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Research Article

On Solutions of a Parabolic Equation with Nonstandard Growth Condition

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A parabolic equation with nonstandard growth condition is considered. A kind of weak solution and a kind of strong solution are introduced, respectively; the existence of solutions is proved by a parabolically regularized method. The stability of weak solutions is based on a natural partial boundary value condition. Two novelty elements of the paper are both the dependence of diffusion coefficient $b(x,t)$ on the time variable t, and the partial boundary value condition based on a submanifold of $\partial\Omega\times(0,T)$. How to overcome the difficulties arising from the nonstandard growth conditions is another technological novelty of this paper.

1. Introduction

In this paper, we are concerned with the initial-boundary value problem of a parabolic equation with nonstandard growth condition

$$u_t = \operatorname{div}\left(b(x,t)|u|^{r(x)-1}\nabla u\right) + \operatorname{div}\left(\vec{a}\left(u^{r(x)},x,t\right)\right) + f(u,x,t), \quad (x,t)\in Q_T=\Omega\times(0,T) \tag{1}$$

where $b(x,t)\geq 0$ is a $C(\overline{QT})$ function, $r(x)>0$ is a $C^1(\bar{\Omega})$ function, $\vec{a}(s,x,t)=\{a_i(s,x,t)\}$, $a_i(s,x,t)\in C(\mathbb{R}\times\overline{Q_T})$, and $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. These kinds of equations can be regarded as nonlinear variation of the classical heat equation, the well-known paradigm to explain the diffusion process. In particular, when $b(x,t)\equiv 1$, $a_i=0$, $r(x)=r$ is a constant; equation (1) is with the type

$$u_t=\Delta u^r+g(\nabla u,u,x,t),\ (x,t)\in Q_T, \tag{2}$$

which includes the so-called porous medium equation (PME), the fast diffusion equation (FDE) when $r>1$, and the slow diffusion equation (SDE) when $r<1$. Correspondingly, if $r^-=\min_{x\in\bar{\Omega}}r(x)>1$, then equation (1) is a degenerate parabolic equation and has a slow diffusion characteristic. If $r^+=\max_{x\in\bar{\Omega}}r(x)<1$, then equation (1) is a singular parabolic equation and has a fast diffusion characteristic.

In fact, equation (2) itself has a wide number of applications, ranging from plasma physics to filtration in porous media, thin films, Riemannian geometry, relativistic physics, and many others. It has at the same time tested as the testing ground from the development of new methods of analytical investigation, since it offers a variety of surprising phenomena that strongly deviate from the heat equation standard, for example, free boundary, limited regularity, mass loss, and extinction or quenching. There exists an abundant literature on these topics, one can refer to [1–10] and the references therein. In this literature, the nonlinearities are with power types that lead to degenerate or singular parabolicity, and we can gather these collectively under the name "the porous medium equation with standard growth conditions." Since, in equation (1), the nonlinearity is with variable exponent, so it is called as "with nonstandard growth condition."

From another perspective, since $b(x,t)\geq 0$, equation (1) is a special case of the usual reaction-diffusion equation

$$u_t=\operatorname{div}\left(b(u,x,t)\nabla u\right)+\operatorname{div}\left(\vec{a}(u,x,t)\right)+f(u,x,t),\ (x,t)\in Q_T. \tag{3}$$

If $b(u,x,t)$ is degenerate in the interior of Ω, this equation is a hyperbolic-parabolic mixed type equation theoretically. To ensure the uniqueness of a weak solution to this equation, the entropy condition and the corresponding entropy solution are required, one can refer to references [11–16] for the details. In this paper, we assume that

$$b(x,t) > 0, (x,t) \in \Omega \times [0,T). \tag{4}$$

As a consequence, equation (1) is a pure nonlinear parabolic equation and has not the hyperbolic characteristic.

The porous medium equation with nonstandard growth conditions was first proposed by Antontsev and Shmarev in [17]. The existence, uniqueness, and localization properties of solutions to equation

$$u_t = \operatorname{div}\left(|u|^{r(x)}\nabla u\right) + g(\nabla u, u, x, t), \tag{5}$$

had been studied. Another property proved in [17] is the finite speed of propagation, which permits one to consider the free boundary problem. The work [18] by Duque etc. implemented a finite element method with adaptive mesh to approximate the solution of the porous medium equation with nonstandard growth conditions in 2D domains with free boundary. The equidistribution principle deduced by de Boor [19] and the moving mesh for partial differential equations by Huang and Russell [20] were used there. Recently, when $b(x,t) = b(x)$ with $b(x) > 0, x \in \Omega$, and

$$b(x) = 0, x \in \partial\Omega, \tag{6}$$

the well-posedness of weak solutions to equation (1) has been studied by the author [21]. Roughly speaking, if $b(x)$ satisfies (6) and

$$\int_\Omega b(x)^{-1}dx < \infty, \tag{7}$$

the uniqueness of weak solution to equation (1) with the initial value

$$u(x,0) = u_0(x), x \in \Omega, \tag{8}$$

has been proved in [21]. This result can be comprehended as that, the degeneracy of diffusion coefficient $b(x)$ on the boundary (6) acts as a role as the usual boundary value condition

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T). \tag{9}$$

In this paper, different from [21], the diffusion coefficient $b(x,t)$ is dependent on the time variable t; moreover, we merely assume that $b(x,t) \geq 0$ satisfies (4) and do not restrict the similar condition as (7). As a compensation, a partial boundary value condition

$$u(x,t) = 0, (x,t) \in \Sigma = \{(x,t) \in \partial\Omega \times (0,T) : b(x,t) > 0\}, \tag{10}$$

is imposed as follows. One can see that, unlike the usual Dirichlet boundary value condition (9), in which $\partial\Omega \times (0,T)$, is a cylinder, $\Sigma \subset \partial\Omega \times (0,T)$ appearing in (10) is just a submanifold and generally can not be expressed as a cylinder type as $\Gamma \times (0,T)$ with $\Gamma \subseteq \partial\Omega$. The first aim of this paper is to study the stability of weak solutions based on this partial boundary value condition. Another aim of this paper is to study the existence of a strong solution to equation (1). The details are provided below. It is well-known that, for the usual porous medium equation

$$u_t = \Delta u^r, (x,t) \in Q_T, \tag{11}$$

there holds

$$\frac{\partial u}{\partial t} \in L^2(Q_T), \tag{12}$$

and u is with the Hölder continuity [5]. However, for a porous medium equation with nonstandard growth conditions, weak solutions in [17, 18, 21–30] do not satisfy the regularity as (12). Thus, in this paper, we will try to make up for some gaps in such a field, considering that the weak solutions of equation (1) have the property (12).

Definition 1. Let $u(x,t)$ be a nonnegative function satisfying

$$\begin{gathered} u \in L^\infty(Q_T), u_t \in L^2\left(0,T; W^{-1,2}(\Omega)\right), \\ \sqrt{b(x,t)}u^{r(x)-1}|\nabla u| \in L^\infty\left(0,T; L^2(\Omega)\right). \end{gathered} \tag{13}$$

If for any function $\varphi \in C_0^1(Q_T)$, there is

$$\begin{aligned} &\iint_{Q_T}\left(-\frac{\partial\varphi}{\partial t}u + b(x,t)|u|^{r(x)-1}\nabla u\nabla\varphi\right)dxdt \\ &\quad + \sum_{i=1}^N \iint_{Q_T} a_i\left(u^{r(x)}, x, t\right)\varphi_{x_i}(x,t)dxdt \\ &= \iint_{Q_T} f(u,x,t)\varphi dxdt, \end{aligned} \tag{14}$$

then $u(x,t)$ is said to be a weak solution of the initial-boundary value problem of equation (1), provided that initial value (6) is true in the sense

$$\lim_{t\to 0}\int_\Omega (u(x,t) - u_0(x))O(x)dx = 0, O(x) \in C_0^\infty(\Omega), \tag{15}$$

and the partial boundary value condition (10) is true in the sense of the trace.

Definition 2. Let $u(x,t)$ be a nonnegative function satisfying (14) and

$$\begin{aligned} &u \in L^\infty(Q_T), \sqrt{u^{r(x)-1}}u_t \\ &\in L^2(Q_T), \sqrt{b(x,t)u^{r(x)-1}}|\nabla u| \\ &\in L^\infty\left(0,T; L^2(\Omega)\right), \end{aligned} \tag{16}$$

then $u(x,t)$ is said to be a strong solution of the initial-boundary value problem of equation (1), provided that initial value (6) is true in the sense of (15), and the partial boundary value condition (10) is true in the sense of the trace.

Since $b(x,t)$ satisfies (4), (16) means that $u_t^{r(x)}$ and $\nabla u^{r(x)}$ exist almost everywhere in QT. This is the reason that we call $u(x,t)$ is a strong solution of equation (1).

From Definition 2, for all $\varphi(x,t) \in L^2(0,T; W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} &\iint_{Q_T} u_t \varphi(x,t)\,dxdt + \iint_{Q_T} \frac{b(x,t)}{r(x)} \nabla u^{r(x)} \nabla \varphi\,dxdt \\ &\quad - \iint_{Q_T} \frac{b(x,t)}{r(x)} u^{r(x)} \log u \nabla r \nabla \varphi\,dxdt \\ &\quad + \sum_{i=1}^{N} \iint_{Q_T} a_i\left(u^{r(x)}, x, t\right) \varphi_{x_i}\,dxdt \\ &\quad = \iint_{Q_T} f(u,x,t)\varphi(x,t)\,dxdt, \end{aligned} \tag{17}$$

which implies that

$$\|u_t\|_{L^2}\left(0,T; W^{-1,2}(\Omega)\right) \le c. \tag{18}$$

Thus, if $u(x,t)$ is a strong solution of equation (1), then it is a weak solution.

Theorem 3. *If $r(x)$ is a $C^1(\bar{\Omega})$ function, $r(x) \ge r - > 1$, $a_i(s,x,t) \in C(\mathbb{R} \times \overline{Q_T})$, $i = 1, 2, \cdots, N$, $u_0(x) \ge 0$ satisfies*

$$0 \le u_0 \in L^\infty(\Omega), \sqrt{b(x,0)} \nabla u_0^{r(x)} \in L^2(\Omega), \tag{19}$$

$0 \le b(x,t) \in C(\overline{Q_T})$ satisfies (4), $f(s,x,t)$ is a continuous function on $\mathbb{R} \times \overline{Q_T}$ and when $s < 0$

$$f(s,x,t) > 0, (x,t) \in \Omega \times (0,T), \tag{20}$$

then the initial boundary value problem of equation (1) has a nonnegative weak solution.

Theorem 4. *If $r(x)$ is a $C^1(\bar{\Omega})$ function, $r(x) \ge r^- > 1$, $a_i(s,x,t) \in C^1(\mathbb{R} \times \overline{Q_T})$, $i = 1, 2, \cdots, N$, $u_0(x) \ge 0$ satisfies (19), $b(\cdot,t) \in C^1[0,T]$ satisfies (4) and*

$$\begin{aligned} &b_t(x,t) \le 0, (x,t) \in \Omega \times (0,T), \\ &\left|\frac{\partial}{\partial s} a_i\left(s^{r(x)}, x, t\right)\right|^2 b(x,t)^{-1} \le c(M), \end{aligned} \tag{21}$$

for any $|s| \le M + 1$, $M = \|u_0(x)\|_{L^\infty(\Omega)}$, $f(s,x,t)$ is a continuous function satisfies (20), then the initial boundary value problem of equation (1) has a nonnegative strong solution.

Throughout this paper, the constant c may be different from one to another, $c(M)$ represents the constant c depends on M, and $c(T)$ represents the constant c depends on T.

Theorem 5. *Let $u(x,t)$ be a weak solution of equation (1) with the initial value (8) and with the partial homogenous value condition (10),*

$$\int_\Omega [\log u]^2 |\nabla r|^2 dx \le c(T). \tag{22}$$

If $0 < r^- \le r(x) \in C^1(\Omega)$, $f(s,x,t)$ is a Lipschitz function on $\mathbb{R} \times \overline{Q_T}$, for every given $t \in [0,T)$, $b(x,t)$ is a differential function on the spatial variables x_i, $i = 1, 2, \cdots, N$, $a_i(s,x,t) \in C(\mathbb{R} \times \overline{Q_T})$ and satisfies

$$\left|a_i\left(u^{r(x)}, x, t\right) - a_i\left(v^{r(x)}, x, t\right)\right| \le c(M) b(x,t), \tag{23}$$

$$\int_\Omega b(x,t)^{-1} |\nabla b|^2 dx \le c(T), \tag{24}$$

then the solution $u(x,t)$ is unique.

One can see that there are functions satisfying condition (24). For example, if $\Gamma \subset \partial\Omega$ is a relatively open set of $\partial\Omega$, and let $d_\Gamma(x) = dist(x,\Gamma)$. Then, for any $f(t) \in C^1[0,T]$,

$$b(x,t) = f(t) d_\Gamma(x)^2 \tag{25}$$

is a function satisfying (24).

We note that, if $r(x) \equiv r$ is a constant, then condition (22) is naturally true. By the way, just as one reviewer has suggested, one can study Theorem 3 and Theorem 4 when $r(x) \ge r^- > 0$ and $f(s,x,t)$ is just a Carathodory function.

2. The Proof of Theorems 3-4

In this section, we give the proof of the existence theorems.

Proof of Theorem 3. By the monotone convergent method [5], since $f(s,x,t) > 0$ when $s < 0$, there is a nonnegative weak solution $u_n(x,t)$ of the following initial-boundary value problem

$$\begin{cases} u_{nt} = \operatorname{div}\left(\left(b(x,t) + \dfrac{1}{n}\right) |u_n|^{r(x)-1} \nabla u\right) + \operatorname{div}\left(a\left(u_n^{r(x)}, x, t\right)\right) + f(u_n, x, t), (x,t) \in Q_T, \\ u_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u_n(x,0) = u_0(x), x \in \Omega, \end{cases} \tag{26}$$

which satisfies

$$u_n \in L^\infty(Q_T), 0 \le u_n(x,t) \le \|u_0(x)\|_{L^\infty(\Omega)} + 1, \tag{27}$$

$$\left(b(x,t) + \frac{1}{n}\right)^{\frac{1}{2}} u_n^{\frac{r(x)-1}{2}} \nabla u_n \in L^2(Q_T). \tag{28}$$

Multiplying (26) by $\phi = u_n^{r(x)}$, then we obtain

$$\begin{aligned}\iint_{Q_T} u_{nt} u_n^{r(x)} dxdt = {} & \iint_{Q_T} \operatorname{div}\left[\left(b(x,t) + \frac{1}{n}\right) u_n^{r(x)-1} \nabla u_n\right] u_n^{r(x)} dxdt \\ & + \sum_{i=1}^N \iint_{Q_T} \frac{\partial a_i\left(u_n^{r(x)}, x, t\right)}{\partial x_i} u_n^{r(x)} dxdt \\ & + \iint_{Q_T} f(u_n, x, t) u_n^{r(x)} dxdt.\end{aligned} \tag{29}$$

In the first place, by a similar calculation as that in [21], we can extrapolate that

$$\iint_{Q_T} \left(b(x,t) + \frac{1}{n}\right) u_n^{r(x)-1} |\nabla u_n|^2 dxdt \le c, \tag{30}$$

$$\int_\Omega \left(b(x,t) + \frac{1}{n}\right) \left|\nabla u_n^{r(x)}\right|^2 dx \le c(T), \tag{31}$$

and show that

$$\left[\left(b(x,t) + \frac{1}{n}\right) u_n^{r(x)-1}\right]^{\frac{1}{2}} \nabla u_n \rightharpoonup b(x,t)^{\frac{1}{2}} u^{r(x)-1} \nabla u, \tag{32}$$

weakly in $L^2(Q_T)$.

In the second place, for all $\varphi(x,t) \in L^2(0, T; W_0^{1,2}(\Omega))$, we have

$$\begin{aligned}&\iint_{Q_T} u_{nt} \varphi(x,t) dxdt + \iint_{Q_T} \frac{b(x,t) + (1/n)}{r(x)} \nabla u_n^{r(x)} \nabla \varphi dxdt \\ &\quad - \iint_{Q_T} \frac{b(x,t) + (1/n)}{r(x)} u_n^{r(x)} \log u_n \nabla r \nabla \varphi dxdt \\ &\quad + \sum_{i=1}^N \iint_{Q_T} a_i\left(u_n^{r(x)}, x, t\right) \varphi x_i dxdt \\ &= \iint_{Q_T} f(u_n, x, t) \varphi(x,t) dxdt,\end{aligned} \tag{33}$$

from this equality, by (27) and (31), it is easy to get

$$\|u_{nt}\|_{L^2}\left(0, T; W^{-1,2}(\Omega)\right) \le c. \tag{34}$$

Moreover, from (27), there is a nonnegative function, $u_n \rightharpoonup u$ weakly star in $L^\infty(Q_T)$. For any $\phi \in C_0^1(\Omega), 0 \le \phi \le 1$, by (34), we can show that

$$\left\|\left(\phi u_n^{r(x)}\right) t\right\|_{L^2\left(0,T;W^{-1,2}(\Omega)\right)} \le c. \tag{35}$$

Since $H_0^s(\Omega) \hookrightarrow W^{1,2}(\Omega)$ when $s > (N/2) + 1$, then $W^{-1,2}(\Omega) \hookrightarrow H^{-s}(\Omega)$. Accordingly,

$$\left\|\left(\phi u_n^{r(x)}\right)_t\right\|_{L^2(0,T;H^{-s}(\Omega))} \le c. \tag{36}$$

In addition, we have

$$\iint_{Q_T} \left|\nabla\left(\phi u_n^{r(x)}\right|^2 dxdt \le c(\phi) \left(1 + \int_0^T \int_{\Omega_\phi} \left|\nabla u_n^{r(x)}\right|^2 dxdt\right) \le c(\phi), \tag{37}$$

where $\Omega_\phi = \operatorname{supp} \phi$. Thus,

$$\left\|\phi u_n^{r(x)}\right\|_{L^2\left(0,T;W_0^{1,2}(\Omega)\right)} \le c. \tag{38}$$

Since $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (\Omega)$, Aubin's compactness theorem in [31] yields that there is a function v such that $\phi u_n^{r(x)} \longrightarrow \phi v$ strongly in $L^2(Q_T)$. Then, $\phi u_n^{r(x)} \longrightarrow \phi v$ almost everywhere in Q_T, by that $u_n \rightharpoonup u$ weakly star in $L^\infty(Q_T)$, we know that $v = u^{r(x)}$. The arbitraries of ϕ yields $u_n \longrightarrow u$ almost everywhere in Q_T.

Thus, since $a_i(s,x,t)$ and $f(u,s,t)$ are continuous functions on $\mathbb{R} \times \bar{Q}_T$), we have

$$a_i(u_n, x, t) \longrightarrow a_i(u, x, t), \text{a.e.in } Q_T, i = 1, 2, \cdots, N, \tag{39}$$

$$f(u_n, x, t) \longrightarrow f(u, x, t), \text{a.e.in } Q_T. \tag{40}$$

Let $n \longrightarrow \infty$ in (29). By (32), (39), and (40), we know u (x,t) satisfies (13) and (14).

The last but not the least, by a process of limit, in a similar way as that in [17], we can choose the test function $\varphi(x,t) = \chi_{[t_1,t_2]}\phi(x)$ in which $\phi(x) \in C_0^\infty(\Omega)$ and $\chi_{[t_1,t_2]}$ is the characteristic function of $[t_1, t_2] \subset (0, T)$. Then

$$\begin{aligned}&\int_{t_1}^{t_2} \int_\Omega \left[b(x,t) u^{r(x)-1} \nabla u \nabla \phi + \sum_{i=1}^N a_i\left(u^{r(x)}, x, t\right) \phi_{x_i} - f(u,x,t)\phi(x)\right] dxdt \\ &= \int_\Omega (u(x,t_2) - u(x,t_1)) \phi(x) dx,\end{aligned} \tag{41}$$

Let $t = t_2$ and $t_1 \longrightarrow 0$. Then we have (15).

Finally, since $b(x,t) > 0$ when $x \in \Sigma$ in (10), by (32), we know that the trace of $u(x,t)$ on Σ is well defined, the details are given in Section 4 as follows. Till now, we have shown that u is a weak solution of equation (1) with the initial value (6) and the partial boundary value condition (10).

Proof of Theorem 4. We consider the regularized problem (26) and obtain (27)–(32) as in the proof Theorem 3.

Let us multiply $u_{nt}^{r(x)}$ on both sides of equation (26), integrate it over $Q_t = \Omega \times (0, t)$. Then

$$\begin{aligned}\iint_{Q_t} r(x) u^{r(x)-1} u_{nt}^2 dxdt = \iint_{Q_t} \operatorname{div}\left(\left(b(x,t) + \frac{1}{n}\right)|u_n|^{r(x)-1}\nabla u_n\right) \\ \cdot u_{nt}^{r(x)} dxdt + \iint_{Q_t} f(u_n, x, t) u_{nt}^{r(x)} dxdt \\ + \sum_{i=1}^{N} \iint_{Q_t} \frac{\partial a_i\left(u_n^{r(x)}, x, t\right)}{\partial x_i} u_{nt}^{r(x)} dxdt.\end{aligned} \tag{42}$$

Since $u_n \geq 0$,

$$\begin{aligned}u_n^{r(x)-1}\nabla u_n = \frac{1}{r(x)}\nabla u_n^{r(x)} - \frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n, \\ \iint_{Q_t} \operatorname{div}\left(\left(b(x,t) + \frac{1}{n}\right)|u_n|^{r(x)-1}\nabla u_n\right) u_{nt}^{r(x)} dxdt \\ = \iint_{Q_t} \operatorname{div}\left(\left(b(x,t) + \frac{1}{n}\right)\left(\frac{1}{r(x)}\nabla u_n^{r(x)} - \frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n\right)\right) u_{nt}^{r(x)} dxdt \\ = -\iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{1}{2r(x)}\frac{\partial}{\partial t}\left|\nabla u_n^{r(x)}\right|^2 dxdt \\ + \iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \frac{\partial}{\partial t}\nabla u_n^{r(x)} dxdt,\end{aligned} \tag{43}$$

we have

$$\begin{aligned}\iint_{Q_t} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt + \iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{1}{2r(x)}\frac{\partial}{\partial t}\left|\nabla u_n^{r(x)}\right|^2 dxdt \\ = \iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \frac{\partial}{\partial t}\nabla u_n^{r(x)} dxdt \\ + \iint_{Q_t} f(u_n, x, t) u_{nt}^{r(x)} dxdt.\end{aligned} \tag{44}$$

Since $b_t(x,t) \leq 0$,

$$\begin{aligned}-\iint_{Q_t} b_t(x,t)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \nabla u_n^{r(x)} dxdt \\ \leq \frac{1}{2}\iint_{Q_t} \frac{-b_t(x,t)}{2r(x)}\left[\left|\nabla u_n^{r(x)}\right|^2 + 4\left(|\nabla r| u_n^{r(x)} \ln u_n\right)^2\right] dxdt \\ \leq \frac{1}{2}\iint_{Q_t} \frac{-b_t(x,t)}{2r(x)}\left|\nabla u_n^{r(x)}\right|^2 dxdt + c,\end{aligned} \tag{45}$$

by the assumption $r(x) \geq r^- > 1$,

$$\begin{aligned}-\iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)}\left[r(x) u_n^{r(x)-1} \ln u_n + u_n^{r(x)-1}\right] u_{nt} dxdt \\ \leq \frac{1}{4}\iint_{Q_t} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt + 4\iint_{Q_t} \frac{u_n^{r(x)-1}}{r(x)} \\ \cdot\left[\left(b(x,t) + \frac{1}{n}\right)\frac{|\nabla r|}{r(x)}\left(r(x) \ln u_n + 1\right)\right]^2 dxdt \\ \leq \frac{1}{4}\iint_{Q_t} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt + c,\end{aligned} \tag{46}$$

we have

$$\begin{aligned}\iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \frac{\partial}{\partial t}\nabla u_n^{r(x)} dxdt \\ = \int_{\Omega} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \nabla u_n^{r(x)} dx\Big|_0^t \\ - \iint_{Q_t} b_t(x,t)\frac{\nabla r}{r(x)} u_n^{r(x)} \ln u_n \nabla u_n^{r(x)} dxdt \\ - \iint_{Q_t} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)}\left[r(x) u_n^{r(x)-1} \ln u_n + u_n^{r(x)-1}\right] u_{nt} dxdt \\ \leq \int_{\Omega} \left(b(x,t) + \frac{1}{n}\right)\frac{\nabla r}{r(x)} u_n^{r(x)+1} \ln u_n \nabla u_n^{r(x)} dx\Big|_0^t \\ + \frac{1}{2}\iint_{Q_t} \frac{-b_t(x,t)}{r(x)}\left|\nabla u_n^{r(x)}\right|^2 dxdt + \frac{1}{4}\iint_{Q_t} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt + c.\end{aligned} \tag{47}$$

By the assumption $|(\partial/\partial u_n) a_i(u_n^{r(x)}, x, t)|^2 b(x,t)^{-1} \leq c$ and

$$\left|\frac{\partial}{\partial x_i} a_i\left(u_n^{r(x)}, x, t\right)\right| \leq c(M) \tag{48}$$

when $|u_n| \leq M + 1$, we have

$$\begin{aligned}\iint_{Q_T} \frac{\partial a_i\left(u_n^{r(x)}, x, t\right)}{\partial x_i} u_{nt}^{r(x)} dxdt \\ = \iint_{Q_T} r(x) u_n^{r(x)-1}\left[\frac{\partial}{\partial u_n} a_i\left(u_n^{r(x)}, x, t\right) u_{nx_i}^{r(x)} \right. \\ \left. + \frac{\partial}{\partial x_i} a_i\left(u_n^{r(x)}, x, t\right)\right] u_{nt} dxdt \\ \leq 4\iint_{Q_T} \left|\frac{\partial}{\partial u_n} a_i\left(u_n^{r(x)}, x, t\right)\right|^2 b(x,t)^{-1} r(x) u_n^{r(x)-1} b(x,t)\left|u_{nx_i}^{r(x)}\right|^2 dxdt \\ + \frac{1}{4}\iint_{Q_T} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt \\ + 4\iint_{Q_T} \left|\frac{\partial}{\partial x_i} a_i\left(u_n^{r(x)}, x, t\right)\right|^2 r(x) u_n^{r(x)-1} dxdt \\ + \frac{1}{4}\iint_{Q_T} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt \\ \leq c + c\iint_{Q_T} b(x,t)\left|\nabla u_n^{r(x)}\right|^2 dxdt + \frac{1}{4}\iint_{Q_T} r(x) u_n^{r(x)-1} u_{nt}^2 dxdt.\end{aligned} \tag{49}$$

Moreover, we have

$$\left|\iint_{Q_T} f(u_n,x,t)u_{nt}dxdt\right| \le c + \frac{1}{4}\iint_{Q_T} r(x)u_n^{r(x)-1}u_{nt}^2dxdt, \tag{50}$$

$$\begin{aligned}\iint_{Q_t}\left(b(x,t)+\frac{1}{n}\right)\frac{1}{2r(x)}\frac{\partial}{\partial t}\left|\nabla u_n^{r(x)}\right|^2 dxdt = \frac{1}{2}\int_\Omega\\ \left(b(x,t)+\frac{1}{n}\right)\frac{1}{r(x)}\left|\nabla u_n^{r(x)}\right|^2 dx|_0^t - \frac{1}{2}\iint_{Q_t}\\ \cdot\frac{b_t(x,t)}{r(x)}\left|\nabla u_n^{r(x)}\right|^2 dxdt.\end{aligned} \tag{51}$$

Now, combining with (44)–(51), we have

$$\iint_{Q_t} r(x)u_n^{r(x)-1}u_{nt}^2dxdt \le c. \tag{52}$$

Then

$$\iint_{Q_t}\left|u_{nt}^{r(x)}\right|^2 dxdt \le c. \tag{53}$$

Since $b(x,t)>0$ when $x\in\Omega$, by Sobolev embedding theorem, (32) and (53) implies that $u_n^{r(x)} \longrightarrow u_1$ a.e. in Q_T. Since $u_n \rightharpoonup u$ weakly star in $L^\infty(Q_T)$, by the uniqueness property of the weak convergence, we know that $u_1 = u^{r(x)}$.

Accordingly, $a_i(s,x,t)\in C^1(\mathbb{R}\times\bar{Q}_T)$, we have

$$\lim_{n\to\infty} a_i\left(u_n^{r(x)},x,t\right) = a_i\left(u^{r(x)},x,t\right), i=1,2,\cdots,N, \tag{54}$$

as well as

$$\lim_{n\to\infty} f\left(u_n^{r(x)},x,t\right) = f\left(u^{r(x)},x,t\right), \tag{55}$$

by that $f(u,x,t)$ is a contunuous function.

Moreover, for any $\varphi(x,t)\in C_0^1(Q_T)$,

$$\begin{aligned}&\lim_{n\to 0}\iint_{Q_T}\left(\sqrt{u_n^{r(x)-1}}u_{nt} - \sqrt{u^{r(x)-1}}u_t\right)\varphi(x,t)dxdt\\ &= \lim_{n\to 0}\iint_{Q_T}\frac{2}{r(x)+1}\left(u_{nt}^{\frac{r(x)-1}{2}+1} - u_t^{\frac{r(x-1)}{2}+1}\right)\varphi(x,t)dxdt\\ &= -\lim_{n\to 0}\iint_{Q_T}\frac{2}{r(x)+1}\left(u_n^{\frac{r(x)-1}{2}+1} - u^{\frac{r(x)-1}{2}+1}\right)\varphi_t(x,t)dxdt = 0.\end{aligned} \tag{56}$$

Thus,

$$\sqrt{u_n^{r(x)-1}}u_{nt} \rightharpoonup \sqrt{u^{r(x)-1}}u_t, \text{ in } L^2(Q_T). \tag{57}$$

Let $n \longrightarrow \infty$ in (29). By (32), (35), (39), (56), and (57), we know $u(x,t)$ satisfies (14) and (16). The initial value (6) and the partial boundary value condition have the same meaning as those in Theorem 3.

3. The Stability Based on the Partial Boundary Value Condition

In this section, we will prove Theorem 5.

Lemma 6. *Let $u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ for any $p>1$, $p'=p/p-1$. For any continuous function $h(s)$, $H(s)=\int_0^s h(s)ds$, a. e. $t_1, t_2\in(0,T)$,*

$$\int_{t_1}^{t_2}\int_\Omega h(u)u_tdxdt = \left[\int_\Omega (H(u)(x,t_2) - H(u)(x,t_1))dx\right]. \tag{58}$$

This Lemma can be seen as a generalization of Lemma 2.2 in [32]. We postpone the reader to it for details.

Theorem 7. *Let $0<r^- \le r(x)\in C^1(\bar{\Omega})$, $f(s,x,t)$ be a Lipschitz function on $\mathbb{R}\times\bar{Q}_T$, $b(x,t)\in C(Q_T)$ with that $b(x,t)>0$ if $x\in\Omega$ and for every given $t\in[0,T)$, $b(x,t)$ be a differential function on the spatial variables x_i, $i=1,2,\cdots,N$, $a_i(s,x,t)\in C(\mathbb{R}\times\bar{Q}_T)$ and satisfy*

$$\int_\Omega b(x,t)^{-1}|\nabla b|^2dx \le c, \tag{59}$$

$$\begin{aligned}&\left|a_i\left(u^{r(x)},x,t\right) - a_i\left(v^{r(x)},x,t\right)\right| \le cg_i(x,t),\\ &\cdot\int_\Omega\left|\frac{b_{x_i}(x,t)g_i(x,t)}{b(x,t)}\right|dx < \infty.\end{aligned} \tag{60}$$

If $u(x,t), v(x,t)$ are two nonnegative weak solutions of equation (1) with the initial value $u_0(x), v_0(x)$, respectively, with the same homogeneous partial boundary value condition

$$u(x,t)=v(x,t)=0, (x,t)\in\Sigma=\{(x,t)\in\partial\Omega\times(0,T): b(x,t)>0\}, \tag{61}$$

and satisfying

$$\int_\Omega [\log u]^2|\nabla r|^2dx \le c, \int_\Omega [\log v]^2|\nabla r|^2dx \le c, \tag{62}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|dx \le c\int_\Omega |u_0(x)-v_0(x)|dx. \tag{63}$$

Proof. Let $g_n(s)=\int_0^s h_n(\tau)d\tau$, $h_n(s)=2n(1-n|s|)_+$. Here and the after, n is a positive integer.

$$\lim_{n\to\infty} g_n(s) = \text{sgn } s, \lim_{n\to\infty} sh_n(s) = 0. \tag{64}$$

Supposed that $\chi_{[\tau,s]}(t)$ is the characteristic function of $[\tau,s]\subset(0,T)$, since $u(x,t)$ and $v(x,t)$ are with the same homogeneous partial boundary value condition (61), we

then choose

$$\chi_{[\tau,s]}(t)g_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right) \tag{65}$$

as the test function. Thus

$$\begin{aligned}
&\int_\tau^s\int_\Omega g_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)\frac{\partial(u-v)}{\partial t}dxdt \\
&\quad+\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left(\nabla u^{r(x)}-\nabla v^{r(x)}\right)\nabla\left(u^{r(x)}-v^{r(x)}\right) \\
&\quad\cdot b(x,t)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt \\
&\quad+\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left(\nabla u^{r(x)}-\nabla v^{(r(x)}\right)\cdot\nabla b\left(u^{r(x)}-v^{r(x)}\right) \\
&\quad\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt \\
&\quad-\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left[u^{r(x)}\log u-v^{r(x)}\log v\right]\nabla r \\
&\quad\cdot\nabla\left(u^{r(x)}-v^{r(x)}\right)b(x,t)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt \\
&\quad-\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left[u^{r(x)}\log u-v^{r(x)}\log v\right]\nabla r \\
&\quad\cdot\nabla b\left(u^{r(x)}-v^{r(x)}\right)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt \\
&\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega\left[a_i\left(u^{r(x)},x,t\right)-b_i\left(v^{r(x)},x,t\right)\right] \\
&\quad\cdot\left[b_{x_i}(x,t)\left(u^{r(x)}-v^{r(x)}\right)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)\right. \\
&\quad\left.+b(x,t)\left(u^{r(x)}-v^{r(x)}\right)_{x_i}h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)\right]dxdt \\
&=\int_\tau^s\int_\Omega[f(u,x,t)-f(v,x,t)]g_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt.
\end{aligned} \tag{66}$$

In the first place

$$\begin{aligned}
&\int_\Omega\frac{b(x,t)^2}{r(x)}\left(\nabla u^{r(x)}-\nabla v^{r(x)}\right)\cdot\nabla\left(u^{r(x)}-v^{r(x)}\right) \\
&\quad\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dx\geq 0.
\end{aligned} \tag{67}$$

In the second place, by the Hölder inequality, using (59) and (64), we obtain

$$\begin{aligned}
&\lim_{n\to 0}\left|\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left(\nabla u^{r(x)}-\nabla v^{(r(x)}\right)\nabla b\left(u^{r(x)}-v^{r(x)}\right)h_n\right. \\
&\quad\left.\cdot\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right| \\
&\leq\lim_{n\to 0}\int_\tau^s\int_\Omega\frac{b(x,t)}{r(x)}\left(\left|\nabla u^{r(x)}\right|+\left|\nabla v^{(r(x))}\right|\right)|\nabla b|\left|u^{r(x)}-v^{r(x)}\right| \\
&\quad\cdot h_n\left(b\left((x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right. \\
&\leq c\lim_{n\to 0}\int_\tau^s\left(\int_\Omega b(x,t)\left(\left|\nabla u^{r(x)}\right|^2+\left|\nabla v^{r(x)}\right|^2\right)dx\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\int_\Omega b(x,t)\left[\frac{|\nabla b|}{b(x,t)}b(x,t)\left(u^{r(x)}-v^{r(x)}\right)h_n\right.\right. \\
&\quad\left.\left.\cdot\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)\right]^2dx\right)^{\frac{1}{2}}dt=0.
\end{aligned} \tag{68}$$

In the third place, for simplism, let us denote that

$$D_n=\left\{\Omega:\left|u^{r(x)}-v^{r(x)}\right|<\frac{1}{n}\right\},D_0=\{x\in\Omega:|u-v|=0\} \tag{69}$$

as a consequence

$$\lim_{n\to\infty}D_n=D_0. \tag{70}$$

Since condition (62) implies

$$\begin{aligned}
&\lim_{n\to 0}\left|\int_\tau^s\int_{D_n}b(x,t)\left[u^{r(x)}\log u^{r(x)}-v^{r(x)}\log v^{r(x)}\right]^2h_n\right. \\
&\quad\left.\cdot\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)^2|\nabla r|^2dxdt\right| \\
&\leq\lim_{n\to 0}\int_\tau^s\int_{D_n}\frac{\left[u^{r(x)}\log u^{r(x)}-v^{r(x)}\log v^{r(x)}\right]^2}{\left|u^{r(x)}-v^{r(x)}\right|^2}|\nabla r|^2dxdt \\
&\leq\int_\tau^s\int_{D_0}[1+r(x)\log v]^2|\nabla r|^2dxdt<\infty,
\end{aligned} \tag{71}$$

using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\tau^s\int_{D_n}b(x,t)\left[u^{r(x)}\log u^{r(x)}-v^{r(x)}\log v^{r(x)}\right]^2h_n \\
&\quad\cdot\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)^2|\nabla r|^2dxdt=0,
\end{aligned} \tag{72}$$

which yields

$$\begin{aligned}
&\lim_{n\to\infty}\int_\tau^s\left|\int_\Omega\frac{b(x,t)}{r(x)}\left[u^{r(x)+1}\log u-v^{r(x)}\log v\right]\cdot\nabla rb(x,t)\nabla\right. \\
&\quad\left.\cdot\left(u^{r(x)}-v^{r(x)}\right)h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right| \\
&\leq\lim_{n\to\infty}\int_\tau^s\left(\int_{D_n}b(x,t)\left|\nabla u^{r(x)}-\nabla v^{r(x)}\right|^2dx\right)^{\frac{1}{2}} \\
&\quad\cdot c\left(\int_{D_n}b(x,t)\left[u^{r(x)}\log u^{r(x)}-v^{r(x)}\log v^{r(x)}\right]^2h_n\right. \\
&\quad\left.\cdot\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)^2|\nabla r|^2dx\right)^{\frac{1}{2}}dt=0.
\end{aligned} \tag{73}$$

In the fourth place, since $|\nabla b| \le c$, using (64) and the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega \frac{b(x,t)}{r(x)}\left[u^{r(x)}\log u - v^{r(x)}\log v\right]\nabla r\cdot\nabla b \cdot\left(u^{r(x)}-v^{r(x)}\right)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right| = 0, \tag{74}$$

is obviously.

In the fifth place, since

$$\left|a_i\left(u^{r(x)},x,t\right)-a_i\left(v^{r(x)},x,t\right)\right| \le cg_i(x,t),\ i=1,2,\cdots,N, \tag{75}$$

satisfies (60), using the Lebesgue dominated convergence theorem, we can extrapolate

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega\left[a_i\left(u^{r(x)},x,t\right)-a_i\left(v^{r(x)},x,t\right)\right]b_{x_i}(x,t) \cdot\left(u^{r(x)}-v^{r(x)}\right)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right|\\ &=\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega\left[a_i\left(u^{r(x)},x,t\right)-a_i\left(v^{r(x)},x,t\right)\right]\frac{b_{x_i}(x,t)}{b(x,t)}b(x,t) \cdot\left(u^{r(x)}-v^{r(x)}\right)\cdot h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right| = 0.\end{aligned} \tag{76}$$

Also by the Hölder inequality, we are easily to obtain

$$\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega\left[a_i\left(u^{r(x)},x,t\right)-a_i\left(v^{r(x)}\right),\mathrm{x},t\right] \cdot b(x,t)\left(u^{r(x)}-v^{r(x)}\right)_{x_i}h_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\right| = 0. \tag{77}$$

In the sixth place,

$$\begin{aligned}&\lim_{n\to\infty}\int_\tau^s\int_\Omega [f(u,x,t)-f(v,x,t)]g_n\left(b(x,t)\left(u^{r(x)}-v^{r(x)}\right)\right)dxdt\\ &\le c\int_\tau^s\int_\Omega|u(x,t)-v(x,t)|dxdt.\end{aligned} \tag{78}$$

At last, by Lemma 6, choosing $p=p'=2$, by (64), we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\tau^s\int_\Omega g_n\left(b\left(u^{r(x)}-v^{r(x)}\right)\right)\frac{\partial(u-v)}{\partial t}dx\\ &=\int_\tau^s\int_\Omega|u(x,s)-v(x,s)|dx-\int_\Omega|u(x,\tau)-v(x,\tau)|dxdt.\end{aligned} \tag{79}$$

Then, letting $n\to\infty$ in (66), we have

$$\begin{aligned}\int_\Omega|u(x,s)-v(x,s)|dx &\le c\int_\Omega|u(x,\tau)-v(x,\tau)|dx\\ &+\int_\tau^s\int_\Omega|u(x,s)-v(x,s)|dx\end{aligned} \tag{80}$$

By the Gronwall inequality, we have

$$\int_\Omega|u(x,s)-v(x,s)|dx \le c\int_\Omega|u(x,\tau)-v(x,\tau)|dx. \tag{81}$$

By the arbitraries of τ, we have

$$\int_\Omega|u(x,s)-v(x,s)|dx \leqslant \int_\Omega|u_0(x)-v_0(x)|dx. \tag{82}$$

At last, since conditions (22), (23), and (24) imply all the assumptions in Theorem 7 are true, we have Theorem 4.

4. The Explanation of the Partial Boundary Value Condition

In [21], condition (7) guarantees

$$\int_\Omega\left|\nabla u^{r(x)}\right|dx<\infty, \tag{83}$$

then we can define the trace of $u^{r(x)}$ on $\partial\Omega$, and the homogeneous boundary value condition (9) is valid. But in this paper, we do not assume

$$\int_\Omega[b(x,t)]^{-1}dx \le c(T), \tag{84}$$

which is similar as condition (7), how to guarantee the partial boundary value condition (10) is true in the sense of the trace? Actually, by the character of $u(x,t)$ defined in Definition 1

$$\sqrt{b(x,t)}|u|^{r(x)-1}|\nabla u| \in L^\infty\left(0,T;L^2(\Omega)\right), \tag{85}$$

it is not difficult to show that

$$b(x,t)\left|\nabla u^{r(x)}\right|^2 \in L^1(Q_T). \tag{86}$$

Now, for a given small constant δ and for any $t\in(0,T)$, if we denote

$$\Omega_{1\delta\lambda t}=\{x\in\Omega: b(x,t)>\delta\}\bigcap\{x\in\Omega: d(x)<\lambda\} \tag{87}$$

where $d(x)=\mathrm{dist}(x,\partial\Omega)$ is the distance function from the boundary $\partial\Omega$, then

$$\lim_{\lambda\to 0}\Omega_{1\delta\lambda t}=\Sigma_{1\delta t}=\{x\in\partial\Omega: b(x,t)>\delta\}. \tag{88}$$

If we denote that

$$\begin{gathered}\Sigma_0=\{(x,t)\in\partial\Omega\times(0,T)\colon b(x,t)=0\},\\ \Sigma=\{(x,t)\in\partial\Omega\times(0,T)\colon b(x,t)>0\},\\ \sum_0\bigcap\Sigma=\partial\Sigma,\end{gathered}\tag{89}$$

then

$$\bigcup_{\delta>0}\Sigma_{1\delta t}\times(0,T)=\Sigma\bigcup\partial\Sigma.\tag{90}$$

Since $b(x,t)$ satisfies (4) and (86), we have

$$\int_{\Omega_{1\delta t}}\left|\nabla u^{r(x)}\right|dx\le c(\delta)\int_{\Omega_{1\delta t}}b(x,t)\left|\nabla v^{r(x)}\right|dx<\infty.\tag{91}$$

Then, we can define the trace of $u^{r(x)}$ on $\Sigma_{1\delta t}$. By the arbitrariness of δ, we can define the trace of $u^{r(x)}$ on $\Sigma_{1\delta t}$. In other words, no matter whether condition (84) is true or not, we can always define the trace of $u^{r(x)}$ (also u) on $\Sigma_{1\delta t}$ due to that $b(x,t)>0$ on Σ, and so the partial homogenous value condition (10) is always valid. Certainly, we should point out that, on the remained part of the boundary

$$\Sigma_1=\partial\Omega\times(0,T)\setminus\Sigma=\{(x,t)\in\partial\Omega\times(0,T)\colon b(x,t)=0\}\tag{92}$$

whether the trace of $u^{r(x)}$ can be defined or not is unknown, unless condition (84) is supplied.

5. Conclusion

Since the beginning of this century, the evolutionary $p(x)$-Laplacian equations $u_t=\operatorname{div}\,(b(u,x,t)|\nabla u|^{p(x)-2}\nabla u)+g(\nabla u,u,x,t)$ have been studied by many mathematicians in [32–37]. So far, however, there has been little discussion about the porous medium equation with nonstandard growth conditions. Therefore, this study makes a major contribution to research in the related fields by imposing a reasonable partial boundary value condition matching up with the equation. The unusual is that this partial boundary value condition is based on a submanifold $\Sigma\subset\partial\Omega\times(0,T)$. In addition, since the equation is with nonstandard growth conditions, how to deal with the nonlinearity becomes difficult, especially when we try to prove that $\sqrt{u^{r(x)-1}}u_t\in L^2(Q_T)$.

Data Availability

There is not any data in this paper.

Conflicts of Interest

The author declares that he has no competing interests.

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