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Research Article

Completeness Theorem for Eigenparameter Dependent Dissipative Dirac Operator with General Transfer Conditions

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This paper deals with a singular (Weyl's limit circle case) non-self-adjoint (dissipative) Dirac operator with eigenparameter dependent boundary condition and finite general transfer conditions. Using the equivalence between Lax-Phillips scattering matrix and Sz.-Nagy-Foiaş characteristic function, the completeness of the eigenfunctions and associated functions of this dissipative operator is discussed.

1. Introduction

Spectral analysis and expansion of eigenfunctions in the fields of differential operators are important parts in the theory of ordinary differential equation boundary value problems. Generally speaking, the spectral parameter appears only in the equation. However, lots of problems in the mathematical physics require that eigenparameter appears not only in the equation but also in the boundary and transfer conditions. As is well known, many complicated physical phenomena with discontinuities can be transferred to operator problems with transfer conditions (also called point interactions, transmission conditions, interface conditions). Various physical applications of such problems arise in the theory of mechanics, heat, and mass transfer problems, etc. (see, for example, [1–23]).

Dirac system plays an important role in the theory of relativistic. Meanwhile, Dirac system with spectral parameter in the boundary conditions describes the behavior of a relativistic particle in an electromagnetic field. When an atomic system is subjected to an external electromagnetic field or a mechanical system to an external force, these would result in the discontinuity of origin system. Such as in geophysical problems, the reflection of transverse waves at the bottom of the earth's crust jumps phenomena due to high-speed ions colliding with atomic systems. These reasons may cause the eigenfunctions in the equations describing the system to have discontinuities, that is to say, operators with transfer conditions [9, 24, 25].

Dissipative operator is an important class of non-self-adjoint operators, which can be traced back to the study of hyperbolic partial differential equations. For example, the telegraph operator equation can be converted into the study of dissipative differential operator [26]. The dissipation of differential operators can be induced by many factors, such as the boundary conditions, coefficients in the equation, etc. In 1970s, Pavlov [27] proposed a new method for spectral analysis of dissipative operators. This method is based on building their self-adjoint dilation and the corresponding functional model of Sz.-Nagy-Foias type. Based on this, one could study spectral properties of the operator using the equivalence between the Lax-Phillips scattering matrix and the characteristic function. This equivalence has been used by many authors [2, 14, 15, 28–31]. Moreover, Behrndt et al. [32] investigated an alternative approach to the construction of the self-adjoint dilation of an m-dissipative operator as well as a connection between the characteristic function and scattering matrix. In this paper, we investigate a class of dissipative discontinuous Dirac operators with two singular endpoints (in Weyl's limit circle case), and one of boundary conditions is linearly dependent on the eigen parameter, and general transfer conditions are imposed on the discontinuous points. Using the equivalence between the Lax-Phillips scattering matrix and the characteristic function, we investigate the discreteness of the spectrum and the completeness of the system consisting of eigenfunctions and associated functions. What calls for special attention is that the transfer conditions in this paper are in the sense of coupled, which are different from the usual, namely, the values of the solutions and their derivatives at the interior discontinuous points are not independent of each other. The transfer conditions in the sense of separated for dissipative Dirac operators have been investigated by Uğurlu [14, 15]. For self-adjoint Sturm–Liouville operators, this kind of problems have been investigated by Mukhtarov [16, 17, 19]. It should be noted that completeness properties and the Riesz basis property for strongly regular boundary value problems for Dirac operators on a finite interval have been established in [33–35].

The arrangement of this paper is as follows: in Section 2, we transfer the considered problem to a maximal dissipative operator $\mathcal{A}_h$. In Section 3, the self-adjoint dilation, incoming and outgoing spectral representations, functional model, and characteristic function of this dissipative operator are derived. Our main results on the completeness theorem are investigated in Section 4.

2. Dissipative Operator

We study the following Dirac system consisting of the equation

$$l(y) := By'(x) + Q(x)y(x) = \lambda W(x)y(x), \quad x \in I, \tag{1}$$

where $\lambda \in \mathbb{C}$, and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{2}$$

$$W(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}, \; Q(x) = \begin{pmatrix} q(x) & q_0(x) \\ q_0(x) & p(x) \end{pmatrix}. \tag{3}$$

We have the following basic hypotheses on the coefficients of Equation (1) and the interval I :

(a_1) $I = \cup_{k=1}^{n+1} I_k, I_k = (\xi_{k-1}, \xi_k), -\infty \le a = \xi_0 < \xi_1 < \cdots < \xi_{n+1} = b \le +\infty$,

(a_2) $W(x) > 0$ holds almost everywhere on I; $Q(x)$ and $W(x)$ are real vector-valued locally integrable and Lebesgue measurable functions on I.

Let $L^2_W(I;\mathcal{C}), \mathcal{C} = \mathbb{C}^2$ be a Hilbert space which consists of all functions $y(x) \in \mathbb{C}^2$ satisfying $\int_I (W(x)y(x), y(x))_{\mathcal{C}} dx < \infty$ and equipped with inner product $(y, z) := \int_I (W(x) y(x), z(x))_{\mathcal{C}} dx$.

Let $\mathcal{A}(y) := W^{-1}(x)l(y)$. Setting $\mathcal{D}_{\max}$ be a set such that for any $y(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in L^2_W(I;\mathcal{C})$, y_1, y_2 are locally absolutely continuous functions on I and $\mathcal{A}(y) \in L^2_W(I;\mathcal{C})$.

For arbitrary vectors $\varphi, \chi \in \mathcal{D}_{\max}$ we have

$$(\mathcal{D}_{\max}\varphi, \chi) - (\varphi, \mathcal{D}_{\max}\chi) = \sum_{j=1}^{n+1} [\varphi, \chi]_{\xi_{j-1}+}^{\xi_j -}, \tag{4}$$

where $[\varphi, \chi]_x := \mathcal{W}[\varphi, \overline{\chi}]_x := \varphi_1(x)\overline{\chi}_2(x) - \varphi_2(x)\overline{\chi}_1(x)$, $x \in I$. Hence, $[\varphi, \chi]_{\xi_0} := \lim_{x\to\xi_0+}[\varphi, \chi]_x$, $[\varphi, \chi]_{\xi_{n+1}} := \lim_{x\to\xi_{n+1}-}[\varphi, \chi]_x$ exist and are finite by Equation (4).

In the sequel, we will always assume that Weyl's limit-circle case holds for the Dirac system (1) at endpoints a and b (see [11, 36]).

Consider the boundary value transfer problem (BVTP)

$$\mathcal{A}(y) = \lambda y, \quad y \in D_{\max}, \; x \in I, \tag{5}$$

$$\delta_1[y,u]_a - \delta_2[y,v]_a = \lambda(\delta_1'[y,u]_a - \delta_2'[y,v]_a), \tag{6}$$

$$L_1 y := [y,u]_b - h[y,v]_b = 0, \tag{7}$$

$$L_{2k} y := y_1(\xi_k+) - \delta_{1k}y_1(\xi_k-) - \delta_{2k}y_2(\xi_k-) = 0, \tag{8}$$

$$L_{3k} y := y_2(\xi_k+) - \delta_{3k}y_1(\xi_k-) - \delta_{4k}y_2(\xi_k-) = 0, \tag{9}$$

where $k = \overline{1,n}$, h is a complex number with $\mathscr{I}h > 0$, λ is a complex parameter, $\delta_1, \delta_2, \delta_1', \delta_2', \delta_{1k}, \delta_{2k}, \delta_{3k}, \delta_{4k}$ are given real numbers, and $\beta = \delta_1'\delta_2 - \delta_1\delta_2' > 0$, $\rho_k = \delta_{1k}\delta_{4k} - \delta_{2k}\delta_{3k} > 0$,

$$\begin{aligned} u(x) &= \{u_1(x), u_2(x), \ldots, u_{n+1}(x)\}, \\ v(x) &= \{v_1(x), v_2(x), \ldots, v_{n+1}(x)\}, \end{aligned} \tag{10}$$

are the solutions of the system

$$\mathcal{A}(y) = 0, \quad x \in I, \tag{11}$$

and satisfy the conditions

$$u_{11}(a) = 1, \; u_{12}(a) = 0, \; v_{11}(a) = 0, \; v_{12}(a) = 1, \tag{12}$$

and transfer conditions (8) and (9), where

$$u_k(x) = \begin{pmatrix} u_{k1}(x) \\ u_{k2}(x) \end{pmatrix}, \; v_k(x) = \begin{pmatrix} v_{k1}(x) \\ v_{k2}(x) \end{pmatrix}, \tag{13}$$

$u_k(x), v_k(x), k = \overline{1,n}$ are parts of $u(x), v(x)$ defined on the interval I_k, respectively. By the property of the Wronskian and (12) we get that

$$\mathcal{W}[u_k, v_k](\xi_k+) = \rho_k \mathcal{W}[u_k, v_k](\xi_k-), \quad k = \overline{1,n}. \tag{14}$$

Hence, any solutions of (11) can be represented as linear combination of $v(x)$ and $u(x)$. Since Weyl's limit circle cases hold for the Dirac system, $u, v \in L^2_W(I;E)$, moreover, $u, v \in \mathcal{D}_{\max}$.

Remark 1. In this article, we assume that $\mathscr{I}h > 0$, with this hypothesis, the problem (5)–(9) can be transferred to the study of dissipative operators. Particularly, when $\mathscr{I}h = 0$, the considered problem can be transferred to the self-adjoint case which is well known. Our interests focus on the non-self-adjoint case, hence, here we assume that $\mathscr{I}h > 0$.

For convenience, the following notations will be used:

$$\begin{aligned} \mathcal{M}_1(y) &:= \delta_1[y,u]_a - \delta_2[y,v]_a, \; \mathcal{M}_2(y) := \delta_1'[y,u]_a - \delta_2'[y,v]_a, \\ \mathcal{N}_1(y) &:= [y,u]_b, \; \mathcal{N}_2(y) := [y,v]_b. \end{aligned} \tag{15}$$

Then, for any $y, z \in \mathcal{D}_{\max}$,

$$[y,z]_a = \frac{1}{\beta}\left[\mathcal{M}_1(y)\overline{\mathcal{M}_2(z)} - \mathcal{M}_2(y)\overline{\mathcal{M}_1(z)}\right], \tag{16}$$

$$[y,z]_x = [y,u]_x[\overline{z},v]_x - [y,v]_x[\overline{z},u]_x, \quad x \in I_1, \tag{17}$$

$$[y,z]_x = \rho_1^{-1}\left([y,u]_x[\bar{z},v]_x - [y,v]_x[\bar{z},u]_x\right), \quad x \in I_2,$$
$$\vdots$$
$$[y,z]_x = (\rho_1\rho_2\cdots\rho_n)^{-1}\left([y,u]_x[\bar{z},v]_x - [y,v]_x[\bar{z},u]_x\right), \quad x \in I_{n+1}. \tag{18}$$

Let $\varphi(x,\lambda) = \{\varphi_1(x,\lambda), \varphi_2(x,\lambda), \ldots, \varphi_{n+1}(x,\lambda)\}$ and $\psi(x,\lambda) = \{\psi_1(x,\lambda), \psi_2(x,\lambda), \ldots, \psi_{n+1}(x,\lambda)\}$ be the solutions of (5) satisfying

$$[\varphi_1,u]_a = \delta_2 - \delta_2'\lambda,\ [\varphi_1,v]_a = \delta_1 - \delta_1'\lambda,\ [\psi_{n+1},u]_b = h,\ [\psi_{n+1},v]_b = 1, \tag{19}$$

and transfer conditions (8) and (9), respectively. Let $\Delta_k(\lambda) = \mathcal{W}[\varphi,\psi]_x$, $x \in I_k$, then simple calculation gives

$$\Delta(\lambda) := \Delta_{n+1}(\lambda) = \rho_n\Delta_n(\lambda) = \cdots = \rho_1\rho_2\cdots\rho_n\Delta_1(\lambda), \tag{20}$$

From (20), we have $\Delta(\lambda)$ is an entire function and the spectrum of BVTP (5)–(9) coincide with the zeros of $\Delta(\lambda)$.

In the following, in order to transfer BVTP (5)–(9) to operator form, a special inner product is introduced in the Hilbert space $H = L_W^2(I;\mathcal{C}) \oplus \mathbb{C}$. To this end, for any $Y, Z \in H$, denote the inner product as

$$\langle Y,Z\rangle_H = \sum_{k=0}^{n}\prod_{i=0}^{k}\rho_i^{-1}\int_{\xi_k+}^{\xi_{k+1}-}\left(W(x)y(x),z(x)\right)_{\mathcal{C}}dx + \frac{1}{\beta}\tilde{y}\bar{\tilde{z}}, \tag{21}$$

where

$$\rho_0 = 1,\ Y(x) = \begin{pmatrix} y(x) \\ \tilde{y} \end{pmatrix},\ Z(x) = \begin{pmatrix} z(x) \\ \tilde{z} \end{pmatrix}. \tag{22}$$

Consider the operator $\mathcal{A}_h$ with domain

$$\mathcal{D}(\mathcal{A}_h) = \left\{Y(x) = \begin{pmatrix} y(x) \\ \tilde{y} \end{pmatrix} \in H \middle| L_1(y) = 0,\ L_{2k}(y) = 0,\ L_{3k}(y) = 0,\ k = \overline{1,n}, \tilde{y} = \mathcal{M}_2(y),\ y \in \mathcal{D}_{\max}\right\}, \tag{23}$$

and acts as

$$\mathcal{A}_hY = \begin{pmatrix} \mathcal{A}(y) \\ \mathcal{M}_1(y) \end{pmatrix}. \tag{24}$$

Therefore, the problem (5)–(9) is transferred into operator form

$$\mathcal{A}_hY = \lambda Y. \tag{25}$$

Theorem 1. *$\mathcal{A}_h$ is maximal dissipative in H.*

Proof. Let $G \in \mathcal{D}(\mathcal{A}_h)$. Then

$$\begin{aligned}\langle \mathcal{A}_hG,G\rangle_H - \langle G,\mathcal{A}_hG\rangle_H &= (\rho_1\rho_2\cdots\rho_n)^{-1}[g,g]_{\xi_n}^{b} \\ &\quad + \sum_{k=2}^{n}\prod_{i=0}^{k-1}\rho_i^{-1}[g,g]_{\xi_{k-1}}^{\xi_k} + [g,g]_a^{\xi_1} \\ &\quad + \frac{1}{\beta}\left[\mathcal{M}_1(g)\overline{\mathcal{M}_2(g)} - \mathcal{M}_2(g)\overline{\mathcal{M}_1(g)}\right].\end{aligned} \tag{26}$$

By (16), we have

$$\left[\mathcal{M}_1(g)\overline{\mathcal{M}_2(g)} - \mathcal{M}_2(g)\overline{\mathcal{M}_1(g)}\right] = \beta[g,g]_a. \tag{27}$$

Using transfer conditions (8) and (9), we have

$$\rho_1[g,g]_{\xi_1-} = [g,g]_{\xi_1+}, \ldots, \rho_n[g,g]_{\xi_n-} = [g,g]_{\xi_n+}. \tag{28}$$

By (7) and (17), one gets that

$$\begin{aligned}[g,g]_b &= (\rho_1\rho_2\cdots\rho_n)^{-1}\left[h\mathcal{N}_2(g)\mathcal{N}_2(\bar{g}) - \bar{h}\mathcal{N}_2(g)\mathcal{N}_2(\bar{g})\right] \\ &= (\rho_1\rho_2\cdots\rho_n)^{-1}\left(h - \bar{h}\right)\left|\mathcal{N}_2(g)\right|^2.\end{aligned} \tag{29}$$

Substituting (27)–(29) into (26) yields that

$$\mathcal{I}\langle \mathcal{A}_hG,G\rangle_H = (\rho_1\rho_2\cdots\rho_n)^{-1}\,\mathcal{I}h\left|\mathcal{N}_2(g)\right|^2. \tag{30}$$

Since $\mathcal{I}h > 0$ and $\rho_i > 0, i = \overline{1,n}$, $\mathcal{A}_h$ is dissipative in H. Moreover, it can be easily proved that

$$(\mathcal{A}_h - \lambda I)\mathcal{D}(\mathcal{A}_h) = H, \quad \mathcal{I}\lambda < 0. \tag{31}$$

Therefore, the result follows. □

Using the same method in [2], it is easy to check that the following lemma holds.

Lemma 1. *For the same eigenvalue λ_0, each chain of eigenvectors and associated vectors of BVTP (5)–(9) corresponds to the chain of eigenvectors and associated vectors $G_0, G_1, \ldots, G_n$ of $\mathcal{A}_h$. In this case, the equality*

$$G_k = \begin{pmatrix} g_k \\ \mathcal{M}_2(g_k) \end{pmatrix}, \quad k = 0,1,\ldots,n, \tag{32}$$

takes place.

3. Scattering Function

In this section, we derive self-adjoint dilation, to this end, the incoming channel $L^2(\mathbb{R}^-)$ and outgoing channel $L^2(\mathbb{R}^+)$ are added, and orthogonal sum $\mathcal{H} = L^2(\mathbb{R}^-) \oplus H \oplus L^2(\mathbb{R}^+)$ is called main Hilbert space, where $\mathbb{R}^- := (-\infty, 0]$ and $\mathbb{R}^+ := [0, +\infty)$.

We consider the operator $\mathcal{B}_h$ in $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{B}_h)$, which is generated by the expression

$$\mathcal{B}(\nu_-,F,\nu_+) = \left(i\frac{d\nu_-}{ds}, \mathcal{A}(F), i\frac{d\nu_+}{dt}\right), \tag{33}$$

$$\begin{aligned}\mathcal{D}(\mathcal{B}_h) = \Bigg\{ \hat{f} = (\nu_-,F,\nu_+) \in \mathcal{H} \Bigg| &[f,u]_b - h[f,v]_b \\ &= \gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\nu_-(0), [f,u]_b - \bar{h}[f,v]_b \\ &= \gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\nu_+(0),\ L_{2k}(f) = 0, L_{3k}(f) = 0, k = \overline{0,1}, \\ &\tilde{f} = \mathcal{M}_2(f), \nu_\pm \in W_2^1(\mathbb{R}^\pm), f(x) \in \mathcal{D}_{\max}, F \in H\Bigg\},\end{aligned} \tag{34}$$

where $F = \begin{pmatrix} f(x) \\ \tilde{f} \end{pmatrix}$, $W_2^1(\mathbb{R}^\pm)$ are Sobolev spaces, and $\gamma^2 := 2\mathcal{I}h, \gamma > 0$.

Theorem 2. *$\mathcal{B}_h$ is self-adjoint in $\mathcal{H}$.*

Proof. Let $\widehat{y} = (\nu_-, Y, \nu_+), \widehat{z} = (\mu_-, Z, \mu_+) \in \mathcal{D}(\mathcal{B}_h)$. Integration by parts yields

$$\begin{aligned}\langle \mathcal{B}_h\widehat{y}, \widehat{z}\rangle_{\mathcal{H}} - \langle \widehat{y}, \mathcal{B}_h\widehat{z}\rangle_{\mathcal{H}} &= \left(\prod_{k=1}^{n}\rho_k\right)^{-1}[y,z]_{\xi_n}^{b} + \sum_{k=2}^{n}\prod_{i=0}^{k-1}\rho_i^{-1}[y,z]_{\xi_{k-1}}^{\xi_k} + [y,z]_a^{\xi_1} \\ &\quad + \frac{1}{\beta}\left[\mathcal{M}_1(y)\overline{\mathcal{M}_2(z)} - \mathcal{M}_2(y)\overline{\mathcal{M}_1(z)}\right] \\ &\quad + i\mu_-(0)\overline{\mu_-}(0) - i\nu_+(0)\overline{\mu_+}(0) \\ &= \left(\prod_{k=1}^{n}\rho_k\right)^{-1}[y,z]_b + i\nu_-(0)\overline{\mu_-}(0) - i\nu_+(0)\overline{\mu_+}(0) \\ &= \left(\prod_{k=1}^{n}\rho_k\right)^{-1}[y,z]_b - \frac{1}{i(\prod_{k=1}^{n}\rho_k)\gamma^2}\left([y,u]_b - h[y,v]_b\right)\left([\overline{z},u]_b - \overline{h}[\overline{z},v]_b\right) \\ &\quad + \frac{1}{i(\prod_{k=1}^{n}\rho_k)\gamma^2}\left([y,u]_b - \overline{h}[y,v]_b\right)\left([\overline{z},u]_b - h[\overline{z},v]_b\right) = 0,\end{aligned} \tag{35}$$

which implies that $\mathcal{B}_h$ is symmetric in $\mathcal{H}$, and $\mathcal{D}(\mathcal{B}_h) \subseteq \mathcal{D}(\mathcal{B}_h^*)$.

In order to prove $\mathcal{B}_h$ is self-adjoint, we only need to show $\mathcal{B}_h^* \subseteq \mathcal{B}_h$. To do this, let $\widehat{y} = (\nu_-, 0, \nu_+) \in D(\mathcal{B}_h)$, $\nu_\pm \in W_2^1(\mathbb{R}^\pm)$, $\nu_\pm(0) = 0$ and $\widehat{z} = (\mu_-, Z, \mu_+) \in D(\mathcal{B}_h^*)$. Simple calculation gives

$$\begin{aligned}\langle \mathcal{B}_h\widehat{y}, \widehat{z}\rangle_{\mathcal{H}} &= \left\langle \left(i\frac{d\nu_-}{ds}, 0, i\frac{d\nu+}{dt}\right), (\mu_-, Z, \mu_+)\right\rangle_{\mathcal{H}} \\ &= \left\langle (\nu_-, 0, \nu_+), \left(i\frac{d\mu_-}{ds}, Z^*, i\frac{d\mu_+}{dt}\right)\right\rangle_{\mathcal{H}},\end{aligned} \tag{36}$$

where $\mu_\pm \in W_2^1(\mathbb{R}^\pm)$, $Z^* \in \mathcal{D}(\mathcal{A}_h^*)$. Analogously, let $\widehat{y} = (0, Y, 0) \in \mathcal{D}(\mathcal{B}_h)$, then we have

$$\mathcal{B}_h^*(\widehat{z}) = \left(i\frac{d\mu_-}{ds}, \mathcal{A}(Z), i\frac{d\mu_+}{dt}\right), \quad \tilde{z} = \mathcal{M}_2(z), \quad g \in \mathcal{D}_{\max}, \tag{37}$$

through integrating by parts with respect to $\langle \mathcal{B}_h\widehat{y}, \widehat{z}\rangle$. Then the equality $\langle \mathcal{B}_h\widehat{y}, \widehat{z}\rangle_{\mathcal{H}} = \langle \widehat{y}, \mathcal{B}_h\widehat{z}\rangle_{\mathcal{H}}$, $\forall \widehat{y} \in \mathcal{D}(\mathcal{B}_h)$ holds by (37).

Through the definition of $\mathcal{D}(\mathcal{B}_h)$, we have

$$L_{2k}(z) = 0, \ L_{3k}(z) = 0, \quad k = \overline{1,n}, \tag{38}$$

and

$$\begin{aligned}&\nu_-(0)\left[\left(\gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2} + \frac{ih(\prod_{k=1}^{n}\rho_k)^{1/2}}{\gamma}\right)[\overline{z},v]_b - \frac{i(\prod_{k=1}^{n}\rho_k)^{1/2}[\overline{z},u]_b}{\gamma}\right] \\ &\quad - \nu_+(0)\left[\frac{ih(\prod_{k=1}^{n}\rho_k)^{1/2}[\overline{z},v]_b}{\gamma} - \frac{i(\prod_{k=1}^{n}\rho_k)^{1/2}[\overline{z},u]_b}{\gamma}\right] \\ &= \left(\prod_{k=1}^{n}\rho_k\right)[i\nu_+(0)\overline{\mu_+}(0) - i\nu_-(0)\overline{\mu_-}(0)].\end{aligned} \tag{39}$$

Comparing the coefficients of $\nu_-(0)$ and $\nu_+(0)$ yields that

$$[z,u]_b - h[z,v]_b = \gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\mu_-(0), \tag{40}$$

$$[z,u]_b - \overline{h}[z,v]_b = \gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\mu_+(0). \tag{41}$$

Therefore, $\mathcal{D}(\mathcal{B}_h^*) \subseteq \mathcal{D}(\mathcal{B}_h)$ by (40) and (41). Above discussion implies $\mathcal{B}_h^* = \mathcal{B}_h$. ☐

Let $U_t = \exp(i\mathcal{B}_h t)$, $t \in \mathbb{R}$, then it is an unitary group. Define the mappings $\mathcal{P}^-$ and $\mathcal{P}^+$ as follows:

$$\begin{aligned}&\mathcal{P}^- : \mathcal{H} \to H, \ \mathcal{P}^+ : H \to \mathcal{H}, \\ &\mathcal{P}^- : (\psi_-, F, \psi_+) \to F, \ \mathcal{P}^+ : F \to (0, F, 0).\end{aligned} \tag{42}$$

Utilizing $U_t, \mathcal{P}^-, \mathcal{P}^+$, then one can construct $\mathcal{Z}_t = \mathcal{P}^- U_t \mathcal{P}^+ (t \ge 0)$, which is a strongly continuous semigroup of nonunitary contractions on $\mathcal{H}$. $\mathcal{B}_h$ is called the self-adjoint dilation of the generator B_h of $\{\mathcal{Z}_t\}$ and

$$B_h F = \lim_{t\to 0+}\frac{1}{it}(\mathcal{Z}_t F - F). \tag{43}$$

It should be noted that B_h is dissipative in H [27, 37].

Theorem 3. *$\mathcal{B}_h$ is the self-adjoint dilation of $\mathcal{A}_h$.*

Proof. We only need to show $B_h = \mathcal{A}_h$. To this end, we consider the following equality

$$(\mathcal{B}_h - \lambda)^{-1}\mathcal{P}^+ F = \widehat{g} := (\mu_-, G, \mu_+), \tag{44}$$

where $F \in H$, $\widehat{g} \in \mathcal{D}(\mathcal{B}_h)$ and $\mathscr{I}\lambda < 0$. Then we have $\mathcal{A}(G) - \lambda G = F$, $\mu_-(s) = \mu_-(0)\exp(-i\lambda s)$ and $\mu_+(t) = \mu_+(0)\exp(-i\lambda t)$. Since $\widehat{g} \in \mathcal{D}(\mathcal{B}_h)$, then $\mu_- \in L^2(\mathbb{R}^-)$, $\mu_-(0) = 0$. Therefore, G satisfies $[g,u]_b - h[g,v]_b = 0$, and $G \in \mathcal{D}(\mathcal{A}_h)$. Since $\mathcal{A}_h$ is a dissipative operator, we get that if $\mathscr{I}\lambda < 0$, then λ is not the eigenvalue of $\mathcal{A}_h$. Hence,

$$(\mathcal{B}_h-\lambda)^{-1}\mathcal{P}^{+}F=\left(0,(\mathcal{A}_h-\lambda)^{-1}F,\left(\prod_{k=1}^{n}\rho_k\right)^{-1/2}\right.$$
$$\left.\gamma^{-1}\left([g,u]_b-\bar{h}[g,v]_b\right)\exp(-i\lambda t)\right). \quad (45)$$

Through $\mathcal{P}^{-}$ we have

$$\mathcal{P}^{-}(\mathcal{B}_h-\lambda)^{-1}\mathcal{P}^{+}=(\mathcal{A}_h-\lambda)^{-1}. \quad (46)$$

On the other hand,

$$\begin{aligned}\mathcal{P}^{-}(\mathcal{B}_h-\lambda)^{-1}\mathcal{P}^{+}&=-i\mathcal{P}^{-}\int_0^{\infty}U_t\ \exp(-i\lambda t)dt\mathcal{P}^{+}\\&=-i\int_0^{\infty}\mathcal{Z}_t\quad \exp(-i\lambda t)dt=(B_h-\lambda)^{-1}.\end{aligned} \quad (47)$$

Hence $B_h=\mathcal{A}_h$ by (46) and (47), and result follows. □

In what follows, let us consider the subspaces $\mathcal{D}_-=\left(L^2(\mathbb{R}^-),0,0\right)$ and $\mathcal{D}_+=\left(0,0,L^2(\mathbb{R}^+)\right)$.

Lemma 2. *The spaces $\mathcal{D}_-$ and $\mathcal{D}_+$ possess the following properties:*

(i) $U_t\mathcal{D}_-\subset\mathcal{D}_-, t\le 0; U_t\mathcal{D}_+\subset\mathcal{D}_+, t\ge 0$;
(ii) $\cap_{t\le 0}U_t\mathcal{D}_-=\cap_{t\ge 0}U_t\mathcal{D}_+=\{0\}$;
(iii) $\overline{\cup_{t\ge 0}U_t\mathcal{D}_-}=\overline{\cup_{t\le 0}U_t\mathcal{D}_+}=\mathcal{H}$;
(iv) $\mathcal{D}_-\perp\mathcal{D}_+$.

Proof. Let $\hat{f}=(0,0,v_+)\in\mathcal{D}_+$, then for $\mathcal{I}\lambda<0$, we have

$$(\mathcal{B}_h-\lambda)^{-1}\hat{f}=\left(0,0,-i\exp(-i\lambda t)\int_0^t\exp(i\lambda s)v_+(s)ds\right)\in\mathcal{D}_+. \quad (48)$$

Hence, if $\hat{\eta}\perp\mathcal{D}_+$ and $\mathcal{I}\lambda<0$, then

$$0=\left\langle(\mathcal{B}_h-\lambda)^{-1}\hat{f},\hat{\eta}\right\rangle_{\mathcal{H}}=-i\int_0^{\infty}\exp(-i\lambda t)\left\langle U_t\hat{f},\hat{\eta}\right\rangle_{\mathcal{H}}dt. \quad (49)$$

This gives that $\left\langle U_t\hat{f},\hat{\eta}\right\rangle_{\mathcal{H}}=0, t\ge 0.$ Hence, for $t\ge 0$, we have $U_t\mathcal{D}_+\subset\mathcal{D}_+$. The similar discussion can be done for $\mathcal{D}_-$, thus the proof of property (i) is finished.

In the following, consider the semigroup of isometries $U_t^+=\mathcal{P}_1^+U_t\mathcal{P}_1, t\ge 0$, where $\mathcal{P}_1^+$ and $\mathcal{P}_1$ are in the form of

$$\begin{aligned}&\mathcal{P}_1^+:\mathcal{H}\to L^2(\mathbb{R}^+),\ \mathcal{P}_1:L^2(\mathbb{R}^+)\to\mathcal{H},\\&\mathcal{P}_1^+:(\mu_-,Y,\mu_+)\to\mu_+,\ \mathcal{P}_1:\mu_+\to(0,0,\mu_+).\end{aligned} \quad (50)$$

The generator M_+ of U_t^+ is

$$M_+\psi=\mathcal{P}_1^+\mathcal{B}_h\mathcal{P}_1v=\mathcal{P}_1^+\mathcal{B}_h(0,0,v_+)=\mathcal{P}_1^+\left(0,0,i\frac{dv}{dt}\right)=i\frac{dv}{dt}, \quad (51)$$

where $v\in W_2^1(\mathbb{R}^+)$ and $v(0)=0$. It is known that the generator of the one-sided shift $\widetilde{U}_t^+$ in $L^2(\mathbb{R}_+)$ is differential operator $i(d/ds)$ satisfying $v(0)=0$. Due to a semigroup is uniquely determined by its the generator, we have $U_t^+=\widetilde{U}_t^+$. Therefore,

$$\bigcap_{t\ge 0}U_t^+\mathcal{D}_+=\left(0,0,\bigcap_{t\ge 0}\tilde{U}_t^+L^2(\mathbb{R}^+)\right)=0. \quad (52)$$

The same proof can be done with respect to $\mathcal{D}_-$. The proof of property (ii) is done.

Let

$$\mathcal{H}^-=\overline{\bigcup_{t\ge 0}U_t\mathcal{D}_-},\ \mathcal{H}^+=\overline{\bigcup_{t\le 0}U_t\mathcal{D}_+}. \quad (53)$$

If $\widetilde{\mathcal{A}_h}$ the restriction of $\mathcal{A}_h$ on a subspace $\widetilde{H}$ is the self-adjoint part, then for $F\in D(\widetilde{\mathcal{A}_h})\cap\widetilde{H}$ we have

$$0=\langle\mathcal{A}_hF,F\rangle_H-\langle F,\mathcal{A}_hF\rangle_H=i\gamma^2\left(\prod_{k=1}^{n}\rho_k\right)^{-1}\left([f,v]_b\right)^2. \quad (54)$$

This gives $[f,v]_b=0$ and $[f,u]_b=0$ by the boundary condition (7). It follows from $L_{2k}(f)=0$ and $L_{3k}(f)=0$ that $F(x)=0$. Utilizing the expansion theorem on in eigenfunctions of the self-adjoint operator $\widetilde{\mathcal{A}_h}$, we have $\widetilde{H}=\{0\}$. Thus $\mathcal{A}_h$ is completely non-self-adjoint in H which results in

$$\mathcal{H}^-+\mathcal{H}^+=\mathcal{H}. \quad (55)$$

Otherwise, there exists a nontrivial subspace $\widehat{H}=\mathcal{H}\ominus(\mathcal{H}^-+\mathcal{H}^+)$ which would invariant with respect to group $\{U_t\}$ and the restriction of $\{U_t\}$ to $\widehat{H}$ were unitary. Thus the restriction of $\mathcal{A}_h$ on $\widehat{H}$ is a self-adjoint operator.

Let $\eta(x,\lambda)=\{\eta_1(x,\lambda),\eta_2(x,\lambda),\dots,\eta_{n+1}(x,\lambda)\}$and $\chi(x,\lambda)=\{\chi_1(x,\lambda),\chi_2(x,\lambda),\cdots,\chi_{n+1}(x,\lambda)\}$be the solutions of Equation (5) satisfying

$$[\eta,v]_a=\frac{\delta_1'}{\delta},\ [\eta,u]_a=\frac{\delta_2'}{\delta}, \quad (56)$$

$$[\chi,v]_a=\delta_1-\delta_1'\lambda,\ [\chi,u]_a=\delta_2-\delta_2'\lambda, \quad (57)$$

and

$$\begin{aligned}L_{2k}\eta(x,\lambda)=0,\\L_{3k}\eta(x,\lambda)=0,\end{aligned}\ \begin{aligned}L_{2k}\chi(x,\lambda)=0,\\L_{3k}\chi(x,\lambda)=0.\end{aligned}\quad \overline{k=1,n}. \quad (58)$$

Setting the vectors

$$W^-(x,s,t,\lambda)=\left(\exp(-i\lambda s),\gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\right.$$
$$\left.\cdot\frac{\beta(\lambda)}{(\zeta(\lambda)+h)[\eta,v]_b}\Upsilon(x,\lambda),\overline{\Lambda}_h(\lambda)\exp(-i\lambda t)\right) \quad (59)$$

and

$$W^+(x,s,t,\lambda)=\left(\Lambda_h(\lambda)\exp(-i\lambda s),\gamma\left(\prod_{k=1}^{n}\rho_k\right)^{1/2}\right.$$
$$\left.\cdot\frac{\beta(\lambda)}{\left(\zeta(\lambda)+\bar{h}\right)[\eta,v]_b}\Upsilon(x,\lambda),\exp(-i\lambda t)\right), \quad (60)$$

where

$$\beta(\lambda):=-\frac{[\eta,v]_b}{[\chi,v]_b},\ \zeta(\lambda):=-\frac{[\eta,u]_b}{[\eta,v]_b},\ \Upsilon(x,\lambda)=\begin{pmatrix}\eta(x,\lambda)\\\beta\end{pmatrix},$$
$$\Lambda_h(\lambda):=\frac{\zeta(\lambda)+h}{\zeta(\lambda)+\bar{h}}. \quad (61)$$

When λ is real, the vectors W^- and W^+ do not belong to the space $\mathcal{H}$. Simple calculation gives that W^- and W^+ satisfy $\mathcal{B}W^{\pm} = \lambda W^{\pm}$ and the boundary-transfer conditions of $\mathcal{B}_h$. For $\hat{g} = (\nu_-, g, \nu_+)$, define the Fourier transformations

$$\mathcal{G}^- : \hat{g} \to \hat{g}_-(\lambda) = \frac{1}{\sqrt{2\pi}} \langle \hat{g}, W^- \rangle_{\mathcal{H}}, \tag{62}$$

$$\mathcal{G}^+ : \hat{g} \to \hat{g}_+(\lambda) = \frac{1}{\sqrt{2\pi}} \langle \hat{g}, W^+ \rangle_{\mathcal{H}}, \tag{63}$$

where ν_-, g, ν_+ are smooth, compactly supported functions. Let $\hat{g} = (\nu_-, 0, 0) \in \mathcal{D}_-$. Then the equality

$$\hat{g}_-(\lambda) = \frac{1}{\sqrt{2\pi}} \langle \hat{g}, W^- \rangle_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \nu_-(s) \exp(i\lambda s) ds \in \mathcal{H}_-^2, \tag{64}$$

holds. Here $\mathcal{H}_{\pm}^2$ are the Hardy classes in $L^2(\mathbb{R})$. Now consider the dense set $\overset{\vee}{H}_-$ in $\mathcal{H}^-$ consisting of all vectors $\hat{y}$ such that $\hat{y}$ is compactly supported in $\mathcal{D}_-$ and $\hat{y} \in \overset{\vee}{H}_-$ if $\hat{y} = U_N \hat{y}_0, \hat{y}_0 = (\psi_-, 0, 0), \psi_- \in C_0^{\infty}(\mathbb{R}^-)$, where $N = N_{\hat{y}}$ is nonnegative number. Then if $\hat{y}, \hat{z} \in \mathcal{H}^-$ we obtain for $N > N_{\hat{y}}$ and $N > N_{\hat{z}}$ that $U_{-N}\hat{y}, U_{-N}\hat{z} \in \mathcal{D}_-$ and their first components belong to $C_0^{\infty}(\mathbb{R}^-)$. Therefore

$$\begin{aligned} \langle \hat{y}, \hat{z} \rangle_{\mathcal{H}} &= \langle U_{-N}\hat{y}, U_{-N}\hat{z} \rangle_{\mathcal{H}} = \langle \mathcal{G}^- U_{-N}\hat{y}, \mathcal{G}^- U_{-N}\hat{z} \rangle_{L^2} \\ &= \langle \exp(-i\lambda N)\mathcal{G}^-\hat{y}, \exp(-i\lambda N)\mathcal{G}^-\hat{z} \rangle_{L^2} = \langle \hat{y}_-, \hat{z}_- \rangle_{L^2}. \end{aligned} \tag{65}$$

Taking closure in (65), then the Parseval equality holds for the entire space $\mathcal{H}^-$. Furthermore, the inversion formula

$$\hat{y} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}_-(\lambda) W^- d\lambda, \tag{66}$$

results from the Parseval equality if all integrals are taken as limits in the mean of the intervals. Consequently,

$$\mathcal{G}^- \mathcal{H}^- = \overline{\bigcup_{t \geq 0} \mathcal{G}^- U_t \mathcal{D}_-} = \overline{\bigcup_{t \geq 0} \exp(-i\lambda t) \mathcal{H}_-^2} = L^2(\mathbb{R}). \tag{67}$$

The analogous argument can be used for $\mathcal{H}^+$. Hence, $\mathcal{H}^+$ and $\mathcal{H}^-$ are isometrically identical with $L^2(\mathbb{R})$, which imply that $\mathcal{H}^- = \mathcal{H}^+ = \mathcal{H}$. The proof of property (iii) follows.

Utilizing the inner product in $\mathcal{H}$ the property (iv) holds. □

Evidently, $\zeta(\lambda)$ is a meromorphic function in $\mathbb{C}$, and has a countable number of poles on $\mathbb{R}$ by the definition of $\zeta(\lambda)$. Moreover, for all $\mathcal{I}\lambda \neq 0, \mathcal{I}\lambda\mathcal{I}\tau(\lambda) < 0$ and for all $\lambda \in \mathbb{C}$, $\overline{\zeta(\lambda)} = \zeta(\overline{\lambda})$ except the real poles of $\tau(\lambda)$.

In the proof of Lemma 2, we have obtained that $\mathcal{G}^-$ ($\mathcal{G}^+$) is the incoming (outgoing) spectral representation for the group $\{U_t\}$, respectively. U_t is transformed into $\exp(i\lambda t)$.

For $\lambda \in \mathbb{R}$ we have $|\Lambda_h(\lambda)| = 1$. Therefore, by utilizing (59)–(61) we have

$$W^- = \overline{S}_h(\lambda) W^+, \quad \lambda \in \mathbb{R}. \tag{68}$$

Thus from (68) we have

$$\hat{g}_-(\lambda) = \Lambda_h(\lambda) \hat{g}_+(\lambda). \tag{69}$$

According to the Lax-Phillips scattering theory [38], the following theorem holds.

Theorem 4. *$\Lambda_h(\lambda)$ is the scattering function of $\{U_t\}$ (also of $\mathcal{B}_h$).*

Using $\mathcal{G}^-$ one get that

$$\mathcal{H} = \mathcal{D}_- \oplus H \oplus \mathcal{D}_+ \to \mathcal{H}_-^2 \oplus H \oplus \Lambda(\lambda)\mathcal{H}_+^2. \tag{70}$$

Therefore,

$$H = \mathcal{H}_+^2 \ominus \Lambda(\lambda)\mathcal{H}_+^2. \tag{71}$$

Because $U_t\kappa$ is unitary equivalent under $\mathcal{G}^-$ to $\exp(i\lambda s)\kappa(\lambda), \kappa = \kappa(\lambda)$. Let $\mathcal{P}$ be the orthogonal projection from $\mathcal{H}_+^2$ onto H, then we have $\widetilde{\mathcal{V}}_t\kappa = \mathcal{P}[\exp(i\lambda s)\kappa(\lambda)](s \geq 0)$ is a semigroup of operators. Hence the generator of $\widetilde{\mathcal{V}}_t$

$$\mathcal{F}\kappa = \lim_{t \to +0} (it)^{-1} (\widetilde{\mathcal{V}}_t\kappa - \kappa), \tag{72}$$

is a maximal dissipative operator on H. The operator $\mathcal{F}$ is called a model dissipative operator (see [2, 27, 37, 38]). Therefore, $\Lambda(\lambda)$ is the characteristic function of the operator N. Since the characteristic function of unitary equivalent dissipative operators coincides (see [38], Chapter VI), therefore, the following result is proved.

Theorem 5. *$\Lambda_h(\lambda)$ is the characteristic function of the operator $\mathcal{A}_h$.*

4. Completeness Theorem

Theorem 6. *The characteristic function $\Lambda_h(\lambda)$ of $\mathcal{A}_h$ is a Blaschke product except for a single value in the upper half-plane.*

Proof. It can be verified that $\Lambda_h(\lambda)$ is an inner function in the upper half-plane. Therefore, $S_h(\lambda)$ can be written as

$$\Lambda_h(\lambda) = \exp(i\lambda d) D(\lambda), \quad d \geq 0, \tag{73}$$

where $D(\lambda)$ is a Blaschke product. By (73), we have

$$|\Lambda_h(\lambda)| \leq \exp(-d\mathcal{J}\lambda), \quad \mathcal{J}\lambda \geq 0. \tag{74}$$

From (61) one gets that

$$\zeta(\lambda) = \frac{h - \overline{h}\Lambda_h(\lambda)}{\Lambda_h(\lambda) - 1}. \tag{75}$$

Utilizing (74), we have

$$\lim_{t \to +\infty} \zeta(it) = h_0. \tag{76}$$

Thus d is zero except for a single point h_0 and this completes the proof. □

Using the results in Sections 2–4, our main results can be stated as follow.

Theorem 7. *If Weyl's limit-circle case holds for the Dirac system (1) at the endpoints a and b. Then the BVTP (5)–(9) has purely discrete spectrum in the open upper half-plane and the possible limit points can only occur at infinity. For all h with* $\mathcal{I}h > 0$*, all eigenvectors and associated vectors of BVTP (5)–(9) are complete in the space H except possibly for a single value* $h = h_0$.

5. Conclusion

Boundary value problem with eigenparameter dependent boundary conditions and with discontinuities inside an interval has been extensively studied since its wide applications in engineering and mathematical physics. In this paper, we investigate a class of dissipative Dirac operators with discontinuities and eigenparameter dependent boundary conditions. For such a problem, we obtain the completeness theorem of this dissipative operator.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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