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Research Article

On Mixed Quermassintegral for Log-Concave Functions

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In this paper, the functional Quermassintegral of log-concave functions in $\mathbb{R}^n$ is discussed. We obtain the integral expression of the ith functional mixed Quermassintegral, which is similar to the integral expression of the ith mixed Quermassintegral of convex bodies.

1. Introduction

Let $\mathscr{K}^n$ be the set of convex bodies (compact convex subsets with nonempty interiors) in $\mathbb{R}^n$, the fundamental Brunn-Minkowski inequality for convex bodies states that for $K, L \in \mathscr{K}^n$, the volume of the bodies and of their Minkowski sum $K + L = \{x + y : x \in K, y \in L\}$ is given by

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1}$$

with equality if and only if K and L are homothetic; namely, they agree up to a translation and a dilation. Another geometric quantity related to the convex bodies K and L is the mixed volume. The most important result concerning the mixed volume is Minkwoski's first inequality:

$$V_1(K, L) := \frac{1}{n} \lim_{t \to 0^+} \frac{V(K+tL) - V(K)}{t} \geq V(K)^{(n-1)/n} V(L)^{1/n}, \tag{2}$$

for $K, L \in \mathscr{K}^n$. In particular, when choosing L to be a unit ball, up to a factor, $V_1(K, L)$ is exactly the perimeter of K, and inequality (2) turns out to be the isoperimetric inequality in the class of convex bodies. The mixed volume $V_1(K, L)$ admits a simple integral representation (see [1, 2]):

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L \, dS_K, \tag{3}$$

where h_L is the support function of L and S_K is the area measure of K.

The Quermassintegrals $W_i(K) (i = 0, 1, \cdots, n)$ of K, which are defined by letting $W_0(K) = V_n(K)$, the volume of K; $W_n(K) = \omega_n$, the volume of the unit ball B_2^n in $\mathbb{R}^n$ and for general $i = 1, 2, \cdots, n-1$,

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{\mathscr{G}_{i,n}} \mathrm{vol}_i \left(K|_{\xi_i} \right) d\mu(\xi_i), \tag{4}$$

where $\mathscr{G}_{i,n}$ is the Grassmannian manifold of i-dimensional linear subspaces of $\mathbb{R}^n$, $d\mu(\xi_i)$ is the normalized Haar measure on $\mathscr{G}_{i,n}$, $K|_{\xi_i}$ denotes the orthogonal projection of K onto the i-dimensional subspaces ξ_i, and vol_i is the i-dimensional volume on space ξ_i.

In the 1930s, Aleksandrov and Fenchel and Jessen (see [3, 4]) proved that for a convex body K in $\mathbb{R}^n$, there exists a regular Borel measure $S_{n-1-i}(K)$ $(i = 0, 1, \cdots, n-1)$ on S^{n-1}, the unit sphere in $\mathbb{R}^n$, for $K, L \in \mathscr{K}^n$, the following representation holds

$$\begin{aligned} W_i(K, L) &= \frac{1}{n-i} \lim_{t \to 0^+} \frac{W_i(K+tL) - W_i(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(u) \, dS_{n-1-i}(K, u). \end{aligned} \tag{5}$$

The quantity $W_i(K,L)$ is called the ith mixed Quermassintegral of K and L.

In the 1960s, the Minkowski addition was extended to the $L^p(p \geq 1)$ Minkowski sum $h^p_{K+_p t\cdot L} = h^p_K + th^p_L$. The extension of the mixed Quermassintegral to the L^p mixed Quermassintegral due to Lutwak [1], the L^p mixed Quermassintegral inequalities, and the L^p Minkowski problem are established. (See [2, 5–13] for more about the L^p Minkowski theory.) The L^p mixed Quermassintegrals are defined by

$$W_{p,i}(K,L) := \frac{p}{n-i}\lim_{t\to 0^+}\frac{W_i(K+_p t\cdot L) - W_i(L)}{t}, \quad (6)$$

for $i = 0, 1, \cdots, n-1$. In particular, for $p = 1$ in (6), it is $W_i(K,L)$, and $W_{p,0}(K,L)$ is denoted by $V_p(K,L)$, which is called the L_p mixed volume of K and L. Similarly, the L^p mixed Quermassintegral has the following integral representation (see [1]):

$$W_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h^p_L(u)\,dS_{p,i}(K,u). \quad (7)$$

The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$ and has Radon-Nikodym derivative $dS_{p,i}(K,\cdot)/dS_i(K,\cdot) = h_K(\cdot)^{1-p}$. In particular, $p = 1$ in (7) yields the representation (5).

Most recently, the interest in the log-concave functions has been considerably increasing, motivated by the analogy properties between the log-concave functions and the volume convex bodies in $\mathcal{K}^n$. The classical Prékopa-Leindler inequality (see [14–18]) firstly shows the connections of the volume of convex bodies and log-concave functions. The Blaschke-Santaló inequality for even log-concave functions is established in [19, 20] by Ball (for the general case, see [21–24]). The mean width for log-concave function is introduced by Klartag and Milman and Rotem [25–27]. The affine isoperimetric inequality for log-concave functions is proved by Avidan et al. [28]. The John ellipsoid for log-concave functions has been establish by Alonso-Gutiérrez et al. [29]; the LYZ ellipsoid for log-concave functions is established by Fang and Zhou [30]. (See [31–37] for more about the pertinent results.)

Let $f = e^{-u}$, $g = e^{-v}$ be log-concave functions, $\alpha, \beta > 0$, the "sum" and "scalar multiplication" of log-concave functions are defined as

$$\alpha\cdot f \oplus \beta\cdot g := e^{-w}, \quad w^* = \alpha u^* + \beta v^*, \quad (8)$$

where w^* denotes as usual the Fenchel conjugate of the convex function ω. The total mass integral $J(f)$ of f is defined by $J(f) = \int_{\mathbb{R}^n} f(x)dx$. In paper [38] of Colesanti and Fragalà, the quantity $\delta J(f,g)$, which is called as the first variation of J at f along g, $\delta J(f,g) = \lim_{t\to 0^+}(J(f\oplus t\cdot g) - J(f))/t$, is discussed. It has been shown that $\delta J(f,g)$ is finite and has the following integral expression:

$$\delta J(f,g) = \int_{\mathbb{R}^n} v^*\,d\mu(f), \quad (9)$$

where $\mu(f)$ is the measure of f on $\mathbb{R}^n$.

Inspired by the paper [38] of Colesanti and Fragalà, in this paper, we define the ith functional Quermassintegrals $W_i(f)$ as the i-dimensional average total mass of f:

$$W_i(f) := \frac{\omega_n}{\omega_{n-i}}\int_{\mathcal{G}_{n-i,n}} J_{n-i}(f)\,d\mu(\xi_{n-i}), \quad i = 0, 1, \cdots, n-1, \quad (10)$$

where $J_i(f)$ denotes the i-dimensional total mass of f defined in Section 4, $\mathcal{G}_{i,n}$ is the Grassmannian manifold of $\mathbb{R}^n$, and $d\mu(\xi_{n-i})$ is the normalized measure on $\mathcal{G}_{i,n}$. Moreover, we define the first variation of W_i at f along g, which is

$$W_i(f,g) = \lim_{t\to 0^+}\frac{W_i(f\oplus t\cdot g) - W_i(f)}{t}. \quad (11)$$

It is a natural extension of the Quermassintegral of convex bodies in $\mathbb{R}^n$; we call it the ith functional mixed Quermassintegral. In fact, if one takes $f = \chi_K$, and dom $(f) = K \in \mathbb{R}^n$, then $W_i(f)$ turns out to be $W_i(K)$, and $W_i(\chi_K, \chi_L)$ equals to $W_i(K,L)$. The main result in this paper is to show that the ith functional mixed Quermassintegral has the following integral expressions.

Theorem 1. *Let $f, g \in \mathcal{A}'$, be integrable functions, $\mu_i(f)$ be the i-dimensional measure of f, and $W_i(f,g)$ be the ith functional mixed Quermassintegral of f and g. Then,*

$$W_i(f,g) = \frac{1}{n-i}\int_{\mathbb{R}^n} h_{g|_{\xi_{n-i}}}\,d\mu_{n-i}(f), \quad i = 0, 1, \cdots, n-1, \quad (12)$$

where $h_{g|_{\epsilon_{n-i}}}$ is the support function of $g|_{\epsilon_{n-i}}$.

The paper is organized as follows: In Section 2, we introduce some notations about the log-concave functions. In Section 3, the projection of a log-concave function onto subspace is discussed. In Section 4, we focus on how we can represent the ith functional mixed Quermassintegral $W_i(f,g)$ similar as $W_i(K,L)$. Owing to the Blaschke-Petkantschin formula and the similar definition of the support function of f, we obtain the integral representation of the ith functional mixed Quermassintegral $W_i(f,g)$.

2. Preliminaries

Let $u : \Omega \to (-\infty, +\infty]$ be a convex function; that is, $u((1-t)x + ty) \leq (1-t)u(x) + tu(y)$ for $t \in (0,1)$, where $\Omega = \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}$ is the domain of u. By the convexity of u, Ω is a convex set in $\mathbb{R}^n$. We say that u is proper if $\Omega \neq \varnothing$, and u is of class $\mathcal{C}^2_+$ if it is twice differentiable on int (Ω), with a positive definite Hessian matrix. In the following, we define the subclass of u:

$$\mathscr{L} = \left\{ u : \Omega \to (-\infty,+\infty] : u \text{ is convex, low semicontinuous, } \lim_{\|x\| \to +\infty} u(x) = +\infty \right\}. \tag{13}$$

Recall that the Fenchel conjugate of u is the convex function defined by

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - u(x) \}. \tag{14}$$

It is obvious that $u(x) + u^*(y) \geq \langle x, y \rangle$ for all $x, y \in \Omega$, and there is an equality if and only if $x \in \Omega$ and y is in the subdifferential of u at x, which means

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x) \rangle. \tag{15}$$

Moreover, if u is a lower semicontinuous convex function, then also u^* is a lower semicontinuous convex function, and $u^{**} = u$.

The infimal convolution of u and v from Ω to $(-\infty, +\infty]$ is defined by

$$u \square v(x) = \inf_{y \in \Omega} \{ u(x-y) + v(y) \}. \tag{16}$$

The right scalar multiplication by a nonnegative real number α is

$$(u\alpha)(x) := \begin{cases} \alpha u\left(\dfrac{x}{\alpha}\right), & \text{if } \alpha > 0, \\ I_{\{0\}}, & \text{if } \alpha = 0. \end{cases} \tag{17}$$

The following proposition below gathers some elementary properties of the Fenchel conjugate and the infimal convolution of u and v, which can be found in [38, 39].

Proposition 2. *Let $u, v : \Omega \to (-\infty,+\infty]$ be convex functions. Then,*

$$(u \square v)^* = u^* + v^* \tag{18}$$

(1) $(u\alpha)^* = \alpha u^*, \alpha > 0$

(2) $\mathrm{dom}\,(u \square v) = \mathrm{dom}\,(u) + \mathrm{dom}\,(v)$

(3) *it holds $u^*(0) = -\inf\,(u)$; in particular, if u is proper, then $u^*(y) > -\infty$; $\inf\,(u) > -\infty$ implies u^* is proper*

The following proposition about the Fenchel and Legendre conjugates is obtained in [39].

Proposition 3 (see [39]). *Let $u : \Omega \to (-\infty,+\infty]$ be a closed convex function, and set $\mathscr{C} := \mathrm{int}\,(\Omega)$, $\mathscr{C}^* := \mathrm{int}\,(\mathrm{dom}\,(u^*))$. Then, $(\mathscr{C}, u)$ is a convex function of Legendre type if and only if $\mathscr{C}^*, u^*$ is. In this case, $(\mathscr{C}^*, u^*)$ is the Legendre conjugate of $(\mathscr{C}, u)$ (and conversely). Moreover, $\nabla u := \mathscr{C} \to \mathscr{C}^*$ is a continuous bijection, and the inverse map of ∇u is precisely ∇u^*.*

A function $f : \mathbb{R}^n \to (-\infty,+\infty]$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and $0 < t < 1$, we have $f((1-t)x + ty) \geq f^{1-t}(x) f^t(y)$. If f is a strictly positive log-concave function on $\mathbb{R}^n$, then there exists a convex function $u : \Omega \to (-\infty,+\infty]$ such that $f = e^{-u}$. The log-concave function is closely related to the convex geometry of $\mathbb{R}^n$. An example of a log-concave function is the characteristic function χ_K of a convex body K in $\mathbb{R}^n$, which is defined by

$$\chi_K(x) = e^{-I_K(x)} = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K, \end{cases} \tag{19}$$

where I_K is a lower semicontinuous convex function, and the indicator function of K is

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if } x \notin K. \end{cases} \tag{20}$$

In the later sections, we also use f to denote f being extended to $\mathbb{R}^n$:

$$\bar{f} = \begin{cases} f, & x \in \Omega, \\ 0, & x \in \dfrac{R^n}{\Omega}. \end{cases} \tag{21}$$

Let $\mathscr{A} = \{ f : \mathbb{R}^n \to (0,+\infty] : f = e^{-u}, u \in \mathscr{L} \}$ be the subclass of f in $\mathbb{R}^n$. The addition and multiplication by nonnegative scalars in $\mathscr{A}$ are defined by the following (see [38]).

Definition 4. Let $f = e^{-u}$, $g = e^{-v} \in \mathscr{A}$, and $\alpha, \beta \geq 0$. The sum and multiplication of f and g *are* defined as

$$\alpha \cdot f \oplus \beta \cdot g = e^{-[(u\alpha) \square (v\beta)]}. \tag{22}$$

That means,

$$(\alpha \cdot f \oplus \beta \cdot g)(x) = \sup_{y \in \mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta}. \tag{23}$$

In particular, when $\alpha = 0$ and $\beta > 0$, we have $(\alpha \cdot f \oplus \beta \cdot g)(x) = g(x/\beta)^{\beta}$; when $\alpha > 0$ and $\beta = 0$, then $(\alpha \cdot f \oplus \beta \cdot g)(x) = f(x/\alpha)^{\alpha}$; finally, when $\alpha = \beta = 0$, we have $(\alpha \cdot f \oplus \beta \cdot g) = I_{\{0\}}$.

The following lemma is obtained in [38].

Lemma 5 (see [38]). *Let $u \in \mathscr{L}$, then there exist constants a and b, with $a > 0$, such that, for $x \in \Omega$,*

$$u(x) \geq a\|x\| + b. \tag{24}$$

Moreover, u^* is proper and satisfies $u^*(y) > -\infty$, $\forall y \in \Omega$.

Lemma 5 grants that $\mathscr{L}$ is closed under the operations of infimal convolution and right scalar multiplication defined in (16) and (17) which are closed.

Proposition 6 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$, and $\alpha, \beta \geq 0$. Then, $u\alpha \Box v\beta$ belongs to the same class as u and v.*

Let $f \in \mathscr{A}$, according to papers of [26, 40], the support function of $f = e^{-u}$ is defined as

$$h_f(x) = (-\log f(x))^* = u^*(x), \tag{25}$$

where u^* is the Legendre transform of u. The definition of h_f is a proper generalization of the support function h_K. In fact, one can easily check $h_{\chi_K} = h_K$. Obviously, the support function h_f share the most of the important properties of support functions h_K. Specifically, it is easy to check that the function $h : \mathscr{A} \to \mathscr{L}$ has the following properties [27]:

(1) *h is a bijective map from $\mathscr{A} \to \mathscr{L}$*

(2) *h is order preserving: $f \leq g$ if and only if $h_f \leq h_g$*

(3) *h is additive: for every $f, g \in \mathscr{A}$, we have $h_{f \oplus g} = h_f + h_g$*

The following proposition shows that h_f is $GL(n)$ covariant.

Proposition 7 (see [30]). *Let $f \in \mathscr{A}$, $A \in GL(n)$ and $x \in \mathbb{R}^n$. Then,*

$$h_{f\circ A}(x) = h_f\left(A^{-t}x\right). \tag{26}$$

Let $u, v \in \mathscr{L}$, denote by $u_t = u\Box vt(t > 0)$, and $f_t = e^{-u_t}$. The following lemmas describe the monotonicity and convergence of u_t and f_t, respectively.

Lemma 8 (see [38]). *Let $f = e^{-u}$, $g = g^{-v} \in \mathscr{A}$. For $t > 0$, set $u_t = u\Box(vt)$ and $f_t = e^{-u_t}$. Assume that $v(0) = 0$, then for every fixed $x \in \mathbb{R}^n$, $u_t(x)$ and $f_t(x)$ are, respectively, pointwise decreasing and increasing with respect to t; in particular, it holds*

$$u_1(x) \leq u_t(x) \leq u(x), f(x) \leq f_t(x) \leq f_1(x) \quad \forall x \in \mathbb{R}^n, \forall t \in [0, 1]. \tag{27}$$

Lemma 9 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$ and, for any $t > 0$, set $u_t := u\Box(vt)$. Assume that $v(0) = 0$, then*

(1) *$\forall x \in \Omega$, $\lim_{t\to 0^+} u_t(x) = u(x)$*

(2) *$\forall E \subset\subset \Omega$, $\lim_{t\to 0^+} \nabla u_t(x) = \nabla u$ uniformly on E*

Lemma 10 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$ and for any $t > 0$, let $u_t := u\Box(vt)$. Then, $\forall x \in \operatorname{int}(\Omega_t)$, and $\forall t > 0$,*

$$\frac{d}{dt}(u_t(x)) = -\psi(\nabla u_t(x)), \tag{28}$$

where $\psi := v^*$.

3. Projection of Functions onto Linear Subspace

Let $\mathscr{G}_{i,n}(0 \leq i \leq n)$ be the Grassmannian manifold of i-dimensional linear subspace of $\mathbb{R}^n$. The elements of $\mathscr{G}_{i,n}$ will usually be denoted by ξ_i, and $\xi_i^\perp$ stands for the orthogonal complement of ξ_i which is a $(n-i)$-dimensional subspace of $\mathbb{R}^n$. Let $\xi_i \in \mathscr{G}_{i,n}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The projection of f onto ξ_i is defined by (see [25, 41])

$$f|_{\xi_i}(x) := \max\left\{f(y) : y \in x + \xi_i^\perp\right\}, \quad \forall x \in \Omega|_{\xi_i}, \tag{29}$$

where $\xi_i^\perp$ is the orthogonal complement of ξ_i in $\mathbb{R}^n$ and $\Omega|_{\xi_i}$ is the projection of Ω onto ξ_i. By the definition of the log-concave function $f = e^{-u}$, for every $x \in \Omega|_{\xi_i}$, one can rewrite (29) as

$$f|_{\xi_i}(x) = \exp\left\{\max\left\{-u(y) : y \in x + \xi_i^\perp\right\}\right\} = e^{-u|_{\xi_i}}(x). \tag{30}$$

Regarding the "sum" and "multiplication" of f, we say that the projection keeps the structure on $\mathbb{R}^n$. In other words, we have the following proposition.

Proposition 11. *Let $f, g \in \mathscr{A}$, $\xi_i \in \mathscr{G}_{i,n}$, and $\alpha, \beta > 0$. Then,*

$$(\alpha \cdot f \oplus \beta \cdot g)|_{\xi_i} = \alpha \cdot f|_{\xi_i} \oplus \beta \cdot g|_{\xi_i}. \tag{31}$$

Proof. Let $f, g \in \mathscr{A}$, let $x_1, x_2, x \in \xi_i$ such that $x = \alpha x_1 + \beta x_2$, then we have

$$\begin{aligned}(\alpha \cdot f \oplus \beta \cdot g)|_{\xi_i}(x) &\geq (\alpha \cdot f \oplus \beta \cdot g)\left(\alpha x_1 + \beta x_2 + \xi_i^\perp\right) \\ &\geq f\left(x_1 + \xi_i^\perp\right)^\alpha g\left(x_2 + \xi_i^\perp\right)^\beta.\end{aligned} \tag{32}$$

Taking the supremum of the second right-hand inequality over all $\xi_i^\perp$, we obtain $(\alpha \cdot f \oplus \beta \cdot g)|_{\xi_i} \geq \alpha \cdot f|_{\xi_i} \oplus \beta \cdot g|_{\xi_i}$. On the other hand, for $x \in \xi_i$, $x_1, x_2 \in \xi_i$ such that $x_1 + x_2 = x$, then

$$\begin{aligned}\left(\alpha \cdot f|_{\xi_i} \oplus \beta \cdot g_{\xi_i}\right)(x) &= \sup_{x_1+x_2=x}\left\{\max\left\{f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)\right\}\max\right. \\ &\quad \left.\cdot\left\{g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right\}\right\} \\ &\geq \sup_{x_1+x_2=x}\left\{\max\left(f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right)\right\} \\ &= \max\left\{\sup_{x_1+x_2=x}\left(f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right)\right\} \\ &= (\alpha \cdot f \oplus \beta \cdot g)|_{\xi_i}(x).\end{aligned} \tag{33}$$

Since $f, g \geq 0$, the inequality $\max\{f \cdot g\} \leq \max\{f\} \cdot \max\{g\}$ holds. So, we complete the proof.

Proposition 12. *Let $\xi_i \in \mathcal{G}_{i,n}$, f and g are functions on $\mathbb{R}^n$, such that $f(x) \leq g(x)$ holds. Then,*

$$f|_{\xi_i} \leq g|_{\xi_i} \tag{34}$$

holds for any $x \in \xi_i$.

Proof. For $y \in x + \xi_i^{\perp}$, since $f(y) \leq g(y)$, then $f(y) \leq \max\{g(y): y \in x + \xi_i^{\perp}\}$. So, $\max\{f(y): y \in x + L_i^{\perp}\} \leq \max\{g(y) : y \in x + \xi_i^{\perp}\}$. By the definition of the projection, we complete the proof.

For the convergence of f, we have the following.

Proposition 13. *Let $\{f_i\}$ be functions such that $\lim_{n\to\infty} f_n = f_0$, $\xi_i \in \mathcal{G}_{i,n}$, then $\lim_{n\to\infty} (f_{n\xi_i}) = f_0|_{\xi_i}$.*

Proof. Since $\lim_{n\to\infty} f_n = f_0$, it means that $\forall \varepsilon > 0$, there exist N_0, $\forall n > N_0$, such that $f_0 - \varepsilon \leq f_n \leq f_0 + \varepsilon$. By the monotonicity of the projection, we have $f_0|_{\xi_i} - \varepsilon \leq f_n|_{\xi_i} \leq f_0|_{\xi_i} + \varepsilon$. Hence, each $\{f_n|_{\xi_i}\}$ has a convergent subsequence; we denote it also by $\{f_n|_{\xi_i}\}$, converging to some $f'_0|_{\xi_i}$. Then, for $x \in \xi_i$, we have

$$f_0|_{\xi_i}(x) - \varepsilon \leq f'_0|_{\xi_i}(x) = \lim_{n\to\infty} \left(f_n|_{\xi_i}\right)(x) \leq f_0|_{\xi_i}(x) + \varepsilon. \tag{35}$$

By the arbitrary of ε, we have $f'_0|_{\xi_i} = f_0|_{\xi_i}$, so we complete the proof.

Combining with Proposition 13 and Lemma 9, it is easy to obtain the following proposition.

Proposition 14. *Let u and v belong both to the same class $\mathcal{L}$ and $\Omega \in \mathbb{R}^n$ be the domain of u, for any $t > 0$, set $u_t = u\square(vt)$. Assume that $v(0) = 0$ and $\xi_i \in \mathcal{G}_{i,n}$, then*

(1) $\forall x \in \Omega|_{\xi_i}$, $\lim_{t\to 0^+} u_t|_{\xi_i}(x) = u|_{\xi_i}(x)$

$$\forall x \in \operatorname{int}\left(\Omega|_{\xi_i}\right), \lim_{t\to 0^+} \nabla u_t|_{\xi_i} = \nabla u|_{\xi_i} \tag{36}$$

Now, let us introduce some facts about the functions $u_t = u\square(vt)$ with respect to the parameter t.

Lemma 15. *Let $\xi_i \in \mathcal{G}_{i,n}$, u and v belong both to the same class $\mathcal{L}$, $u_t := u\square(vt)$ and Ω_t be the domain of u_t ($t > 0$). Then, for $x \in \Omega_t|_{\xi_i}$,*

$$\frac{d}{dt}\left(u_t|_{\xi_i}\right)(x) = -\psi\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right), \tag{37}$$

where $\psi := v^*|_{\xi_i}$.

Proof. Set $D_t := \Omega_t|_{\xi_i} \subset \xi_i$, for fixed $x \in \operatorname{int}(D_t)$, the map $t \to \nabla(u_t|_{\xi_i})(x)$ is differentiable on $(0, +\infty)$. Indeed, by the definition of Fenchel conjugate and the definition of projection u, it is easy to see that $(u_{\xi_i})^* = u^*|_{\xi_i}$ and $(u\square ut)|_{\xi_i} = u|_{\xi_i}\square ut|_{\xi_i}$ hold. Proposition 6 and the property of the projection grant the differentiability. Set $\varphi := u^*|_{\xi_i}$ and $\psi := v^*|_{\xi_i}$, and $\varphi_t = \varphi + t\psi$, then φ_t belongs to the class $\mathcal{C}_+^2$ on ξ_i. Then, $\nabla^2\varphi_t = \nabla^2\varphi + t\nabla^2\psi$ is nonsingular on ξ_i. So, the equation

$$\nabla\varphi(y) + t\nabla\psi(y) - x = 0 \tag{38}$$

locally defines a map $y = y(x, t)$ which is of class $\mathcal{C}^1$. By Proposition 3, we have $\nabla(u_t|_{\xi_i})$ is the inverse map of $\nabla\varphi_t$, that is, $\nabla\varphi_t(\nabla(u_t|_{\xi_i}(x)) = x$, which means that for every $x \in \operatorname{int}(D_t)$ and every $t > 0$, $t \to \nabla(u_t|_{\xi_i})$ is differentiable. Using equation (15) again, we have

$$u_t|_{\xi_i}(x) = \left\langle x, \nabla\left(u_t|_{\xi_i}\right)(x)\right\rangle - \varphi_t\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right), \quad \forall x \in \operatorname{int}(D_t). \tag{39}$$

Moreover, note that $\varphi_t = \varphi + t\psi$, we have

$$\begin{aligned} u_t\big|_{\xi_i}(x) &= \left\langle x, \nabla\left(u_{t\xi_i}\right)(x)\right\rangle - \varphi\left(\nabla\left(u_{t\xi_i}\right)(x)\right) - t\psi\left(\nabla\left(u_{t\xi_i}\right)(x)\right) \\ &= u_t\big|_{\xi_i}\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right) - t\psi\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right). \end{aligned} \tag{40}$$

Differential the above formal we obtain, $d/dt(u_t|_{\xi_i})(x) = -\psi(\nabla(u_t|_{\xi_i})(x))$. Then, we complete the proof of the result.

4. Functional Quermassintegrals of Log-Concave Function

A function $f \in \mathcal{A}$ is nondegenerate and integrable if and only if $\lim_{\|x\|\to+\infty} u(x)/\|x\| = +\infty$. Let $\mathcal{L}' = \{u \in \mathcal{L} : u \in \mathcal{C}_+^2(\mathbb{R}^n), \lim_{\|x\|\to+\infty} u(x)/\|x\| = +\infty\}$, and $\mathcal{A}' = \{f : \mathbb{R}^n \to (0,+\infty]: f = e^{-u}, u \in \mathcal{L}'\}$. Now, we define the ith total mass of f.

Definition 16. Let $f \in \mathcal{A}'$, $\xi_i \in \mathcal{G}_{i,n}(i = 1, 2, \cdots, n-1)$, and $x \in \Omega|_{\xi_i}$. The ith total mass of f is defined as

$$J_i(f) := \int_{\xi_i} f\Big|_{\xi_i}(x)dx, \tag{41}$$

where $f|_{\xi_i}$ is the projection of f onto ξ_i defined by (29) and dx is the i-dimensional volume element in ξ_i.

Remark 17.

(1) The definition of $J_i(f)$ follows the i-dimensional volume of the projection a convex body. If $i=0$, we defined $J_0(f) := \omega_n$, the volume of the unit ball in $\mathbb{R}^n$, for the completeness

(2) When taking $f=\chi_K$, the characteristic function of a convex body K, one has $J_i(f)=V_i(K)$, the i-dimensional volume in ξ_i

Definition 18. Let $f \in \mathscr{A}'$. Set $\xi_i \in \mathscr{G}_{i,n}$ be a linear subspace and for $x \in \Omega|_{\xi_i}$, the ith functional Quermassintegrals of f (or the i-dimensional mean projection mass of f) are defined as

$$W_{n-i}(f) := \frac{\omega_n}{\omega_i}\int_{\mathscr{G}_{i,n}} J_i(f)d\mu(\xi_i), \quad i=1,2,\cdots,n, \tag{42}$$

where $J_i(f)$ is the ith total mass of f defined by (41) and $d\mu(\xi_i)$ is the normalized Haar measure on $\mathscr{G}_{i,n}$.

Remark 19.

(1) The definition of $W_i(f)$ follows the definition of the i th Quermassintegrals $W_i(K)$, that is, the ith mean total mass of f on $\mathscr{G}_{i,n}$. Also, in a recent paper [42], the authors give the same definition by defining the Quermassintegral of the support set for the quasiconcave functions

(2) When i equals to n in (42), we have $W_0(f)=\int_{\mathbb{R}^n} f(x)dx = J(f)$, the total mass function of f defined by Colesanti and Fragalá [38]. Then, we can say that our definition of $W_i(f)$ is a natural extension of the total mass function of $J(f)$

(3) From the definition of the Quermassintegrals $W_i(f)$, the following properties are obtained (see also [42]):

$$\text{Positivity::} 0 \le W_i(f) \le +\infty \tag{43}$$

(i) Monotonicity: $W_i(f) \le W_i(g)$, if $f \le g$

(ii) Generally speaking, $W_i(f)$ has no homogeneity under dilations. That is, $W_i(\lambda\cdot f)=\lambda^{n-i}W_i(f^\lambda)$, where $\lambda\cdot f(x)=\lambda f(x/\lambda)$, $\lambda>0$

Definition 20. Let $f, g \in \mathscr{A}'$, $\oplus$, and $\cdot$ denote the operations of "sum" and "multiplication" in $\mathscr{A}'$. $W_i(f)$ and $W_i(g)$ are, respectively, the ith Quermassintegrals of f and g. Whenever the following limit exists,

$$W_i(f,g) = \frac{1}{(n-i)}\lim_{t\to 0^+}\frac{W_i(f\oplus t\cdot g)-W_i(f)}{t}, \tag{44}$$

we denote it by $W_i(f,g)$ and call it as the first variation of W_i at f along g, or the ith functional mixed Quermassintegrals of f and g.

Remark 21. Let $f=\chi_K$ and $g=\chi_L$, with $K, L \in \mathscr{K}^n$. In this case $W_i(f\oplus t\cdot g)=W_i(K+tL)$, then $W_i(f,g)=W_i(K,L)$. In general, $W_i(f,g)$ has no analog properties of $W_i(K,L)$; for example, $W_i(f,g)$ is not always nonnegative and finite.

The following is devoted to proving that $W_i(f,g)$ exists under the fairly weak hypothesis. First, we prove that the first i-dimensional total mass of f is translation invariant.

Lemma 22. *Let $\xi_i \in \mathscr{G}_{i,n}$, $f=e^{-u}, g=e^{-v}\in\mathscr{A}'$. Let $c=\inf u|_{\xi_i} =: u(0)$, $d=\inf v|_{\xi_i} := v(0)$, and set $\tilde{u}_i(x)=u|_{\xi_i}(x)-c$, $\tilde{v}_i(x)=v|_{\xi_i}(x)-d$, $\tilde{\varphi}_i(y)=(\tilde{u}_i)^*(y)$, $\tilde{\psi}_i(y)=(\tilde{v}_i)^*(y)$, $\tilde{f}_i=e^{-\tilde{u}_i}$, $\tilde{g}_i=e^{-\tilde{v}_i}$, and $\tilde{f}_t|_i=\tilde{f}\oplus t\cdot\tilde{g}$. Then, if $\lim_{t\to0^+}((J_i(\tilde{f}_t)-J_i(\tilde{f}))/t)=\int_{\xi_i}\tilde{\psi}_i d\mu_i(\tilde{f})$ holds, then we have $\lim_{t\to0^+}((J_i(f_t)-J_i(f))/t)=\int_{\xi_i}\psi_i d\mu_i(f)$.*

Proof. By the construction, we have $\tilde{u}_i(0)=0$, $\tilde{v}_i(0)=0$, and $\tilde{v}_i\ge0$, $\tilde{\varphi}_i\ge0$, $\tilde{\psi}_i\ge0$. Further, $\tilde{\psi}_i(y)=\psi_i(y)+d$, and $\tilde{f}_i=e^c f_i$. So,

$$\lim_{t\to0^+}\frac{J_i\left(\tilde{f}_t\right)-J_i\left(\tilde{f}\right)}{t}=\int_{\xi_i}\tilde{\psi}_i d\mu_i\left(\tilde{f}\right)=e^c\int_{\xi_i}\psi_i d\mu_i(f)+de^c\int_{\xi_i}d\mu_i(f). \tag{45}$$

On the other hand, since $f_i\oplus t\cdot g_i=e^{-(c+dt)}(\tilde{f}_i\oplus t\cdot\tilde{g}_i)$, we have, $J_i(f\oplus t\cdot g)=e^{-(c+dt)}J_i(\tilde{f}_i\oplus t\cdot\tilde{g}_i)$. By derivation of both sides of the above formula, we obtain

$$\begin{aligned}\lim_{t\to0^+}\frac{J_i(f\oplus t\cdot g)-J_i(f)}{t} &= -de^{-c}\lim_{t\to0^+}J_i\left(\tilde{f}_i\oplus t\tilde{g}_i\right)dx+e^{-c}\lim_{t\to0^+}\\ &\quad\cdot\left[\frac{J_i\left(\tilde{f}_t\right)-J_i\left(\tilde{f}\right)}{t}\right]=-de^{-c}J_i\left(\tilde{f}_i\right)\\ &\quad+\int_{\xi_i}\psi_i d\mu_i(f)+d\int_{\xi_i}d\mu_i(f)\\ &=\int_{\xi_i}\psi_i d\mu_i(f).\end{aligned} \tag{46}$$

So, we complete the proof.

Theorem 23. *Let $f, g \in \mathscr{A}'$, with $-\infty \le \inf(\log g) \le +\infty$ and $W_i(f)>0$. Then, $W_j(f,g)$ is differentiable at f along g, and it holds*

$$W_j(f,g)\in[-k,+\infty], \tag{47}$$

where $k=\max\{d,0\}W_i(f)$.

Proof. Let $\xi_i \in \mathcal{G}_{i,n}$, since $u|_{\xi_i} := -\log(f_{\xi_i}) = -(\log f)|_{\xi_i}$ and $v|_{\xi_i} := -\log(g_{\xi_i}) = -(\log f)|_{\xi_i}$. By the definition of f_t and Proposition 11, we obtain $f_t|_{\xi_i} = (f \oplus t \cdot g)|_{\xi_i} = f|_{\xi_i} \oplus t \cdot g|_{\xi_i}$. Notice that $v|_{\xi_i}(0) = v(0)$, set $d := v(0)$, $\tilde{v}|_{\xi_i}(x) := v|_{\xi_i}(x) - d$, $\tilde{g}|_{\xi_i}(x) := e^{-\tilde{v}|_{\xi_i}(x)}$, and $\tilde{f}_t|_{\xi_i} := f|_{\xi_i} \oplus t \cdot \tilde{g}|_{\xi_i}$. Up to a translation of coordinates, we may assume $\inf(v) = v(0)$. Lemma 8 says that for every $x \in \xi_i$,

$$f|_{\xi_i} \leq \tilde{f}_t\Big|_{\xi_i} \leq \tilde{f}_1\Big|_{\xi_i}, \quad \forall x \in \mathbb{R}^n, \forall t \in [0,1]. \tag{48}$$

Then, there exists $\tilde{f}|_{\xi_i}(x) := \lim_{t\to 0^+} \tilde{f}_t\big|_{\xi_i}(x)$. Moreover, it holds $\tilde{f}|_{\xi_i}(x) \geq f|_{\xi_i}(x)$ and $\tilde{f}_t|_{\xi_i}$ is pointwise decreasing as $t \to 0^+$. Lemma 5 and Proposition 6 show that $f|_{\xi_i} \oplus t \cdot \tilde{g}|_{\xi_i} \in \mathcal{A}'$, $\forall t \in [0,1]$. Then, $J_i(f) \leq J_i(\tilde{f}_t) \leq J_i(\tilde{f}_1)$, $-\infty \leq J_i(f)$, $J_i(\tilde{f}_1) < \infty$. Hence, by monotonicity and convergence, we have $\lim_{t\to 0^+} W_i(\tilde{f}_t) = W_i(\tilde{f})$. In fact, by definition, we have $\tilde{f}_t|_{\xi_i}(x) = e^{-\inf\{u|_{\xi_i}(x-y) + tv|_{\xi_i}(y/t)\}}$,

$$-\inf\left\{u|_{\xi_i}(x-y) + tv|_{\xi_i}\left(\frac{y}{t}\right)\right\} \leq -\inf u|_{\xi_i}(x-y) - t\inf v|_{\xi_i}\left(\frac{y}{t}\right). \tag{49}$$

Note that $-\infty \leq \inf(v|_{\xi_i}) \leq +\infty$, then $-\inf u|_{\xi_i}(x-y) - t\inf v|_{\xi_i}(y/t)$ is a continuous function of variable t, then

$$\tilde{f}\Big|_{\xi_i}(x) := \lim_{t\to 0^+} \tilde{f}_t|_{\xi_i}(x) = f\big|_{\xi_i}(x). \tag{50}$$

Moreover, $W_i(\tilde{f}_t)$ is a continuous function of $(t \in [0,1])$; then, $\lim_{t\to 0^+} W_i(\tilde{f}_t) = W_i(f)$. Since $f_t|_{\xi_i} = e^{-dt}\tilde{f}|_{\xi_i}(x)$, we have

$$\frac{W_i(f_t) - W_i(f)}{t} = W_i(f)\frac{e^{-dt} - 1}{t} + e^{-dt}\frac{W_i\left(\tilde{f}_t\right) - W_i(f)}{t}. \tag{51}$$

Notice that, $\tilde{f}_t|_{\xi_i} \geq f|_{\xi_i}$, we have the following two cases, that is, $\exists t_0 > 0 : W_i(\tilde{f}_{t_0}) = W_i(f)$ or $W_i(\tilde{f}_t) = W_i(f)$, $\forall t > 0$.

For the first case, since $W_i(\tilde{f}_t)$ is a monotone increasing function of t, it must hold $W_i(\tilde{f}_t) = W_i(f)$ for every $t \in [0, t_0]$. Hence, we have $\lim_{t\to 0^+}(W_i(f_t) - W_i(f))/t = -dW_i(f)$; the statement of the theorem holds true.

In the latter case, since $\tilde{f}_t|_{\xi_i}$ is an increasing nonnegative function, it means that $\log(W_i(\tilde{f}_t))$ is an increasing concave function of t. Then, $\exists(\log(W_i(\tilde{f}_t)) - \log(W_i(f)))/t \in [0,+\infty]$. On the other hand, since

$$\log'\left(W_i\left(\tilde{f}_t\right)\right)\Big|_{t=0} = \lim_{t\to 0^+}\frac{\log\left(W_i\left(\tilde{f}_t\right)\right) - \log(W_i(f))}{W_i\left(\tilde{f}_t\right) - W_i(f)} = \frac{1}{W_i(f)}. \tag{52}$$

Then,

$$\lim_{t\to 0^+}\frac{W_i\left(\tilde{f}_t\right) - W_i(f)}{\log\left(W_i\left(\tilde{f}_t\right)\right) - \log(W_i(f))} = W_i(f) > 0. \tag{53}$$

From above, we infer that $\exists\lim_{t\to 0^+}(W_i(\tilde{f}_t) - W_i(f))/t \in [0,+\infty]$. Combining the above formulas, we obtain

$$\lim_{t\to 0^+}\frac{W_i(f_t) - W_i(f)}{t} \in [-\max\{d, 0\}W_i(f), +\infty]. \tag{54}$$

So, we complete the proof.

In view of the example of the mixed Quermassintegral, it is natural to ask whether, in general, $W_i(f, g)$ has some kind of integral representation.

Definition 24. Let $\xi_i \in \mathcal{G}_{i,n}$ and $f = e^{-u} \in \mathcal{A}'$. Consider the gradient map $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$, the Borel measure $\mu_i(f)$ *on* ξ_i is defined by

$$\mu_i(f) := \frac{\left(\nabla u|_{\xi_i}\right)_\#}{\|x\|^{n-i}}\left(f|_{\xi_i}\right). \tag{55}$$

Recall that the following Blaschke-Petkantschin formula is useful.

Proposition 25 (see [43]). *Let $\xi_i \in \mathcal{G}_{i,n}(i = 1, 2, \cdots, n)$ be linear subspace of $\mathbb{R}^n$ and f be a nonnegative bounded Borel function on $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f(x)dx = \frac{\omega_n}{\omega_i}\int_{\mathcal{G}_{i,n}}\int_{\xi_i} f(x)\|x\|^{n-i}dx\,d\mu(\xi_i). \tag{56}$$

Now, we give a proof of Theorem 1.

Proof of Theorem 1. By the definition of the ith Quermassintegral of f, we have

$$\frac{W_i(f_t) - W_i(f)}{t} = \frac{\omega_n}{\omega_{n-i}}\int_{\mathcal{G}_{n-i,n}}\frac{J_{n-i}(f_t) - J_{n-i}(f)}{t}d\mu(\xi_{n-i}). \tag{57}$$

Let $t > 0$ be fixed, take $C \subset\subset \Omega|_{\xi_{n-i}}$, and by reduction $0 \in \operatorname{int}(\Omega)|_{\xi_{n-i}}$, we have $C \subset\subset \Omega|_{\xi_{n-i}}$, by Lemma 15, we obtain

$$\lim_{h\to 0}\frac{J_{n-i}(f_{t+h})(x)-J_{n-i}(f_t(x))}{h}=\int_{\xi_{n-i}}\psi\left(\nabla u_t \mid_{\xi_{n-i}}(x)\right)f_t\Big|_{\xi_{n-i}}(x)dx, \tag{58}$$

where $\psi = h_{g|_{\xi_{n-i}}} = \nu|^*_{\xi_{n-i}}$. Then, we have

$$\begin{aligned}\lim_{h\to 0}\frac{W_i(f_{t+h})-W_i(f_t)}{h} &= \frac{\omega_n}{\omega_{n-i}}\int_{\mathscr{G}_{n-i,n}}\int_{\xi_{n-i}}\frac{\psi\left(\nabla u_t \mid_{\xi_{n-i}}(x)\right)f_t\big|_{\xi_{n-i}}(x)}{\|x\|^{n-i}} \\ &\quad \cdot \|x\|^{n-i}dxd\mu(\xi_{n-i}), \\ &= \int_{\mathbb{R}^n}\frac{\psi\left(\nabla u_t \mid_{\xi_{n-i}}(x)\right)f_t\big|_{\xi_{n-i}}(x)}{\|x\|^{n-i}}dx \\ &= \int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_t).\end{aligned} \tag{59}$$

So, we have $W_i(f_{t+h})-W_i(f_t)=\int_0^t\{\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)\}ds$. The continuity of ψ implies $\lim_{s\to 0^+}\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)ds=\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f)ds$. Therefore,

$$\begin{aligned}\lim_{t\to 0^+}\frac{W_i(f_t)-W_i(f)}{t} &= \frac{d}{dt}W_i(f_t)\Big|_{t=0^+}=\lim_{s\to 0^+}\frac{d}{dt}W_i(f_t)\Big|_{t=s} \\ &= \lim_{s\to 0^+}\frac{d}{dt}\int_0^t\left\{\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)\right\}ds \\ &= \int_{\mathbb{R}^n}\psi d\mu_{n-i}(f).\end{aligned} \tag{60}$$

Since $\psi = h_{g|_\xi}$, we have

$$W_i(f,g)=\frac{1}{n-i}\lim_{t\to 0^+}\frac{W_i(f_t)-W_i(f)}{t}=\frac{1}{n-i}\int_{\mathbb{R}^n}h_{g|_{\xi_{n-i}}}d\mu_{n-i}(f). \tag{61}$$

So, we complete the proof.

Remark 26. From the integral representation (12), the ith functional mixed Quermassintegral is linear in its second argument, with the sum in $\mathscr{A}'$, *for* $f, g, h\in\mathscr{A}'$, then we have $W_i(f, g\oplus h)=W_i(f,g)+W_i(f,h)$.

Data Availability

No data were used to support this study.

Disclosure

This paper is presented as Arxiv in the following link: https://arxiv.org/abs/2003.11367.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of this manuscript.

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