Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8459405, 6 pages
https://doi.org/10.1155/2020/8459405

Research Article

On Fekete–Szegö Problems for Certain Subclasses of Analytic Functions Defined by Differential Operator Involving q-Ruscheweyh Operator

Abdullah Alsoboh and Maslina Darus

Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Malaysia

Correspondence should be addressed to Maslina Darus; maslina@ukm.edu.my

Received 5 July 2019; Accepted 28 August 2019; Published 8 January 2020

Academic Editor: Mitsuru Sugimoto

In this paper, we define a new derivative operator involving q-Ruscheweyh differential operator using convolution. Using this new operator, we introduce two new classes of analytic functions and obtain the Fekete–Szegö inequalities.

1. Introduction

The applications of q-calculus are important and pivotal as they contributed worthy of noticed expansion in geometric function theory. In 1908, Jackson, was the first mathematician to develop the application of q-calculus in a systematic way [1, 2]. Then, Aral and Gupta [3] proposed q-analogue of Baskakov and Durrmeyer operator depending on q-calculus; Mohammed and Darus [4] defined a new operator involving the q-hypergeometric function. Some other applications of q-calculus are studied by the authors [5, 6] and Elhaddad et al. [7]. Recently, many mathematicians have worked intensively in this field (see [8–12]) and obtained various results.

Let $\mathscr{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i, \quad (z \in \mathbb{E}), \tag{1}$$

which are analytic in $\mathbb{E} = \{z : z \in \mathbb{C}, |z| < 1\}$ and we denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ that are consisted of one-to-one (univalent) functions in $\mathbb{E}$.

The convolution of functions f as in (1.1) and $\Gamma(z) = z + \sum_{i=2}^{\infty} \gamma_i z^i$ will be:

$$(f * \Gamma)(z) = (\Gamma * f)(z) = z + \sum_{i=2}^{\infty} a_i \gamma_i z^i. \tag{2}$$

Let $f(z)$ and $\tau(z)$ are analytic functions in $\mathbb{E}$, then we say that f is subordinate to τ denoted by $f(z) \prec \tau(z)$ in $\mathbb{E}$, if there exists a Schwarz function $\xi(z)$ which is analytic in $\mathbb{E}$ with $\xi(0) = 0$ and $|\xi(z)| < 1$ $(z \in \mathbb{E})$ such that $f(z) = \tau(\xi(z)) \forall z \in \mathbb{E}$.

Now, we give some notations and definitions of the principal terms of q-calculus by assuming $0 < q < 1$, as follows:

(1) The q-number $[i]_q$ is defined by:

$$[i]_q = \begin{cases} \frac{1-q^i}{1-q} & (i \in \mathbb{C}\backslash\mathbb{N}), \\ 1 + q + q^2 + \ldots + q^{i-1} & (i \in \mathbb{N}). \end{cases} \tag{3}$$

(2) The q-factorial $[i]!$ is defined by:

$$[i]_q! = \begin{cases} [i]_q [i-1]_q \ldots [2]_q [1]_q, & i = 2, 3, 4, \ldots \\ 1, & i = 1. \end{cases} \tag{4}$$

(3) The q-derivative of a function f is defined by:

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z - qz}, & z \in \mathbb{C}^*, \\ f'(0), & z = 0, \end{cases} \tag{5}$$

where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ will be used throughout the article.

As f given by (1), then

$$\partial_q f(z) = 1 + \sum_{i=2}^{\infty} [i]_q a_i z^{i-1}. \tag{6}$$

The authors in [6] introduced a q-differential operator $\mathbb{D}^m_{q,\mu,\delta,\kappa,\lambda} f(z)$ by:

$$\mathbb{D}^m_{q,\mu,\delta,\kappa,\lambda} f(z) = z + \sum_{i=2}^{\infty} \left((k-\lambda)(\delta-\mu)\left([i]_q - 1\right) + 1\right)^m a_i z^i, \tag{7}$$

where $(\delta, \kappa, \lambda, \mu \geq 0)$, $\kappa > \lambda, \delta > \mu, m \in \mathbb{N}_0$.

In 2014, Aldweby and Darus in [9] introduced q-Ruscheweyh operator $\mathscr{R}^{\vartheta}_q f(z)$ by:

$$\mathscr{R}^{\vartheta}_q f(z) = z + \sum_{i=2}^{\infty} \frac{[\vartheta - 1 + i]_q!}{[\vartheta]_q![i-1]_q!} a_i z^i, \quad (\vartheta \geq 0). \tag{8}$$

Using convolution on $f \in \mathscr{A}$, we define new differential operator $\mathscr{D}^{m,\vartheta}_{q,\mu,\delta,\kappa,\lambda} : \mathscr{A} \to \mathscr{A}$ by:

$$\begin{aligned} \mathcal{D}^{m,\vartheta}_{q,\mu,\delta,\kappa,\lambda} f(z) &= \mathbb{D}^m_{q,\mu,\delta,\kappa,\lambda} f(z) * \mathcal{R}^{\vartheta}_q f(z) \\ &= z + \sum_{i=2}^{\infty} \Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([i]_q\right) a_i z^i, \end{aligned} \tag{9}$$

where

$$\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([i]_q\right) = \left((k-\lambda)(\delta-\mu)\left([i]_q - 1\right) + 1\right)^m \frac{[\vartheta - 1 + i]_q!}{[\vartheta]_q![i-1]q!}. \tag{10}$$

Remarks.

(i) When $\vartheta = 0$, $\kappa = 1$, $\lambda = 0$, $\delta = 1$, and $\mu = 0$, we get *q-Sălăgean* differential operator introduced in [13].

(ii) When $\vartheta = 0$, $\kappa = 1$, $\lambda = 0$, $\delta = 1$, $\mu = 0$, and $q \to 1^-$, we get *Sălăgean* differential operator introduced in [14].

(iii) When $\vartheta = 0$, $\beta = 1$, $\kappa = 0$, $\delta = 1$, $\mu = 0$, and $q \to 1^-$, then we get Al-Oboudi differential operator introduced in [15].

(iv) When $\vartheta = 0$ and $q \to 1^-$, we get Ramadan and Darus operator introduced in [16].

(v) When $\vartheta = 0$, we get Alsoboh and Darus operator introduced in [6].

(vi) When $m = 0$, then we get q-Ruscheweyh operator introduced in [9].

(vii) When $m = 0$ and $q \to 1^-$, then we have Ruscheweyh operator introduced in [17].

(viii) When $m = 0, \delta = 1, \mu = 0, \lambda = 0$, and $q \to 1^-$, we get $D^n_\lambda f(z)$ [18].

Many sublasses of analytic functions have been introduced by many different authors, for example, Ma and Minda [19], Ravichandran et al. [20], Seoudy and Aouf [12] and others. These works have inspired our introduction of the new subclasses $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi)$ and $\mathscr{C}^{m,\varrho}_{q,\varrho}(\Phi)$ of $\mathscr{A}$, involving the differential operator $\mathscr{D}^{m,\vartheta}_{q,\mu,\delta,\kappa,\lambda} f(z)$ and the principle of subordination as:

Definition 1. Let $\mathscr{P}$ be the subclass of functions Φ which are analytic and univalent in $\mathbb{E}$ and for which $\Phi(\mathbb{E})$ is convex with $\Phi(0) = 1$ and $\Re(\Phi(z)) < 0$ for $z \in \mathbb{E}$. A function $f \in \mathscr{A}$ is said to be in the class $f \in \mathcal{S}^{m,\vartheta}_{q,\varrho}(\Phi)$ if it satisfies the subordination condition:

$$1 + \frac{1}{\varrho}\left(\frac{z\partial_q \mathcal{D}^{m,\vartheta}_q f(z)}{\mathcal{D}^{m,\vartheta}_q f(z)} - 1\right) \prec \Phi(z), \quad (\varrho \in \mathbb{C}^* \Phi \in \mathscr{P}). \tag{11}$$

Definition 2. A function $f \in \mathscr{A}$ is said to be in the class $f \in \mathscr{C}^{m,\vartheta}_{q,\varrho}(\Phi)$ if it satisfies the subordination condition:

$$1 + \frac{1}{\varrho}\left(\frac{\partial_q\left(z\partial_q \mathscr{D}^{m,\vartheta}_q f(z)\right)}{\partial_q \mathscr{D}^{m,\vartheta}_q f(z)} - 1\right) \prec \Phi(z), \quad (\varrho \in \mathbb{C}^*; \Phi \in \mathscr{P}). \tag{12}$$

Note that:

(i) When $m = 0$ and $\vartheta = 0$, then $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi) =: \ell_{q,\varrho}(\phi)$ and $\mathscr{C}^{m,\vartheta}_{q,\varrho}(\Phi) =: \mathscr{C}_{q,\varrho}(\phi)$ [12].

(ii) When $m = 0$, $\vartheta = 0$, and $q \to 1^-$, then $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi) =: \ell_\varrho(\phi)$ and $\mathscr{C}^{m,\vartheta}_{q,\varrho}(\Phi) =: \mathscr{C}_\varrho(\phi)$ [20].

(iii) When $m = 0$, $\vartheta = 0$, and $q \to 1^-$, then $\mathscr{S}^{m,\vartheta}_{q,1}(\Phi) =: \mathscr{S}^*(\phi)$, and $\mathscr{C}^{m,\vartheta}_{q,1}(\Phi) =: \mathscr{C}(\phi)$ [19].

2. Main Results

In this section, we obtain the Fekete–Szegö inequalities for the subclasses $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi)$ and $\mathscr{C}^{m,\vartheta}_{q,\varrho}(\Phi)$, by assuming $(\delta, \kappa, \lambda, \mu \geq 0)$, $\kappa > \lambda, \delta > \mu, m \in \mathbb{N}_0, \varrho \in \mathbb{C}^*$ and $\Phi \in \mathscr{P}$. In order to prove our results, we use the following lemmas of Ma and Minda [19].

Lemma 3. *If $\mathscr{P}(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is a function with positive real part in the open unit disk $\mathbb{E}$ and $\sigma \in \mathbb{C}$, then*

$$\left|c_2 - \sigma c_1^2\right| \leq 2 \max\{1; |2\sigma - 1|\}. \tag{13}$$

Lemma 4. *If $\mathscr{P}(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is a function with positive real part in $\mathbb{E}$ and $\sigma \in \mathbb{C}$, then*

$$\left|c_2 - \sigma c_1^2\right| \leq \begin{cases} -4\sigma + 2, & \sigma \leq 0, \\ 2, & 0 \leq \sigma \leq 1, \\ 4\sigma - 2, & \sigma \geq 1. \end{cases} \tag{14}$$

To get our results, we use the similar methods studied by Alsoboh and Darus [5], Elhaddad and Darus [11], and Seoudy and Aouf [12].

Theorem 5. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $(B_1 \neq 0)$, and f given by (1) belongs to $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi)$, then*

$$\begin{aligned} \left|a_3 - \sigma a_2^2\right| &\leq \frac{\varrho B_1}{2\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)} \\ &\max\left\{1; \left|\frac{B_2}{B_1} + \frac{\varrho B_1}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{\left([2]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)^2}\sigma\right)\right|\right\}. \end{aligned} \tag{15}$$

Proof. Let $f \in \mathscr{S}^*_{q,n}(\Phi)$, then there is a Schwarz function ξ which is analytic in $\mathbb{E}$ with $\xi(0) = 0$, and $|\xi(z)| < 1$ in $\mathbb{E}$, such that

$$\frac{z\partial_q D_q^{m,\vartheta} f(z)}{D_q^{m,\vartheta} f(z)} = \Phi(\xi(z)). \tag{16}$$

Now, we define the function $\mathscr{P}(z)$ with $Re(P(z)) > 0$ and $P(0) = 1$ by:

$$\mathscr{P}(z) = \frac{1+\xi(z)}{1-\xi(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots. \tag{17}$$

Since $\xi(z)$ is a Schwarz function, therefore,

$$\begin{aligned}\Phi(\xi(z)) = \Phi\left(\frac{\mathscr{P}(z)-1}{\mathscr{P}(z)+1}\right) &= 1 + \frac{1}{2}B_1 p_1 z \\ &+ \frac{1}{2}\left(B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{2}B_2 p_1^2\right)z^2 + \dots.\end{aligned} \tag{18}$$

Now, substitute (18) in (16), we get

$$\begin{aligned}1+\frac{1}{\varrho}\left(\frac{z\partial_q \mathscr{D}_q^{m,\vartheta} f(z)}{\mathscr{D}_q^{m,\vartheta} f(z)} - 1\right) &= 1 + \frac{1}{2}B_1 p_1 z \\ &+ \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \dots.\end{aligned} \tag{19}$$

From Equation (19), we get

$$a_2 = \frac{B_1 p_1 \varrho}{2\left([2]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)}, \tag{20}$$

and

$$a_3 = \frac{\varrho B_1}{2\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\left\{p_2 - \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1\varrho}{[2]_q - 1}\right]p_1^2\right\}. \tag{21}$$

Therefore,

$$a_3 - \sigma a_2^2 = \frac{\varrho B_1}{2\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\left\{p_2 - \nu p_1^2\right\}, \tag{22}$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1\varrho}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{\left([2]_q - 1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)^2\right)}\sigma\right)\right], \tag{23}$$

by applying Lemma 3, we get our result. This completes the proof. □

Next, we prove the following theorem for the subclass $C^{m,\vartheta}_{q,\varrho}(\Phi)$.

Theorem 6. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $(B_1 \neq 0)$, *and f given by (1) belongs to* $C^{m,\vartheta}_{q,\varrho}(\Phi)$, *then*

$$\begin{aligned}\left|a_3 - \sigma a_2^2\right| &\leq \frac{\varrho B_1}{2[3]_q\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}.\max \\ &\left\{1; \left|\frac{B_2}{B_1} + \frac{\varrho B_1}{[2]_q - 1}\left(1 - \frac{[3]_q\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{[2]_q^2\left([2]_q - 1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}\sigma\right)\right|\right\}.\end{aligned} \tag{24}$$

Proof. Let $f \in \mathscr{C}^*_{q,n}(\Phi)$, then there is a Schwarz function $\xi(z)$, analytic in $\mathbb{E}$ with $\xi(0) = 0$, and $|\xi(z)| < 1$ in $\mathbb{E}$ such that

$$\frac{\partial_q\left(z\partial_q \mathscr{D}_q^{m,\vartheta} f(z)\right)}{\partial_q \mathscr{D}_q^{m,\vartheta} f(z)} = \Phi(\xi(z)). \tag{25}$$

From the equations (17), (18), and last equality, we have

$$\begin{aligned}1+\frac{1}{\varrho}\left(\frac{\partial_q\left(z\partial_q \mathscr{D}_q^{m,\vartheta} f(z)\right)}{\partial_q \mathscr{D}_q^{m,\vartheta} f(z)} - 1\right) &= 1 + \frac{1}{2}B_1 p_1 z \\ &+ \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \dots.\end{aligned} \tag{26}$$

From Equation (26), we can find

$$a_2 = \frac{B_1 p_1 \varrho}{2[2]_q\left([2]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)} \tag{27}$$

and

$$a_3 = \frac{\varrho B_1}{2[3]_q\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}[3]_q}\left\{p_2 - \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_{1\rho}}{[2]_q - 1}\right]p_1^2\right\}. \tag{28}$$

Therefore,

$$a_3 - \sigma a_2^2 = \frac{\varrho B_1}{2[3]_q\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\left\{p_2 - \nu_1 p_1^2\right\}, \tag{29}$$

where

$$\begin{aligned}\nu_1 = \frac{1}{2}\Bigg[1 - \frac{B_2}{B_1} - \frac{B_1\varrho}{[2]_q - 1} \\ \left(1 - \frac{[3]_q\left([3]_q - 1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{\left([2]_q\right)^2\left([2]_q - 1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}\sigma\right)\Bigg],\end{aligned} \tag{30}$$

by applying Lemma 3, we get our result and this completes the proof. □

When $m = 0$ and $\vartheta = 0$ in previous two theorems, we obtain the result for functions in the classes $\ell_{q,\varrho}(\phi)$ and $\mathscr{C}_{q,\varrho}(\phi)$, respectively, which improved by Seoudy and Aouf [12]:

Corollary 7. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $(B_1 \neq 0)$, *and f given by (1) belong to* $\ell_{q,\varrho}(\phi)$, *then*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\left|B_1\varrho\right|}{[3]_q - 1}\max\left\{1; \left|\frac{B_2}{B_1} + \frac{\varrho B_1}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\sigma\right)\right|\right\}. \tag{31}$$

The result is sharp.

Corollary 8. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $(B_1 \neq 0)$, *and* f *given by (1) belong to* $\ell_{q,\varrho}(\phi)$, *then*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\left|B_1 \varrho\right|}{[3]_q\,[3]_q - 1} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\varrho B_1}{[2]_q - 1}\left(1 - \frac{[3]_q\left([3]_q - 1\right)}{\left([2]_q\right)^2\left([2]_q - 1\right)}\sigma\right)\right|\right\}. \tag{32}$$

The result is sharp.

For $m = 0, \vartheta = 0$ and taking $q \to 1^-$ in Theorem 5, we obtain the following corollary.

Corollary 9 (see [20]). *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $(B_1 \neq 0)$, *and* f *given by (1) belong to* $\ell_{\varrho}(\phi)$, *then*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\left|B_1 \varrho\right|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + \varrho B_1 (1 - 2\sigma)\right|\right\}. \tag{33}$$

The result is sharp.

Next, by using Lemma 4, we obtain the following theorems:

Theorem 10. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $B_1 > 0$, $B_2 \geq 0$ *and* f *given by (1) belong to* $S_{q,\varrho}^{m,\vartheta}(\Phi)$, *then*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\varrho B_2}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} + \frac{\varrho^2 B_1^2}{[2]_q - 1}\left(\frac{1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} - \frac{\sigma}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}\right), & \sigma \leq \chi_1, \\ \frac{\varrho B_1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}, & \chi_1 \leq \sigma \leq \chi_2, \\ \frac{\varrho^2 B_1^2}{[2]_q - 1}\left(\frac{\sigma}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2} - \frac{1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}\right) - \frac{\varrho B_2}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}, & \sigma \leq \chi_2, \end{cases} \tag{34}$$

where

$$\chi_1 = \frac{\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}{\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)\right)}\left(\frac{\left([2]_q - 1\right)^2\left(B_1 + B_2\right) + \left([2]_q - 1\right)\varrho B_1^2}{\left([3]_q - 1\right)\varrho B_1^2}\right), \tag{35}$$

$$\chi_2 = \frac{\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}{\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)\right)}\left(\frac{\left(B_2 - B_1\right)\left([2]_q - 1\right)^2 + \varrho B_1^2\left([2]_q - 1\right)}{\varrho B_1^2\left([3]_q - 1\right)}\right). \tag{36}$$

Proof. Applying Lemma 4 to the Equations (22) and (23), we have three cases:

Case (1): If $\sigma \leq \chi_1$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\varrho B_1}{2\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}\{2 - 4\nu\} \leq \frac{\varrho B_1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} \left\{\frac{B_2}{B_1} + \frac{B_1 \varrho}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}\sigma\right)\right\}. \tag{37}$$

Therefore,

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\varrho B_2}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} + \frac{\varrho^2 B_1^2}{[2]_q - 1}\left(\frac{1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} - \frac{\sigma}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}\right). \tag{38}$$

Case (2): If $\chi_1 \leq \sigma \leq \chi_2$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\varrho B_1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}. \tag{39}$$

Case (3): If $\sigma \geq \chi_2$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\varrho B_1}{2\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}\{4\nu - 2\} \leq -\frac{\varrho B_1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} \left\{\frac{B_2}{B_1} + \frac{B_1 \varrho}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}\sigma\right)\right\}, \tag{40}$$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{\varrho^2 B_1^2}{[2]_q - 1}\left(\frac{\sigma}{\left([2]_q - 1\right)\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2} - \frac{1}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}\right) - \frac{\varrho B_2}{\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}. \tag{41}$$

This completes our proof. □

Theorem 11. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ *with* $B_1 > 0$, $B_2 \geq 0$ *and* f *given by (1) belong to* $\mathscr{C}_{q,\varrho}^{m,\vartheta}(\Phi)$, *then*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\varrho B_2}{[3]_q\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} + \frac{\varrho^2 B_1^2}{\left([2]_q - 1\right)^2}\left(\frac{[2]_q - 1}{[3]_q\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)} - \frac{\sigma}{\left([2]_q\right)^2\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2}\right), & \sigma \leq \psi_1, \\ \frac{\varrho B_1}{[3]_q\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}, & \psi_1 \leq \sigma \leq \psi_2, \\ \frac{\varrho^2 B_1^2}{\left([2]_q - 1\right)^2\left([3]_q - 1\right)}\left(\frac{\sigma\left([3]_q - 1\right)}{\left([2]_q\right)^2\left(\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([2]_q\right)\right)^2} - \frac{[2]_q - 1}{[3]_q\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}\right) - \frac{\varrho B_2}{[3]_q\left([3]_q - 1\right)\Omega_{\kappa,\lambda,\delta,\mu}^{m,\vartheta}\left([3]_q\right)}, & \sigma \geq \psi_2, \end{cases} \tag{42}$$

where

$$\psi_1 = \frac{\left([2]_q\right)^2\left([2]_q-1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2\left(\varrho B_1^2+\left(B_2-B_1\right)\left([2]_q-1\right)\right)}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)\varrho B_1^2}, \tag{43}$$

$$\psi_2 = \frac{\left([2]_q\right)^2\left([2]_q-1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2\left(\varrho B_1^2+\left(B_2+B_1\right)\left([2]_q-1\right)\right)}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)\varrho B_1^2}. \tag{44}$$

Proof. Applying Lemma 4 to the Equations (29) and (30), then we have

Case (1): If $\sigma \le \psi_1$

$$\begin{aligned}
\left|a_3-\sigma a_2^2\right| &\le \frac{\varrho B_1}{2[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\{2-4\nu\},\\
&\le \frac{\varrho B_1}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\\
&\quad\left\{\frac{B_2}{B_1}+\frac{B_1\varrho}{[2]_q-1}\left(1-\frac{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{\left([2]_q\right)^2\left([2]_q-1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}\sigma\right)\right\},\\
&\le \frac{\varrho B_2}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}+\frac{\varrho^2B_1^2}{\left([2]_q-1\right)^2}\\
&\quad\left(\frac{[2]_q-1}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}-\frac{\sigma}{\left([2]_q\right)^2\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}\right).
\end{aligned} \tag{45}$$

Case (2): If $\psi_1 \le \sigma \le \psi_2$

$$\left|a_3-\sigma a_2^2\right| \le \frac{\varrho B_1}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}. \tag{46}$$

Case (3): If $\sigma \ge \psi_2$

$$\begin{aligned}
\left|a_3-\sigma a_2^2\right| &\le \frac{\varrho B_1}{2[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\{4\nu-2\},\\
\left|a_3-\sigma a_2^2\right| &\le -\frac{\varrho B_1}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\\
&\quad\left\{\frac{B_2}{B_1}+\frac{B_1\varrho}{[2]_q-1}\left(1-\frac{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}{\left([2]_q\right)^2\left([2]_q-1\right)\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}\sigma\right)\right\},\\
&\le \frac{\varrho^2B_1^2}{\left([2]_q-1\right)^2\left([3]_q-1\right)}\\
&\quad\left(\frac{\sigma\left([3]_q-1\right)}{\left([2]_q\right)^2\left(\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([2]_q\right)\right)^2}-\frac{[2]_q-1}{[3]_q\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}\right)\\
&\quad-\frac{\varrho B_2}{[3]_q\left([3]_q-1\right)\Omega^{m,\vartheta}_{\kappa,\lambda,\delta,\mu}\left([3]_q\right)}.
\end{aligned} \tag{47}$$

This completes the proof. □

When $m = 0, \vartheta = 0$ in Theorems 5 and 6, we obtain the Fekete–Szegö inequalities for the subclasses $\ell_{q,\varrho}(\phi)$ and $\mathscr{C}_{q,\varrho}(\phi)$, respectively [12].

Corollary 12. *Let* $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots$ *with* $B_1 > 0$, $B_2 \ge 0$ *and* f *given by (1) belong to* $\ell_{q,\varrho}(\phi)$, *with* $\varrho > 0$ *then*

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases}
\frac{\varrho B_2}{\left([3]_q-1\right)}+\frac{\varrho^2B_1^2}{[2]_q-1}\left(\frac{1}{\left([3]_q-1\right)}-\frac{\sigma}{\left([2]_q-1\right)}\right) & \text{if } \sigma \le \chi_1\\
\frac{\varrho B_1}{\left([3]_q-1\right)} & \text{if } \chi_1 \le \sigma \le \chi_2\\
\frac{\varrho^2B_1^2}{[2]_q-1}\left(\frac{\sigma}{\left([2]_q-1\right)}-\frac{1}{\left([3]_q-1\right)}\right)-\frac{\varrho B_2}{\left([3]_q-1\right)} & \text{if } \sigma \ge \chi_2
\end{cases} \tag{48}$$

where

$$\chi_1 = \frac{\left([2]_q-1\right)^2\left(B_1+B_2\right)+\left([2]_q-1\right)\varrho B_1^2}{\left([3]_q-1\right)\varrho B_1^2}, \tag{49}$$

$$\chi_2 = \frac{\left(B_2-B_1\right)\left([2]_q-1\right)^2+\varrho B_1^2\left([2]_q-1\right)}{\varrho B_1^2\left([3]_q-1\right)}. \tag{50}$$

The result is sharp.

Corollary 13. *Let* $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots$ *with* $B_1 > 0$, $B_2 \ge 0$ *and* f *given by (1) belong to* $\mathscr{C}_{q,\varrho}(\phi)$, *with* $\varrho > 0$ *then*

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases}
\frac{\varrho B_2}{[3]_q\left([3]_q-1\right)}+\frac{\varrho^2B_1^2}{[3]_q\left([2]_q-1\right)\left([3]_q-1\right)}\\
-\frac{\varrho^2B_1^2\sigma}{\left([2]_q\right)^2\left([2]_q-1\right)^2} & \textit{if } \sigma \le \psi_1\\
\frac{\varrho B_1}{[3]_q\left([3]_q-1\right)} & \textit{if } \psi_1 \le \sigma \le \psi_2\\
\frac{\varrho^2B_1^2\sigma}{\left([2]_q\right)^2\left([2]_q-1\right)^2}-\frac{\varrho B_2}{[3]_q\left([3]_q-1\right)}\\
-\frac{\varrho^2B_1^2}{[3]_q\left([2]_q-1\right)\left([3]_q-1\right)} & \textit{if } \sigma \ge \psi_2
\end{cases} \tag{51}$$

where

$$\psi_1 = \frac{\left([2]_q\right)^2\left([2]_q-1\right)\left(\varrho B_1^2+\left(B_2-B_1\right)\left([2]_q-1\right)\right)}{[3]_q\left([3]_q-1\right)\varrho B_1^2}, \tag{52}$$

$$\psi_2 = \frac{\left([2]_q\right)^2\left([2]_q-1\right)\left(\varrho B_1^2+\left(B_2+B_1\right)\left([2]_q-1\right)\right)}{[3]_q\left([3]_q-1\right)\varrho B_1^2}. \tag{53}$$

The result is sharp.

3. Conclusion

Fekete–Szegö problems have always been the main interests of many researchers in geometric function theory. Many studies related to Fekete-Szego revolved around classes of analytic normalised univalent functions. In this particular work, the Fekete–Szegö inequality is obtained for functions in more general classes denoted by $\mathscr{S}^{m,\vartheta}_{q,\varrho}(\Phi)$ and $\mathscr{C}^{m,\vartheta}_{q,\varrho}(\Phi)$, respectively, using a new differential operator associated with q-calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Funding

The work here is supported by UKM grant: GUP-2019-032.

Acknowledgments

We would like to thank the referee for the suggestions given to improve the content of the article.

References

[1] F. H. Jackson, "On q-functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[2] F. H. Jackson, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[3] A. Aral and V. Gupta, "Generalized q-baskakov operators," *Mathematica Slovaca*, vol. 61, no. 4, pp. 619–634, 2011.

[4] A. Mohammed and M. Darus, "A generalized operator involving the q-hypergeometric function," *Journal of Mathematics*, vol. 65, no. 4, pp. 454–465, 2013.

[5] A. Alsoboh and M. Darus, "On Fekete-Szego problem associated with q-derivative operator," *Journal of Physics: Conference Series*, vol. 1212, p. 012003, 2019.

[6] A. Alsoboh and M. Darus, "New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points," *International Journal of Mathematics and Computer Science*, vol. 14, no. 4, pp. 761–773, 2019.

[7] S. Elhaddad, H. Aldweby, and M. Darus, "Some properties on a class of harmonic univalent functions defined by q-analogue of Ruscheweyh operator," *Journal of Mathematical Analysis*, vol. 9, no. 4, pp. 28–35, 2018.

[8] I. Aldawish and M. Darus, "Starlikeness of q-differential operator involving quantum calculus," *Korean Journal of Mathematics*, vol. 22, no. 4, pp. 699–709, 2014.

[9] H. Aldweby and M. Darus, "Some subordination results on q-analogue of Ruscheweyh differential operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 958563, 6 pages, 2014.

[10] H. Aldweby and M. Darus, "Coefficient estimates of classes of q-starlike and q-convex functions," *Advanced Studies in Contemporary Mathematics*, vol. 26, no. 1, pp. 21–26, 2016.

[11] S. Elhaddad and M. Darus, "On Fekete–Szegö problems for a certain subclass defined by q-analogue of Ruscheweyh operator," *Journal of Physics: Conference Series*, vol. 1212, p. 012002, 2019.

[12] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q-starlike and q-convex functions of complex order," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 135–145, 2016.

[13] M. Govindaraj and S. Sivasubramanian, "On a class of analytic functions related to conic domains involving q-calculus," *Analysis Mathematica*, vol. 43, no. 3, pp. 475–487, 2017.

[14] G. S. Salagean, "Subclasses of univalent functions," in *Complex Analysis—Fifth Romanian–Finnish Seminar. Lecture Notes in Mathematics*, vol. 1013, pp. 362–372, Springer, Berlin, Heidelberg, 1983.

[15] F. M. Al-Oboudi, "On univalent functions defined by a generalized Sălăgean operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, Article ID 172525, 8 pages, 2004.

[16] S. F. Ramadan and M. Darus, "Univalence of an integral operator defined by generalized operators," *International Journal of Computational and Mathematical Sciences*, vol. 4, no. 8, pp. 1117–1119, 2010.

[17] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.

[18] K. Al-Shaqsi and M. Darus, "On univalent functions with respect to k-symmetric points given by a generalised Ruscheweyh derivatives operator," *Journal of Analysis and Applications*, vol. 7, no. 1, pp. 53–61, 2009.

[19] W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., pp. 157–169, Int. Press, 1992.

[20] V. Ravichandran, Y. Polatoglu, M. Bolcal, and A. Sen, "Certain subclasses of starlike and convex functions of complex order," *Hacettepe Journal of Mathematics and Statistics*, vol. 34, no. 2005, pp. 9–15, 2005.