2\beta - \alpha - 1)v_1 + (\beta - 2\alpha)v_2 \in \mathcal{P} \quad (6)$$

or

$$(2\beta - \alpha - 1)v_1 + \beta v_3 \in \mathcal{P} \quad (7)$$

or

$$(\beta - 2\alpha)v_2 + \beta v_3 \in \mathcal{P}. \quad (8)$$

For (6), if  $v_1 = 0$  and  $v_2 \neq 0$ , then  $(\beta - 2\alpha)v_2 \in \mathcal{P}$ . Thus, there exists  $S : \mathcal{E} \rightarrow \mathcal{E}$  defined by  $S(v_2) = \eta v_2$  ( $\eta = (\beta - 2\alpha)$  is a scalar) such that  $v_1 \leq S(v_2)$ . Now, if  $v_2 = 0$  and  $v_1 \neq 0$ , then,  $(2\beta - \alpha - 1)v_1 \in \mathcal{P}$ . So, there exists  $S : \mathcal{E} \rightarrow \mathcal{E}$  defined by  $S(v_1) = \eta v_1$  ( $\eta = (2\beta - \alpha - 1)$  is a scalar) such that  $v_2 \leq S(v_1)$ , for some  $\alpha \in (0, 1/3)$ . For if both  $v_1 \neq 0$  and  $v_2 \neq 0$ , then we get an absurdity.

For (7), if  $v_3 = 0$  and  $v_1 \neq 0$ , then  $(2\beta - \alpha - 1)v_1 \in \mathcal{P}$ . Thus, there exists  $S : \mathcal{E} \rightarrow \mathcal{E}$  defined by  $S(v_1) = \eta v_1$  ( $\eta = (2\beta - \alpha - 1)$  is a scalar) such that  $v_3 \leq S(v_1)$ . Now, if  $v_3 \neq 0$  and  $v_1 = 0$ , then,  $\beta v_3 \in \mathcal{P}$ . So, there exists  $S : \mathcal{E} \rightarrow \mathcal{E}$  defined by  $S(v_3) = \eta v_3$  ( $\eta = \beta$  is a scalar) such that  $v_1 \leq S(v_3)$ . For if both  $v_1 \neq 0$  and  $v_3 \neq 0$ , then we get an absurdity. Similar arguments hold for (8).

( $\mathcal{L}_3$ ) Let  $v \in \mathcal{E}$  be such that  $\|v\| > 0$  and consider,  $0 \leq \mathcal{L}(v, 0, 0, v, v, 0)$  then  $(1 + \alpha - \beta)v \in \mathcal{P}$ , which holds whenever  $\|v\| > 0$ .

Similarly, the operators  $\mathcal{L} : \mathcal{E}^6 \rightarrow \mathcal{E}$  defined by

$$(1) \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha\{v_5 + v_6\}; \alpha \in [1, \infty)$$

- (2)  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_3 - \alpha v_5 - \beta v_6; \alpha + \beta < 1, \alpha, \beta > 0$
- (3)  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha\{v_3 + v_4\} - (1 - \alpha)\beta \max\{v_5, v_6\}; \alpha \in [0, 1)$  and  $\beta \in [1/2, 1)$
- (4)  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha v_2$  for some  $\alpha \in [0, 1)$
- (5)  $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - k(v_3 + v_4)$  for some  $k \in [0, 1/2)$  are members of  $\mathcal{E}$

The following remark is essential in the sequel.

*Remark 8.* If  $S \in B(\mathcal{E}, \mathcal{E})$ , the Neumann series  $I + S + S^2 + \dots + S^n + \dots$  converges if  $\|S\|_1 < 1$  and diverges otherwise. Also, if  $\|S\|_1 < 1$ , then there exists  $\lambda > 0$  such that  $\|S\|_1 < \lambda < 1$  and  $\|S^n\|_1 \leq \lambda^n < 1$ .

#### 4. New Results

Recently, Popa [16] has employed implicit type contractive condition on self-mapping to obtain some fixed point theorems. Ran and Reurings [9] have presented an analog of Banach fixed point theorem for monotone self-mappings in an ordered metric space. Huang and Zhang [24] introduced the idea of cone metric spaces and obtained analogs of Banach fixed point theorem, Kannan fixed point theorem, and Chatterjea fixed point theorem in cone metric spaces. In this section, we prove some fixed point results for ordered implicit relations in a cone rectangular metric space which improves the results in [9, 16, 24]. We derive these results under two different partial orders: one defined in underlying set and the other in real Banach space.

**Theorem 9.** Let  $(\mathbb{Y}, d_{cr})$  be a complete cone rectangular metric space and  $\mathcal{P} \subset \mathcal{E}$  be a cone. Let  $f : \mathbb{Y} \rightarrow \mathbb{Y}$ . If there exist  $T \in B(\mathcal{E}, \mathcal{E})$ , identity operator  $I : \mathcal{E} \rightarrow \mathcal{E}$  and  $\mathcal{L} \in \mathcal{E}$  such that, for all comparable elements  $\sigma, \kappa \in \mathbb{Y}$

$$(I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa) \text{ implies } \mathcal{L} \cdot (d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr}(\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \leq 0 \quad (9)$$

and

- (1) there exists  $\sigma_0 \in \mathbb{Y}$  such that  $\sigma_0 \mathcal{R} f(\sigma_0)$
- (2) for all  $\sigma, \kappa \in \mathbb{Y}$ ,  $\sigma \mathcal{R} \kappa$  implies  $f(\sigma) \mathcal{R} f(\kappa)$
- (3) for every  $\{\sigma_n\} \subseteq \mathbb{Y}$ ,  $d_{cr}(\sigma_n, \sigma_{n+1}) \leq T(d_{cr}(\sigma_{n-1}, \sigma_n))$
- (4) for a sequence  $\{\sigma_n\}$  with  $\sigma_n \rightarrow x^*$  whose all sequential terms are comparable, we have  $\sigma_n \mathcal{R} x^*$  for all  $n \in \mathbb{N}$  and  $d_{cr}(x^*, f(x^*)) \leq d_{cr}(x^*, f^2(x^*))$

Then,  $x^* = f(x^*)$ .

*Proof.* Let  $\sigma_0 \in \mathbb{Y}$  be as assumed in (1). We construct a sequence  $\{\sigma_n\}$  by  $f(\sigma_{n-1}) = \sigma_n$  starting with  $\sigma_0 \in \mathbb{Y}$ . Then,  $\sigma_0 \mathcal{R} \sigma_1$ . By assumption (2), we have  $\sigma_1 \mathcal{R} \sigma_2, \sigma_2 \mathcal{R} \sigma_3, \dots, \sigma_{n-1} \mathcal{R} \sigma_n$ . For  $\sigma = \sigma_0$  and  $\kappa = \sigma_1$ , we have by (9).

$$(I - T)^2(I + T)(d_{cr}(\sigma_0, f(\sigma_0))) = (I - T)^2(I + T)(d_{cr}(\sigma_0, \sigma_1)) \leq d_{cr}(\sigma_0, \sigma_1) \text{ implies } \mathcal{L} \times (d_{cr}(f(\sigma_0), f(\sigma_1)), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_0, f(\sigma_0)), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}(\sigma_0, f^2(\sigma_1)), d_{cr}(\sigma_1, f^2(\sigma_0))) \leq 0, \quad (10)$$

that is,

$$\mathcal{L}(d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_3), d_{cr}(\sigma_1, \sigma_2)) \leq 0. \quad (11)$$

By (dR3), we have

$$d_{cr}(\sigma_0, \sigma_3) \leq d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3), \quad (12)$$

and so we rewrite (11) employing condition  $(\mathcal{L}_1)$  as follows:

$$\mathcal{L}(d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2)) \leq 0, \quad (13)$$

and thus, by  $(\mathcal{L}_2)$ , there exists an order preserving operator  $S \in B(\mathcal{E}, \mathcal{E})$  with  $\|S\|_1 < 1$  such that

$$d_{cr}(\sigma_1, \sigma_2) \leq S(d_{cr}(\sigma_0, \sigma_1)), \quad (14)$$

$$d_{cr}(\sigma_2, \sigma_3) \leq S(d_{cr}(\sigma_1, \sigma_2)).$$

Now, put  $\sigma = \sigma_1$  and  $\kappa = \sigma_2$  in (9) to have

$$(I - T)^2(I + T)(d_{cr}(\sigma_1, f(\sigma_1))) = (I - T)^2(I + T)(d_{cr}(\sigma_1, \sigma_2)) \leq d_{cr}(\sigma_1, \sigma_2) \text{ implies } \mathcal{L} \times (d_{cr}(f(\sigma_1), f(\sigma_2)), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}(\sigma_2, f(\sigma_2)), d_{cr}(\sigma_1, f(\sigma_3)), d_{cr}(\sigma_2, f(\sigma_2))) \leq 0, \quad (15)$$

that is,

$$\mathcal{L}(d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_4), d_{cr}(\sigma_2, \sigma_3)) \leq 0. \quad (16)$$

By (dR3), we have

$$d_{cr}(\sigma_1, \sigma_4) \leq d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3) + d_{cr}(\sigma_3, \sigma_4), \quad (17)$$

and  $(\mathcal{L}_1)$  implies

$$\begin{aligned} \mathcal{L}(d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2) \\ + d_{cr}(\sigma_2, \sigma_3) + d_{cr}(\sigma_3, \sigma_4), d_{cr}(\sigma_2, \sigma_3)) \leq 0. \end{aligned} \quad (18)$$

By  $(\mathcal{L}_2)$ , there exists  $S \in B(\mathcal{E}, \mathcal{E})$  with  $\|S\|_1 < 1$  such that

$$d_{cr}(\sigma_3, \sigma_4) \leq S(d_{cr}(\sigma_2, \sigma_3)) \leq S^2(d_{cr}(\sigma_1, \sigma_2)) \leq S^3(d_{cr}(\sigma_0, \sigma_1)). \quad (19)$$

By continuing this pattern, we can construct a sequence  $\{\sigma_n\}$  such that  $\sigma_n \mathcal{R} \sigma_{n+1}$  with  $\sigma_{n+1} = f(\sigma_n)$ , and

$$\begin{aligned} (I - T)^2(I + T)(d_{cr}(\sigma_{n-1}, f(\sigma_{n-1}))) \\ = (I - T)^2(I + T)(d_{cr}(\sigma_{n-1}, \sigma_n)) \\ \leq d_{cr}(\sigma_{n-1}, \sigma_n) \text{ implies } d_{cr}(\sigma_n, \sigma_{n+1}) \\ \leq S(d_{cr}(\sigma_{n-1}, \sigma_n)) \leq S^2(d_{cr}(\sigma_{n-2}, \sigma_{n-1})) \\ \leq \dots \leq S^n(d_{cr}(\sigma_0, \sigma_1)). \end{aligned} \quad (20)$$

For  $m, n \in \mathbb{N}$  with  $m > n$ , we have by Remark 8

$$\begin{aligned} d_{cr}(\sigma_n, \sigma_m) &\leq d_{cr}(\sigma_n, \sigma_{n+1}) + d_{cr}(\sigma_{n+1}, \sigma_{n+2}) + \dots + \\ &\quad + d_{cr}(\sigma_{m-1}, \sigma_m) \\ &\leq S^n(d_{cr}(\sigma_0, \sigma_1)) + S^{n+1}(d_{cr}(\sigma_0, \sigma_1)) \\ &\quad + \dots + S^{m-1}(d_{cr}(\sigma_0, \sigma_1)) \\ &= (S^n + S^{n+1} + \dots + S^{m-1})(d_{cr}(\sigma_0, \sigma_1)), \\ &\leq \{S^n(1 + S + \dots + S^{m-n-1} + \dots)\}(d_{cr}(\sigma_0, \sigma_1)) \\ &= \{S^n(I - S)^{-1}\}(d_{cr}(\sigma_0, \sigma_1)). \end{aligned} \quad (21)$$

Since  $\|S\|_1 < 1$ , so,  $S^n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $\lim_{n \rightarrow \infty} d_{cr}(\sigma_n, \sigma_m) = 0$  which implies that  $\{\sigma_n\}$  is a Cauchy sequence in  $\mathbb{Y}$ . Since  $(\mathbb{Y}, d_{cr})$  is a complete cone rectangular metric space, so, there exists  $x^* \in \mathbb{Y}$  such that  $\sigma_n \rightarrow x^*$  as  $n \rightarrow \infty$ . Equivalently, there exists a natural number  $N_2$  such that

$$d_{cr}(\sigma_n, x^*) \ll c \text{ for all } n \geq N_2. \quad (22)$$

We claim that

$$(I - T)^2(I + T)(d_{cr}(\sigma_n, f(\sigma_n))) \leq d_{cr}(\sigma_n, x^*). \quad (23)$$

We assume against our claim that

$$\begin{aligned} (I - T)^2(I + T)(d_{cr}(\sigma_n, f(\sigma_n))) > d_{cr}(\sigma_n, x^*), (I - T)^2(I + T) \\ \cdot (d_{cr}(\sigma_{n+2}, f(\sigma_{n+2}))) > d_{cr}(\sigma_{n+2}, x^*) \text{ for some } n \in \mathbb{N}. \end{aligned} \quad (24)$$

By (dR3), (9), and assumption (3), we have

$$\begin{aligned} d_{cr}(\sigma_n, f(\sigma_n)) &\leq d_{cr}(\sigma_n, x^*) + d_{cr}(x^*, \sigma_{n+2}) + d_{cr} \\ &\quad \cdot (\sigma_{n+2}, \sigma_{n+1}) < (I - T)^2(I + T)(d_{cr}(\sigma_n, f(\sigma_n))) + (I - T)^2 \\ &\quad \cdot (I + T)(d_{cr}(\sigma_{n+2}, f(\sigma_{n+2}))) + T(d_{cr}(\sigma_n, f(\sigma_n))) < (I - T)^2 \\ &\quad \cdot (I + T)(d_{cr}(\sigma_n, f(\sigma_n))) + (I - T)^2(I + T)T^2 \\ &\quad \cdot (d_{cr}(\sigma_n, f(\sigma_n))) + T(d_{cr}(\sigma_n, f(\sigma_n))), (I - T) \\ &\quad \cdot (d_{cr}(\sigma_n, f(\sigma_n))) < (I - T)^2(I + T)(I + T^2) \\ &\quad \cdot (d_{cr}(\sigma_n, f(\sigma_n)))d_{cr}(\sigma_n, f(\sigma_n)) < (I - T)^{-1}(I - T)^2(I + T) \\ &\quad \cdot (I + T^2)(d_{cr}(\sigma_n, f(\sigma_n))) < (I - T^4)d_{cr}(\sigma_n, f(\sigma_n)). \end{aligned} \quad (25)$$

Thus,  $T^4(d_{cr}(\sigma_n, f(\sigma_n))) < 0$ , which is an absurdity. Hence, for each  $n \geq 1$ , we have

$$(I - T)^2(I + T)(d_{cr}(\sigma_n, f(\sigma_n))) \leq d_{cr}(\sigma_n, x^*). \quad (26)$$

Assume that  $\|d_{cr}(x^*, f(x^*))\| > 0$ . As  $\sigma_{n-1} \mathcal{R} \sigma_n$  and by the assumption (4), we have  $\sigma_n \mathcal{R} x^*$  for all  $n \in \mathbb{N}$  and then by (9), we get

$$\mathcal{L} \left( \begin{array}{c} d_{cr}(f(\sigma_n), f(x^*)), d_{cr}(\sigma_n, x^*), d_{cr}(\sigma_n, f(\sigma_n)), d_{cr}(x^*, f(x^*)), \\ d_{cr}(\sigma_{n-1}, f^2(x^*)), d_{cr}(f^2(\sigma_n), f(\sigma_n)) \end{array} \right) \leq 0. \quad (27)$$

Letting  $n \rightarrow \infty$  and in view of assumption (4) and (26), we have

$$\begin{aligned} \mathcal{L}(d_{cr}(x^*, f(x^*)), d_{cr}(x^*, x^*), d_{cr}(x^*, x^*), d_{cr}(x^*, f(x^*)), d_{cr} \\ \cdot (x^*, f^2(x^*)), d_{cr}(x^*, x^*)) \leq 0, \mathcal{L}(d_{cr}(x^*, f(x^*)), 0, 0, d_{cr} \\ \cdot (x^*, f(x^*)), d_{cr}(x^*, f^2(x^*)), 0) \leq 0. \end{aligned} \quad (28)$$

By  $(\mathcal{L}_1)$ , we have

$$\mathcal{L}(d_{cr}(x^*, f(x^*)), 0, 0, d_{cr}(x^*, f(x^*)), d_{cr}(x^*, f(x^*)), 0) \leq 0. \quad (29)$$

This is a contradiction to  $(\mathcal{L}_3)$ . Thus,  $\|d_{cr}(x^*, f(x^*))\| = 0$ . Hence,  $d_{cr}(x^*, f(x^*)) = 0$ . It follows from (dR1) that  $x^* = f(x^*)$ .

**Theorem 10.** Let  $(\mathbb{Y}, d_{cr})$  be a complete cone rectangular metric space and  $f$  be a self-mapping on  $\mathbb{Y}$ . If for all comparable elements  $\sigma, \kappa \in \mathbb{Y}$ , there exist  $T \in B(\mathcal{E}, \mathcal{E})$ , identity operator  $I : \mathcal{E} \rightarrow \mathcal{E}$  and  $\mathcal{L} \in \mathcal{L}$  such that

$$\begin{aligned} (I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa) \text{ implies } \mathcal{L} \\ \cdot (d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \\ \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \leq 0, \end{aligned} \quad (30)$$

and

- (1) there exists  $\sigma_0 \in \mathbb{Y}$  such that  $f(\sigma_0) \mathcal{R} \sigma_0$
- (2) for any  $\sigma, \kappa \in \mathbb{Y}$ ,  $\sigma \mathcal{R} \kappa$  implies  $f(\kappa) \mathcal{R} f(\sigma)$
- (3) for every  $\{\sigma_n\} \subseteq \mathbb{Y}$ ,  $d_{cr}(\sigma_n, \sigma_{n+1}) \leq T(d_{cr}(\sigma_{n-1}, \sigma_n))$
- (4) for a sequence  $\{\sigma_n\}$  with  $\sigma_n \rightarrow x^*$  whose all sequential terms are comparable, we have  $\sigma_n \mathcal{R} x^*$  for all  $n \in \mathbb{N}$  and  $d_{cr}(x^*, f(x^*)) \leq d_{cr}(x^*, f^2(x^*))$

Then,  $f$  has a fixed point in  $\mathbb{Y}$ .

*Proof.* Let  $\sigma_0$  be in  $\mathbb{Y}$  as assumed in (1). Define the sequence  $\{\sigma_n\}$  by  $\sigma_n = f(\sigma_{n-1})$  for all  $n$ . Since  $\sigma_1 = f(\sigma_0) \mathcal{R} \sigma_0$  and by assumption (2)  $\sigma_1 = f(\sigma_0) \mathcal{R} f(\sigma_1) = \sigma_2$ , repeated application of assumption (2) gives us either  $\sigma_n \mathcal{R} \sigma_{n-1}$  or  $\sigma_{n-1} \mathcal{R} \sigma_n$  for each  $n$ . By (30), we have for  $\sigma = \sigma_1$  and  $\kappa = \sigma_0$

$$\begin{aligned}
& (I - T)^2(I + T)(d_{cr}(f(\sigma_0), \sigma_0) = (I - T)^2(I + T) \\
& \cdot (d_{cr}(\sigma_1, \sigma_0))) \leq d_{cr}(\sigma_1, \sigma_0) \text{ implies } \mathcal{L} \\
& \cdot (d_{cr}(f(\sigma_1), f(\sigma_0)), d_{cr}(\sigma_1, \sigma_0), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr} \\
& \cdot (\sigma_0, f(\sigma_0)), d_{cr}(\sigma_1, f^2(\sigma_0)), d_{cr}(\sigma_0, f^2(\sigma_1))) \leq 0 \Rightarrow \mathcal{L} \\
& \cdot (d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr} \\
& \cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_3)) \leq 0.
\end{aligned} \tag{31}$$

By (dR3), we have

$$d_{cr}(\sigma_0, \sigma_3) \leq d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3), \tag{32}$$

and then using  $\mathcal{L}_1$ , we obtain

$$\begin{aligned}
& \mathcal{L}(d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr} \\
& \cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3)) \leq 0.
\end{aligned} \tag{33}$$

By  $(\mathcal{L}_2)$ , there exists an order preserving operator  $S \in B(\mathcal{E}, \mathcal{E})$  with  $\|S\|_1 < 1$  such that

$$\begin{aligned}
& d_{cr}(\sigma_1, \sigma_2) \leq S(d_{cr}(\sigma_0, \sigma_1)), \\
& d_{cr}(\sigma_2, \sigma_3) \leq S(d_{cr}(\sigma_1, \sigma_2)).
\end{aligned} \tag{34}$$

Using (2)  $f(\sigma_0) \mathcal{R} f(\sigma_1)$ , take  $\sigma = \sigma_1$  and  $\kappa = \sigma_2$  in (30), we have

$$\begin{aligned}
& (I - T)^2(I + T)(d_{cr}(\sigma_1, f(\sigma_1))) = (I - T)^2(I + T) \\
& \cdot (d_{cr}(\sigma_1, \sigma_2)) \leq d_{cr}(\sigma_1, \sigma_2) \text{ implies } \mathcal{L}(d_{cr}(f(\sigma_1), f(\sigma_2)), d_{cr} \\
& \cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}(\sigma_2, f(\sigma_2)), d_{cr}(\sigma_1, f^2(\sigma_2)), d_{cr} \\
& \cdot (\sigma_2, f^2(\sigma_1))) \leq 0 \Rightarrow \mathcal{L}(d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2), d_{cr} \\
& \cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_4), d_{cr}(\sigma_2, \sigma_3)) \leq 0.
\end{aligned} \tag{35}$$

By (dR3),  $(\mathcal{L}_1)$  and  $(\mathcal{L}_2)$ , we get

$$d_{cr}(\sigma_3, \sigma_4) \leq S(d_{cr}(\sigma_2, \sigma_3)) \leq S^2(d_{cr}(\sigma_1, \sigma_2)) \leq S^3(d_{cr}(\sigma_0, \sigma_1)). \tag{36}$$

By continuing the pattern, we construct a sequence  $\{\sigma_n\}$  such that

$$d_{cr}(\sigma_n, \sigma_{n+1}) \leq S(d_{cr}(\sigma_{n-1}, \sigma_n)) \leq S^2(d_{cr}(\sigma_{n-2}, \sigma_{n-1})) \leq \dots \leq S^n(d_{cr}(\sigma_0, \sigma_1)). \tag{37}$$

Hence, by the same reasoning as in the proof of Theorem 9, we have  $x^* = f(x^*)$ .

**Theorem 11.** Let  $(\mathbb{Y}, d_{cr})$  be a complete cone rectangular metric space and  $f$  be a monotone self-mapping on  $\mathbb{Y}$ . If for all comparable elements  $\sigma, \kappa \in \mathbb{Y}$ , there exist  $T \in B(\mathcal{E}, \mathcal{E})$ , identity operator  $I : \mathcal{E} \rightarrow \mathcal{E}$ , and  $\mathcal{L} \in \mathcal{Z}$  such that

$$\begin{aligned}
& (I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa) \text{ implies } \mathcal{L} \\
& \cdot (d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \\
& \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \leq 0,
\end{aligned} \tag{38}$$

and

- (1) there exists  $\sigma_0 \in \mathbb{Y}$  such that either  $\sigma_0 \mathcal{R} f(\sigma_0)$  or  $f(\sigma_0) \mathcal{R} \sigma_0$
- (2) for every  $\{\sigma_n\} \subseteq \mathbb{Y}$ ,  $d_{cr}(\sigma_n, \sigma_{n+1}) \leq T(d_{cr}(\sigma_{n-1}, \sigma_n))$
- (3) for a sequence  $\{\sigma_n\}$  with  $\sigma_n \rightarrow x^*$  whose all sequential terms are comparable, we have  $\sigma_n \mathcal{R} x^*$  for all  $n \in \mathbb{N}$  and  $d_{cr}(x^*, f(x^*)) \leq d_{cr}(x^*, f^2(x^*))$

Then,  $f$  has a fixed point in  $\mathbb{Y}$ .

*Proof.* This proof follows the same pattern as given in the previous two proofs, so, we omit it.

*Remark 12.*

- (1) In Theorem 9, Theorem 10, and Theorem 11, uniqueness of the fixed point of  $f$  can be attained by assuming that for every pair of elements  $\sigma, \kappa \in X$ , there exists either an upper bound or lower bound of  $\sigma, \kappa$
- (2) The cone is taken as nonnormal in the above theorems

**Theorem 13.** Let  $(\mathbb{Y}, d_{cr})$  be a complete cone rectangular metric space and  $f$  be a monotone self-mapping on  $\mathbb{Y}$ . If for all comparable elements  $\sigma, \kappa \in \mathbb{Y}$ , there exist  $T \in B(\mathcal{E}, \mathcal{E})$ ,

identity operator  $I : \mathcal{E} \rightarrow \mathcal{E}$ , and  $\mathcal{L} \in \mathcal{G}$  such that

$$(I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa) \text{ implies } d_{cr}(f(\sigma), f(\kappa)) \leq S(d_{cr}(\sigma, \kappa)). \tag{39}$$

Moreover, if

- (1) there exists  $\sigma \in \mathbb{Y}$  such that  $\sigma_0 \mathcal{R} f(\sigma_0)$  or  $f(\sigma_0) \mathcal{R} \sigma_0$
- (2) for a sequence  $\{\sigma_n\}$  with  $\sigma_n \rightarrow x^*$  whose all sequential terms are comparable, we have  $\sigma_n \mathcal{R} x^*$  for all  $n \in \mathbb{N}$  and  $d_{cr}(x^*, f(x^*)) \leq d_{cr}(x^*, f^2(x^*))$

Then, there exists  $x^* \in \mathbb{Y}$  such that  $x^* = f(x^*)$ .

*Proof.* Define  $S(v) = qv$  for all  $v \in \mathcal{E}$  and  $q \in [0, 1)$ , then  $S \in B(\mathcal{E}, \mathcal{E})$  with  $\|S\|_1 < 1$  also define implicit relation by

$$\mathcal{L}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - \alpha \sigma_2 \text{ for some } \alpha \in [0, 1). \tag{40}$$

The proof follows by the application of Theorem 9.

The following examples illustrate the main theorem.

*Example 4.* Let  $\mathcal{E} = (\mathbb{R}^3, \|\cdot\|)$  be a real Banach space. With  $\|\sigma\| = \max\{(|\sigma_1|, |\sigma_2|, |\sigma_3|)\}$ , then  $(\mathcal{E}, \|\cdot\|)$  is a real Banach space. Define the partial order  $\leq$  on  $\mathcal{E}$  by

$$(\sigma, \xi, \nu) \geq 0 \Leftrightarrow \sigma \geq 0, \xi \geq 0, \nu \geq 0. \tag{41}$$

Define  $\mathcal{P} = \{(\sigma, \xi, \nu) \in \mathbb{R}^3 : \sigma, \xi, \nu \geq 0\}$ , then, it is a cone in  $\mathcal{E}$ .

Let  $\mathbb{Y} = \{(0, 0, 0), (1/4, 0, 0), (1/4, 1/4, 0), (1/4, 1/4, 1/4)\} \subset \mathcal{E}$ , define order on  $\mathbb{Y}$  by  $\mathcal{R} = \leq$  and define  $f : \mathbb{Y} \rightarrow \mathbb{Y}$  such that

$$f(\sigma) = \begin{cases} (0, 0, 0) & \text{if } \sigma \in \left\{ (0, 0, 0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, 0, 0\right) \right\}, \\ \left(\frac{1}{4}, \frac{1}{4}, 0\right) & \text{if } \sigma = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \end{cases} \tag{42}$$

Define the function  $d_{cr} : \mathbb{Y} \times \mathbb{Y} \rightarrow \mathcal{E}$  by

$$d_{cr}(\sigma_1, \sigma_2) = \begin{cases} (0, 0, 0) & \text{if } \sigma_1 = \sigma_2, \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) & \text{if } \sigma_1, \sigma_2 \in \left\{ (0, 0, 0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \right\}, \\ \left(\frac{1}{4}, 0, 0\right) & \text{if } \sigma_1 \neq \sigma_2 \text{ cannot be both at a time in } \left\{ (0, 0, 0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \right\}. \end{cases} \tag{43}$$

It is a cone rectangular metric space but not a cone metric space. The assumptions (1)–(4) in Theorem 9 can be verified immediately. Define  $T(\sigma) = \sigma/3$ , then,  $T \in B(\mathcal{E}, \mathcal{E})$  with  $\|T\|_1 \leq 1$ . Now, we verify condition (9). For if  $\sigma = (0, 0, 0)$  and  $\kappa = (1/4, 0, 0)$ , then

$$\begin{aligned} d_{cr}(\sigma, f(\sigma)) &= d_{cr}(f(\sigma), f(\kappa)) = d_{cr}(\sigma, f^2(\kappa)) = (0, 0, 0), \\ d_{cr}(\kappa, f^2(\sigma)) &= \left(\frac{1}{4}, 0, 0\right) = d_{cr}(\sigma, \kappa), \end{aligned} \tag{44}$$

and thus,

$$\begin{aligned} (I - T)^2(I + T)d_{cr}(\sigma, f(\sigma)) &= (0, 0, 0), \alpha[d_{cr}(\sigma, f^2(\kappa)) + d_{cr} \\ &\cdot (\kappa, f^2(\sigma))] = \alpha \left[ (0, 0, 0) + \left(\frac{1}{4}, 0, 0\right) \right] = \alpha \left(\frac{1}{4}, 0, 0\right) : \alpha \geq 1. \end{aligned} \tag{45}$$

For if  $\sigma = (1/4, 0, 0)$  and  $\kappa = (1/4, 1/4, 0)$ , then  $d_{cr}(\sigma, f(\sigma)) = d_{cr}(\sigma, \kappa) = d_{cr}(\sigma, f^2(\kappa)) = (1/4, 0, 0)$ ;  $d_{cr}(\kappa, f^2(\sigma)) = (1/$

$4, 1/4, 1/4)$ ;  $d_{cr}(f(\sigma), f(\kappa)) = (0, 0, 0)$ ;  $(I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) = 16/27(1/4, 0, 0)$  and

$$\alpha[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma))] = \alpha \left[ \left(\frac{1}{4}, 0, 0\right) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \right] = \alpha \left(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\right). \tag{46}$$

Finally, for  $\sigma = (1/4, 1/4, 0)$  and  $\kappa = (1/4, 1/4, 1/4)$ , we have  $d_{cr}(\sigma, f(\sigma)) = d_{cr}(f(\sigma), f(\kappa)) = d_{cr}(\sigma, f^2(\kappa)) = d_{cr}(\sigma, \kappa) = d_{cr}(\kappa, f^2(\sigma)) = (1/4, 1/4, 1/4)$ ;  $(I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) = 16/27(1/4, 1/4, 1/4)$  and

$$\alpha[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma))] = \alpha \left[ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \right] = \alpha \left(\frac{2}{4}, \frac{2}{4}, \frac{2}{4}\right). \tag{47}$$

Define the function  $\mathcal{L}$  by

$$\begin{aligned} \mathcal{L}(d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \\ \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) &= d_{cr} \\ \cdot (f(\sigma), f(\kappa)) - \alpha[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma))]. \end{aligned} \tag{48}$$

As a consequence, we have  $(I - T)^2(I + T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa)$  implies

$$\begin{aligned} & \mathcal{L}(d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \\ & \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \leq 0. \end{aligned} \quad (49)$$

Thus, this example explains Theorem 9 well. Note that  $(0, 0, 0)$  is a fixed point of  $f$ .

*Example 5.* Let  $\mathcal{E} = C_R^1[1, 2]$  and  $\|g\|_{\mathcal{E}} = \|g\|_{\infty} + \|g'\|_{\infty}$ , where  $\|g\|_{\infty} = \sup_{t \in [1, 2]} |g(t)|$  and  $\mathcal{P} = \{g(t) \in \mathcal{E} : g(t) > 0, t \in [1, 2]\}$ . For each  $K \geq 1$ , take  $g(t) = t$  and  $\kappa(t) = t^{2K}$ . By definition  $\|g\| = 2$  and  $\|\kappa\| = 2K + 1$ . Define the partial order  $\leq$  by

for any  $g, \kappa \in \mathcal{E}$ ;  $g \leq \kappa \Leftrightarrow g(t) \leq \kappa(t) \forall t \in [1, 2]$ .

Then,  $g \leq \kappa$  implies  $K\|g\| \leq \|\kappa\|$ . Hence,  $\mathcal{P}$  is a nonnormal cone. Define  $T : \mathcal{E} \rightarrow \mathcal{E}$  such that

$$(Tg)(t) = \frac{1}{2} \int_1^t g(s) ds. \quad (50)$$

Then,  $T \in B(\mathcal{E}, \mathcal{E})$ . Let  $\mathbb{Y} = \{1, 2, 3\}$  and  $f : \mathbb{Y} \rightarrow \mathbb{Y}$  such that  $f(1) = f(2) = 1$  and  $f(3) = 2$ ,  $f$  is monotone with respect to usual order ( $\leq$ ) and also assumption (1)–(2) in Theorem 13 are verified. Define  $d_{cr}$  by

$$d_{cr}(\sigma, \varsigma)(t) = \begin{cases} 0 & \text{if } \sigma = \varsigma, \\ \frac{e^t}{3} & \text{if } \sigma, \varsigma \in \{1, 2\}, \\ e^t & \text{otherwise.} \end{cases} \quad (51)$$

The vector-valued function  $d_{cr}$  is a cone rectangular metric space but not a cone metric space. We are left to verify the contractive condition only. For this, if  $\sigma = 1$  and  $\varsigma = 2$ , then

$$\begin{aligned} d_{cr}(\sigma, f(\sigma)) &= 0; \\ d_{cr}(f(\sigma), f(\varsigma)) &= 0; \\ d_{cr}(\varsigma, f(\varsigma)) &= \frac{e^t}{3}, \end{aligned} \quad (52)$$

and for  $\sigma = 2, \varsigma = 3$ , we have

$$\begin{aligned} d_{cr}(\sigma, \varsigma) &= e^t; \\ d_{cr}(\sigma, f(\sigma)) &= \frac{e^t}{3}; \\ d_{cr}(f(\sigma), f(\varsigma)) &= \frac{e^t}{3}; \\ d_{cr}(\varsigma, f(\varsigma)) &= e^t. \end{aligned} \quad (53)$$

Thus,

$$\begin{aligned} (I - T)(I - T^2)(d_{cr}(\sigma, f(\sigma))) &= \frac{e^t}{3} \left[ 1 - \frac{t}{2} \left( 1 - \frac{t^2}{4} \right) \right], \\ (Td_{cr}(\sigma, \varsigma))(t) &= \frac{te^t}{2}, \quad (Td_{cr}(\varsigma, f(\varsigma)))(t) = \frac{te^t}{2}. \end{aligned} \quad (54)$$

As  $t \in [1, 2]$  so  $t^2 \leq 4 \Rightarrow t^2/4 \leq 1$  and  $1 - t^2/4 \geq 0$ . Hence,

$$(I - T)(I - T^2)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \varsigma) \text{ implies } d_{cr} \cdot (f(\sigma), f(\varsigma)) \leq T(d_{cr}(\sigma, \varsigma)). \quad (55)$$

Note that 1 is a fixed point of  $f$ .

## 5. A Homotopy Result

In what follows, we derive a homotopy result by applying Theorem 13.

**Theorem 14.** Let  $(\mathcal{E}, \|\cdot\|)$  be a real Banach space and  $\mathcal{P} \subset \mathcal{E}$  be a cone. Let  $(\mathbb{Y}, d_{cr})$  be a rectangular cone metric space and  $U \subset \mathbb{Y}$  is open. Assume that there exists  $T \in B(\mathcal{E}, \mathcal{E})$  and  $T(\mathcal{P}) \subset \mathcal{P}$ . Let the operator  $h : \bar{U} \times [0, 1] \rightarrow \mathbb{Y}$  satisfy ((39)) and the condition (1) of Theorem 13 in the first variable and

- (1)  $\sigma \neq h(\sigma, \theta)$  for every  $\sigma \in \partial U$  ( $\partial U$  denotes the boundary of  $U$  in  $\mathbb{Y}$ )
- (2) there exists  $M \geq 0$  such that

$$\|d_{cr}(h(\sigma, \theta), h(\sigma, \mu))\| \leq M|\theta - \mu| \quad (56)$$

for every  $\sigma \in \bar{U}$  and  $\mu, \theta \in [0, 1]$

- (3) For some  $\sigma \in U$ , if there exists  $\kappa$  with  $\|d_{cr}(\sigma, \kappa)\| \leq r$ , then  $\sigma \mathcal{R} \kappa$ , where  $r$  is radius of open ball in  $U$

If  $h(\cdot, 0)$  has a fixed point in the open set  $U$ , then  $h(\cdot, 1)$  also has a fixed point in the open set  $U$ .

*Proof.* Let

$$B = \{\theta \in [0, 1] \mid \sigma = h(\sigma, \theta) \text{ for some } \sigma \in U\}. \quad (57)$$

Define the relation  $\leq$  in  $\mathcal{E}$  by  $u \leq v$  if and only if  $\|u\| \leq \|v\|$  for all  $u, v \in \mathcal{E}$ . Next,  $0 \in B$ , since  $h(\cdot, 0)$  has a fixed point in the open set  $U$ . So  $B$  is nonempty. Since  $d_{cr}(\sigma, h(\sigma, \theta)) = d_{cr}(\sigma, \kappa)$ ,  $(I - T)^2(I + T)(d_{cr}(\sigma, h(\sigma, \theta))) \leq d_{cr}(\sigma, \kappa)$  for all  $\sigma \mathcal{R} \kappa$ , then by Theorem 13, we have

$$d_{cr}(h(\sigma, \theta), h(\kappa, \theta)) \leq S(d_{cr}(\sigma, \kappa)). \quad (58)$$

Firstly, we show that  $B$  is closed in  $[0, 1]$ . For this, let  $\{\theta_n\}_{n=1}^{\infty} \subseteq B$  with  $\theta_n \rightarrow \theta \in [0, 1]$  as  $n \rightarrow \infty$ . It is necessary to prove that  $\theta \in B$ . Since  $\theta_n \in B$  for  $n \in \mathbb{N}$ , there exists  $\sigma_n \in U$  with  $\sigma_n = h(\sigma_n, \theta_n)$ . Since  $h(\sigma, \cdot)$  is monotone, so, for  $n, m$



$\in \mathbb{N}$ , we have  $\sigma_m \mathcal{R} \sigma_n$ . Since

$$(I - T)^2(I + T)(d_{cr}(\sigma_n, h(\sigma_m, \theta_m))) = (I - T)^2 \cdot (I + T)(d_{cr}(\sigma_n, \sigma_m)) \leq d_{cr}(\sigma_n, \sigma_m), \quad (59)$$

we have

$$\begin{aligned} d_{cr}(h(\sigma_n, \theta_m), h(\sigma_m, \theta_m)) &\leq S(d_{cr}(\sigma_n, \sigma_m)), d_{cr} \\ &\cdot (\sigma_n, \sigma_{m+1}) = d_{cr}(h(\sigma_n, \theta_n), h(\sigma_{m+1}, \theta_{m+1})) \leq d_{cr} \\ &\cdot (h(\sigma_n, \theta_n), h(\sigma_n, \theta_m)) + d_{cr}(h(\sigma_n, \theta_m), h \\ &\cdot (\sigma_n, \theta_{m+1})) + d_{cr}(h(\sigma_n, \theta_{m+1}), h(\sigma_{m+1}, \theta_{m+1})), \|d_{cr} \\ &\cdot (\sigma_n, \sigma_m)\| \leq M|\theta_n - \theta_m| + M|\theta_m - \theta_{m+1}| + \|S \\ &\cdot (d_{cr}(\sigma_n, \sigma_m))\|, \|d_{cr}(\sigma_n, \sigma_m)\| \leq \frac{M}{1 - \|S\|} \\ &\cdot [|\theta_n - \theta_m| + |\theta_m - \theta_{m+1}|]. \end{aligned} \quad (60)$$

Since  $\{\theta_n\}_{n=1}^\infty$  is a RMS Cauchy sequence in  $[0, 1]$ , we have

$$\lim_{n, m \rightarrow \infty} d_{cr}(\sigma_n, \sigma_m) = 0, \quad (61)$$

that is,  $d_{cr}(\sigma_n, \sigma_m) \ll c$ , whenever  $n, m \rightarrow \infty$ . Hence,  $\{\sigma_n\}$  is a Cauchy sequence in  $\mathbb{Y}$ . Since  $\mathbb{Y}$  is a complete cone rectangular metric space, there exists  $\sigma \in \bar{U}$  with  $\lim_{n \rightarrow \infty} d_{cr}(\sigma_n, \sigma) \ll c$ . Hence,  $\sigma_n \mathcal{R} \sigma$  for all  $n \in \mathbb{N}$ . Now consider

$$\begin{aligned} d_{cr}(\sigma_n, h(\sigma, \theta)) &= d_{cr}(h(\sigma_n, \theta_n), h(\sigma, \theta)) \leq d_{cr} \\ &\cdot (h(\sigma_n, \theta_n), h(\sigma_n, \theta)) + d_{cr}(h(\sigma_n, \theta), h \\ &\cdot (\sigma_{n+1}, \theta)) + d_{cr}(h(\sigma_{n+1}, \theta), h(\sigma, \theta)), \|d_{cr} \\ &\cdot (\sigma_n, h(\sigma, \theta))\| \leq M|\theta_n - \theta| + M|\theta_{n+1} - \theta| + \|S \\ &\cdot (d_{cr}(\sigma_{n+1}, \sigma))\|. \end{aligned} \quad (62)$$

So we have

$$\lim_{n \rightarrow \infty} d_{cr}(\sigma_n, h(\sigma, \theta)) = 0. \quad (63)$$

Thus,  $d_{cr}(\sigma, h(\sigma, \theta)) = 0$ . Hence,  $\theta \in B$  and so  $B$  is RMS closed in  $[0, 1]$ .

Next, we show that  $B$  is open in  $[0, 1]$ . For this, let  $\theta_2 \in B$ . Then, we have the existence of  $\sigma_2 \in U$  with  $h(\sigma_2, \theta_2) = \sigma_2$ . Since  $U$  is open, there exists  $r > 0$  such that  $B(\sigma_2, r) \subseteq U$ . Now, assume

$$l = d_{cr}(\sigma_2, \partial U) = \inf \{d_{cr}(\sigma_2, \xi) : \xi \in \partial U\}. \quad (64)$$

Then,  $r = l > 0$ . Fix  $\varepsilon > 0$  with  $\varepsilon < ((1 - \|T\|)l)/2M$ . Let  $\theta \in (\theta_1 - \varepsilon, \theta_1 + \varepsilon)$  and  $\theta_1 \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$ . Then,

$$\sigma \in B(\sigma_2, r) = \{\sigma \in X : \|d_{cr}(\sigma, \sigma_2)\| \leq r\}, \text{ as } \sigma \mathcal{R} \sigma_2. \quad (65)$$

Consider

$$\begin{aligned} d_{cr}(h(\sigma, \theta), \sigma_2) &= d_{cr}(h(\sigma, \theta), h(\sigma_2, \theta_2)) \\ &\leq d_{cr}(h(\sigma, \theta), h(\sigma, \theta_1)) + d_{cr} \\ &\cdot (h(\sigma, \theta_1), h(\sigma, \theta_2)) + d_{cr} \\ &\cdot (h(\sigma, \theta_2), h(\sigma_2, \theta_2)), \|d_{cr} \\ &\cdot (h(\sigma, \theta), \sigma_2)\| \leq M|\theta_1 - \theta| + M|\theta_2 - \theta_1| + \|S \\ &\cdot (d_{cr}(\sigma_2, \sigma))\| \leq M\varepsilon + M\varepsilon + \|S\|l = 2M\varepsilon + \|S\|l < l. \end{aligned} \quad (66)$$

Thus, for every fixed  $\theta \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$ ,  $h(\cdot, \theta) : B(\bar{\sigma}, r) \rightarrow B(\bar{\sigma}, r)$  has a fixed point in  $\bar{U}$  and can be deduced by applying Theorem 13. But this fixed point should be in  $U$  as in the previous case. Hence,  $\theta \in B$  for any  $\theta \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$  and so  $B$  is open in  $[0, 1]$ . Thus, we showed that  $B$  is RMS open as well as RMS closed in  $[0, 1]$  and by connectedness,  $B = [0, 1]$ . Hence,  $h(\cdot, 1)$  has a fixed point in  $U$ .

### Data Availability

No data were used to support this study.

### Conflicts of Interest

All authors declare that they have no competing interests.

### Authors' Contributions

All authors contributed equally to this work.

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