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Research Article

Inclusion Relations between α-Modulation Spaces and Triebel–Lizorkin Spaces

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In this paper, we obtain conditions of the inclusion relations between α-modulation spaces and Triebel–Lizorkin spaces.

1. Introduction

The modulation space $M^s_{p,q}$ was first introduced by Feichtinger [1] in 1983 by the short-time Fourier transform. modulation space has a close relationship with the topics of time-frequency analysis (see [2]), and it has been regarded as a appropriate space for the study of partial differential equations (see [3–5]).

The α-modulation space is introduced by Gröbner [6] to link Besov and modulation spaces by the parameter $0 \le \alpha \le 1$. One can find some basic properties about α-modulation spaces in [7, 8]. Among many features of the α-modulation spaces, an interesting subject is the inclusion between α-modulation and function spaces, have been concerned by many authors to this topic, see [8–11]. As applications, α-modulation spaces are quite recently applied to the field of partial differential equations. In [12], Misiolek and Yoneda proved locally ill-posedness of the Euler equations in the frame of α-modulation spaces. Furthermore, Han and Wang [13] proved a global well-posedness for the nonlinear Schrödinger equations on α-modulation spaces, and also in [14] studied the Cauchy problem for the derivative nonlinear Schrödinger equation on α-modulation spaces.

Remark 1. Modulation spaces are special α-modulation spaces in the case $\alpha = 0$, so our theorems also works well in the special case $\alpha = 0$.

In this research, we are interested in studying the inclusion relations between α-modulation spaces $M^{s,\alpha}_{p,q}$ and Triebel–Lizorkin spaces $F_{p,r}$ for $p \le 1$, which greatly improve and extend the results for the inclusion relations between local Hardy spaces and α-modulation spaces obtained by Kato in [10].

2. Preliminaries

The notation $X \lesssim Y$ denotes the statement that $X \le CY$ with a positive constant C that may depend on n, α, p, q, s, r. The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, and the notation $X \simeq Y$ stands for $X = CY$. For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \max_{i=1,2,\dots,n}|k_i|$, $|k| = |k_1| + \cdots + |k_n|$ and $< k >:= (1 + |k|^2)^{1/2}$.

Let $S := S(\mathbb{R}^n)$ be the Schwartz space and $S' := S'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in S(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x.\xi} dx \quad \text{and} \qquad (1)$$
$$\mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x.\xi} d\xi.$$

We give some definitions and properties of sequences.

Definition 2. Let $0 < p, q \le \infty, s \in \mathbb{R}, \alpha \in [0, 1)$. Let $\{a_k\}_{k\in\mathbb{Z}^n}$ be a sequence, we denote its $\ell^{s,\alpha}_p$ (quasi-) norm

$$\|\{a_k\}\|_{\ell^{s,\alpha}_p} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^n} |a_k|^p \langle k\rangle^{sp/1-\alpha}\right)^{1/p}; & 0 < p < \infty, \\ \sup_{k\in\mathbb{Z}^n} |a_k| \langle k\rangle^{s/1-\alpha}; & p = \infty, \end{cases} \qquad (2)$$

and let $\ell_p^{s,\alpha}$ be the (quasi-) Banach space of sequences whose $\ell_p^{s,\alpha}$ (quasi-) norm is finite.

Let $\{b_j\}_{j\in\mathbb{N}}$ be a sequence, we denote its $\ell_p^{s,1}$ (quasi-) norm

$$\left\|\{b_j\}\right\|_{\ell_p^{s,1}} = \begin{cases} \left(\sum_{j\in\mathbb{N}_0} |b_j|^p 2^{jsp}\right)^{1/p}; & 0 < p < \infty, \\ \sup_{j\in\mathbb{N}_0} |b_j| 2^{js}; & p = \infty, \end{cases} \tag{3}$$

and let $\ell_p^{s,1}$ be the (quasi-) Banach space of sequences whose $\ell_p^{s,1}$ (quasi-) norm is finite.

Let $\{c_k\}_{k\in\mathbb{Z}^n}$ be a sequence, we denote its $\ell_p^{s,0}$ (quasi-) norm

$$\left\|\{c_k\}\right\|_{\ell_p^{s,0}} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^n} |c_k|^p \langle k\rangle^{sp}\right)^{1/p}; & 0 < p < \infty, \\ \sup_{k\in\mathbb{Z}^n} |c_k| \langle k\rangle^{s}; & p = \infty, \end{cases} \tag{4}$$

and let $\ell_p^{s,0}$ be the (quasi-) Banach space of sequences whose $\ell_p^{s,0}$ (quasi-) norm is finite.

We recall some embedding lemmas about sequences defined above.

Lemma 3 (sharpness of embedding, for uniform decomposition). *Suppose* $0 < p, q \le \infty$, $s_1, s_2 \in \mathbb{R}$. *Then*

$$\ell_p^{s_1,0} \subset \ell_q^{s_2,0}, \tag{5}$$

holds if and only if

$$\begin{cases} s_2 \le s_1, \\ \frac{1}{q} + \frac{s_2}{n} < \frac{1}{p} + \frac{s_1}{n}, \end{cases} \quad \text{or} \quad \begin{cases} s_1 = s_2, \\ p = q. \end{cases} \tag{6}$$

Lemma 4 (sharpness of embedding, for dyadic decomposition). *Suppose* $0 < p, q \le \infty$, $s_1, s_2 \in \mathbb{R}$. *Then.*

$$\ell_p^{s_1,1} \subset \ell_q^{s_2,1}, \tag{7}$$

holds if and only if

$$s_2 < s_1 \text{ or } \begin{cases} s_1 = s_2, \\ \frac{1}{q} \le \frac{1}{p}. \end{cases} \tag{8}$$

Lemma 5 (sharpness of embedding, for α-decomposition). *Suppose* $0 < p, q \le \infty$, $s_1, s_2 \in \mathbb{R}$, $\alpha \in [0,1)$. *Then*

$$\ell_p^{s_1,\alpha} \subset \ell_q^{s_2,\alpha}, \tag{9}$$

holds if and only if

$$\frac{1-\alpha}{q} + \frac{s_2}{n} < \frac{1-\alpha}{p} + \frac{s_1}{n} \quad \text{or} \quad \begin{cases} s_1 = s_2, \\ \frac{1}{q} \le \frac{1}{p}. \end{cases} \tag{10}$$

We recall some definitions of the function spaces treated in this paper.

Suppose that $c > 0$ and $C > 0$ are two appropriate constants, which relate to the space dimension n, and a Schwartz functions sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ satisfies

$$\begin{cases} \left|\eta_k^\alpha(\xi)\right| \ge 1, \quad if \left|\xi - \langle k\rangle^{\alpha/1-\alpha} k\right| < c\langle k\rangle^{\alpha/1-\alpha}; \\ \operatorname{supp} \eta_k^\alpha \subset \left\{\xi : \left|\xi - \langle k\rangle^{\alpha/1-\alpha} k\right| < C\langle k\rangle^{\alpha/1-\alpha}\right\}; \\ \sum_{k\in\mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n; \\ \left|D^\delta \eta_k^\alpha(\xi)\right| \lesssim \langle k\rangle^{-(\alpha|\delta|/1-\alpha)}, \quad \forall \xi \in \mathbb{R}^n, \delta \in \left(\mathbb{Z}^+ \bigcup \{0\}\right)^n. \end{cases} \tag{11}$$

Then $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ constitutes a smooth decomposition of $\mathbb{R}^n$. The frequency decomposition operators associated with the above function sequence are defined by

$$\Box_k^\alpha := \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F}, \tag{12}$$

for $k \in \mathbb{Z}^n$. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$, and $\alpha \in [0,1)$. Then the α-modulation space associated with the above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq/1-\alpha} \left\|\Box_k^\alpha f\right\|_{L_p}^q\right)^{1/q} < \infty \right\}, \tag{13}$$

with the usual modifications when $q = \infty$. For simplicity, we denote $M_{p,q}^s = M_{p,q}^{s,0}$, $M_{p,q} = M_{p,q}^{0,0}$, and $\eta_k(\xi) = \eta_k^0(\xi)$.

Remark 6. We recall that the above definition is independent of the choice of exact η_k^α (see [8], proposition 2.3). Also, for sufficiently small $\delta > 0$, one can construct a function sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ such that $\eta_k^\alpha(\xi)$=1 and $\eta_k^\alpha(\xi)\eta_\ell^\alpha(\xi) = 0$ if $k \ne \ell$, when ξ lies in the ball $B\left(\langle k\rangle^{\alpha/1-\alpha} k, \langle k\rangle^{\alpha/1-\alpha}\delta\right)$ (see [15, 9, Appendix A]).

To define the Bosov spaces and Triebel–Lizorkin spaces, we introduce the dyadic decomposition of $\mathbb{R}^n$. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $\{\xi : |\xi| < (3/2)\}$ and be equal to 1 on the ball $\{\xi : |\xi| \le (4/3)\}$. For integers $j \in \mathbb{N}$, we define the Littewood–Paley operators

$$\widehat{\Delta_j f} = \left(\varphi\left(2^{-j}\xi\right) - \varphi\left(2^{-j+1}\xi\right)\right)\hat{f}(\xi), \tag{14}$$

$$\widehat{\Delta_0 f} = \varphi(\xi) f(\xi). \tag{15}$$

Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then the Besov spaces is defined by

$$B_{p,q}^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^\infty 2^{jsq} \left\|\Delta_j f\right\|_{L_p}^q\right)^{1/q} < \infty \right\}. \tag{16}$$

Let $0 < p < \infty$, $0 < r \le \infty$, and $s \in \mathbb{R}$. Then the Triebel–Lizorkin spaces is defined by

$$F_{p,r}^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{F_{p,r}^s} = \left\| \left(\sum_{j=0}^\infty 2^{jsr} \left|\Delta_j f\right|^r\right)^{1/r} \right\|_{L_p} < \infty \right\}. \tag{17}$$

Let $\mathbb{Q}^n$ be the collection of all cubes Q_{vk} in $\mathbb{R}^n$ with sides parallel to the axes, centered at $2^{-v}k$, and with side length 2^{-v}, where $k \in \mathbb{Z}^n$ and $v \in \mathbb{N}_0$.

Let Q be a cube in $\mathbb{R}^n$ and $m > 0$, then mQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length m times the side length of Q. We write $(v,k) < (v',k')$ if $v \geq v'$ and

$$Q_{v,k} \subset 2Q_{v',k'} \quad \text{with} \quad Q_{v,k}, 2Q_{v',k'} \in \mathbb{Q}^n. \tag{18}$$

Let $a \in \mathbb{R}$, then $a_+ = \max(a,0)$ and $[a]$ stands for the largest integer less than or equal to a.

Definition 7 (see [16]). Let $s \in \mathbb{R}$, $0 < p \leq 1 < r \leq \infty$. Let K and L be integers with

$$K \geq ([s]+1)_+ \text{and} \quad L \geq \max\left\{\left[n\left(\frac{1}{p}-1\right)-s\right], -1\right\}. \tag{19}$$

(1) The (complex-valued) function $a(x)$ is called a s-atom if

$$\operatorname{supp} a \subset 5Q, \tag{20}$$

for some $Q = Q_{0,k} \in \mathbb{Q}^n$ and

$$\left|D^\alpha a(x)\right| \leq 1 \text{ for } \quad |\alpha| \leq K. \tag{21}$$

(2) Let $Q = Q_{v,k} \in \mathbb{Q}^n$. The (complex-valued) function $a(x)$ is called a (Q,s,p,r)-atom if (20) is satisfied,

$$\left|D^\alpha a(x)\right| \leq |Q|^{-(1/r)+(s/n)-(|\alpha|/n)} \quad \text{for} \quad |\alpha| \leq K, \tag{22}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0 \quad \text{for} \quad |\beta| \leq L. \tag{23}$$

(3) The distribution $g \in S'$ is called an (s,p,r)-atom if

$$g = \sum_{(\mu,l)<(v,k)} d_{\mu l} a_{\mu l}(x) \quad \left(\text{convergence in } F^s_{p,r}\right), \tag{24}$$

for some $v \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$, where $a_{\mu l}(x)$ is a $(Q_{\mu l}, s, p, r)$-atom and $d_{\mu l}$ are complex numbers with

$$\left(\sum_{(\mu,l)<(v,k)} \left|d_{\mu l}\right|^r\right)^{1/r} \leq \left|Q_{v,k}\right|^{(1/r)-(1/p)}, \tag{25}$$

with usual modification if $q = \infty$.

Lemma 8 (see [16]). *Let $s \in \mathbb{R}$, $0 < p \leq 1 < r \leq \infty$. Let K and L be fixed integers satisfying (19). Then $f \in S'$ is an element of $F^s_{p,r}$ if and only if it can be represented as*

$$f = \sum_{j=1}^{\infty} \left(\mu_j a_j + \lambda_j g_j\right) \quad (\text{convergence in } S'), \tag{26}$$

where a_j are s-atoms, g_j are (s,p,r)-atoms, μ_j and λ_j are complex numbers with

$$\left(\sum_{j=1}^{\infty} \left|\mu_j\right|^p + \left|\lambda_j\right|^p\right)^{1/p} \lesssim \|f\|_{F^s_{p,r}}. \tag{27}$$

We also give the following lemma for inclusion relations between Besov and α-modulation spaces [8].

Lemma 9. *Let $0 < p,q \leq \infty$, and $s \in \mathbb{R}$. Then the following tow statement are true:*

(1) $M^{s,\alpha}_{p,q} \subset B_{p,q} \iff s \geq 0 \vee \left[n(\alpha-1)\left(\frac{1}{p}-\frac{1}{q}\right)\right] \vee \left[n(\alpha-1)\left(1-\frac{1}{p}-\frac{1}{q}\right)\right]$.

(2) $B_{p,q} \subset M^{s,\alpha}_{p,q} \iff s \leq 0 \wedge \left[n(\alpha-1)\left(\frac{1}{p}-\frac{1}{q}\right)\right] \wedge \left[n(\alpha-1)\left(1-\frac{1}{p}-\frac{1}{q}\right)\right]$.

Lemma 10 (Young's inequality).

(1) *Let* $0 < p \leq 1, R > 0, \operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g} \subseteq B(x,R) \subseteq \mathbb{R}^n$. We have

$$\left\|f * g\right\|_{L_p} \leq CR^{n((1/p)-1)} \|f\|_{L_p} \|g\|_{L_p}. \tag{28}$$

for all $f, g \in S(\mathbb{R}^n)$ and $R > 0$, where C independent of $x \in \mathbb{R}^n$.

(2) *Let $1 \leq p,q,r \leq \infty$ satisfy $1 + (1/q) = (1/p) + (1/r)$. Then we have*

$$\left\|f * g\right\|_{L_q} \lesssim \|f\|_{L_p} \|g\|_{L_r}. \tag{29}$$

The following Bernstein multiplier theorem will be used in our proof.

Lemma 11 (Bernstein multiplier theorem). *Let $0 < p \leq 1$, $\partial^\delta f \in L_2$ for $|\delta| \leq [n(1-\alpha)((1/p)-(1/2))]+1$. Then,*

$$\left\|\mathcal{F}^{-1} f\right\|_{L_p} \lesssim \sum_{|\delta| \leq [n(1-\alpha)((1/p)-(1/2))]+1} \left\|\partial^\delta f\right\|_{L_2}. \tag{30}$$

3. Main Results

Now, we state our main results as follows.

Theorem 12. *Let $0 < p \leq 1$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha < 1$. Then, $M^{s,\alpha}_{p,q}(\mathbb{R}^n) \subset F_{p,r}(\mathbb{R}^n)$ holds if and only if either of the following conditions is satisfied.*

(1) $p \geq q, s \geq 0, \frac{1}{r} \leq \frac{1}{q}$;

(2) $p < q, s > n(1-\alpha)\left(\frac{1}{p}-\frac{1}{q}\right)$.

Theorem 13. *Let $0 < p \leq 1$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha < 1$. Then, $F_{p,r}(\mathbb{R}^n) \subset M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ holds if and only if either of the following conditions is satisfied.*

(1) $p > q, s < -n(1-\alpha)\left(\frac{1}{p}+\frac{1}{q}-1\right)$;

(2) $p \leq q, s \leq -n(1-\alpha)\left(\frac{1}{p}+\frac{1}{q}-1\right)$.

We prove the following two propositions used for the proof of the Theorem 12.

Proposition 14. *Let $0 < p < \infty$, $0 < q, r \le \infty$, $s \in \mathbb{R}$, and $0 \le \alpha < 1$. Then we have*

(1) $M^{s,\alpha}_{p,q} \subset F_{p,r} \Rightarrow \ell_q^{(s/1-\alpha),0} \subset \ell_p^{0,0}$;
(2) $F_{p,r} \subset M^{s,\alpha}_{p,q} \Rightarrow \ell_p^{0,0} \subset \ell_q^{(s/1-\alpha),0}$.

Proof. Take f to be a smooth function whose Fourier transform has compact small support, denote $\widehat{f_k} = \widehat{f}(. - k)$. We define

$$H_N = \sum_{k\in\mathbb{Z}^n} c_k f_k^N; \quad f_k^N(x) = f_k(x - kN). \tag{31}$$

For a truncated (only finite nonzero items) nonnegative sequence $\{c_k\}_{k\in\mathbb{Z}^n}$, where N is some large integer.

By the definition of α-modulation space $M^{s,\alpha}_{p,q}$, we have

$$\begin{aligned}\|H_N\|_{M^{s,\alpha}_{p,q}} &= \left(\sum_{k\in\mathbb{Z}^n} \|\square_k^\alpha H_N\|_{L_p}^q \langle k\rangle^{sq/1-\alpha}\right)^{1/q} \\ &\sim \left(\sum_{k\in\mathbb{Z}^n} \|c_k f_k^N\|_{L_p}^q \langle k\rangle^{sq/1-\alpha}\right)^{1/q} \\ &\sim \left(\sum_{k\in\mathbb{Z}^n} c_k{}^q \|f\|_{L_p}^q \langle k\rangle^{sq/1-\alpha}\right)^{1/q} \\ &\sim \left(\sum_{k\in\mathbb{Z}^n} c_k{}^q \langle k\rangle^{sq/1-\alpha}\right)^{1/q} \sim \|c_k\|_{\ell_q^{(s/1-\alpha),0}}.\end{aligned} \tag{32}$$

On the other hand, we use the orthogonality of $\{f_k^N\}_{k\in\mathbb{Z}^n}$ as $N \to \infty$, we obtain

$$\begin{aligned}\|H_N\|_{F_{p,r}} \sim \|H_N\|_{L_p} &= \left(\int_{\mathbb{R}^n} \left|\sum_{k\in\mathbb{Z}^n} c_k f_k^N\right|^p dx\right)^{1/p} \\ &\xrightarrow{N\to\infty} \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n} |c_k f_k|^p dx\right)^{1/p} \\ &\sim \left(\sum_{k\in\mathbb{Z}^n} c_k^p dx\right)^{1/p} \sim \|c_k\|_{\ell_p^{0,0}}.\end{aligned} \tag{33}$$

Hence,

$$\|H_N\|_{M^{s,\alpha}_{p,q}} \sim \|c_k\|_{\ell_q^{(s/1-\alpha),0}}, \quad \lim_{N\to\infty} \|H_N\|_{F_{p,r}} \sim \|c_k\|_{\ell_p^{0,0}}, \tag{34}$$

Thus, we obtain $\ell_q^{(s/1-\alpha),0} \subset \ell_p^{0,0}$, if $M^{s,\alpha}_{p,q} \subset F_{p,r}$ hold.

On the other hand, we obtain $\ell_p^{0,0} \subset \ell_q^{(s/1-\alpha),0}$, if $F_{p,r} \subset M^{s,\alpha}_{p,q}$ hold. □

Proposition 15. *Let $0 < p < \infty$, $0 < q, r \le \infty$, $s \in \mathbb{R}$ and $0 \le \alpha < 1$. Then we have*

$$M^{s,\alpha}_{p,q} \subset F_{p,r} \Rightarrow \ell_q^{0,\alpha} \subset \ell_r^{0,\alpha}. \tag{35}$$

Proof. Take f to be a nonzero smooth function whose Fourier transform has small support, such that $\square_k^\alpha f_k = f_k$ and $\square_\ell^\alpha f_k = 0$ if $k \ne \ell$, where we denote $\widehat{f_k} = \widehat{f}\big((\xi - \langle k\rangle^{\alpha/(1-\alpha)}k)/\langle k\rangle^{\alpha/(1-\alpha)}\big)$. Denote

$$E = \sum_{k\in\mathbb{Z}^n} a_k f_k. \tag{36}$$

For a truncated (only finite nonzero items) nonnegative sequence $\{a_k\}_{k\in\mathbb{Z}^n}$. We have

$$\begin{aligned}\|E\|_{F_{p,r}} &= \left\|\left(\sum_{k\in\mathbb{Z}^n} |\Delta_j E|^r\right)^{1/r}\right\|_{L_p} \sim \left\|\left(\sum_{k\in\mathbb{Z}^n} |a_k f_k|^r\right)^{1/r}\right\|_{L_p} \\ &\sim \left\|\left(\sum_{k\in\mathbb{Z}^n} a_k{}^r |f_k|^r\right)^{1/r}\right\|_{L_p} \sim \|a_k\|_{\ell_r^{0,\alpha}}.\end{aligned} \tag{37}$$

On the other hand,

$$\begin{aligned}\|E\|_{M^{0,\alpha}_{p,q}} &= \left(\sum_{k\in\mathbb{Z}^n} \|\square_k^\alpha E\|_{L_p}^q\right)^{1/q} = \left(\sum_{k\in\mathbb{Z}^n} \|a_k f_k\|_{L_p}^q\right)^{1/q} \\ &= \left(\sum_{k\in\mathbb{Z}^n} a_k{}^q \|f\|_{L_p}^q\right)^{1/q} \sim \|a_k\|_{\ell_q^{0,\alpha}}.\end{aligned} \tag{38}$$

Hence,

$$\|E\|_{M^{0,\alpha}_{p,q}} \sim \|a_k\|_{\ell_q^{0,\alpha}}, \quad \|E\|_{F_{p,r}} \sim \|a_k\|_{\ell_r^{0,\alpha}}, \tag{39}$$

Thus, we obtain $\ell_q^{0,\alpha} \subset \ell_r^{0,\alpha}$, if $M^{0,\alpha}_{p,q} \subset F_{p,r}$. □

Proof of Theorem 12. We divide this proof into two parts.

Necessary. For $p \ge q$, using Proposition 14 and Lemma 3 to deduce $\ell_q^{(s/1-\alpha),0} \subset \ell_p^{0,0}$, which implies $s \ge 0$.

On the other hand, we use Proposition 15 and Lemma 5 to deduce $\ell_q^{0,\alpha} \subset \ell_r^{0,\alpha}$, which implies $(1/r) \le (1/q)$.

For $q \ge p$, using Proposition 14 and Lemma 3 to deduce $\ell_q^{(s/1-\alpha),0} \subset \ell_p^{0,0}$, which implies $s > n(1-\alpha)((1/p) - (1/q))$.

Sufficiency. For $p \ge q$. We have $(1/r) \le (1/q)$, and then $F_{p,q} \subset F_{p,r}$. using Lemma 9, we obtain $M^{s,\alpha}_{p,q} \subset B_{p,q}$. Thus we deduce that

$$M^{s,\alpha}_{p,q} \subset B_{p,q} \subset F_{p,q} \subset F_{p,r}. \tag{40}$$

Which is the desired conclusion.

For $p < q$. Use $M^{0,\alpha}_{p,p} \subset F_{p,p}$ obtained above to deduce that

$$M^{n(1-\alpha)((1/p)-(1/q))+2\varepsilon,\alpha}_{p,q} \subset M^{\varepsilon,\alpha}_{p,p} \subset F^{\varepsilon}_{p,p} \subset F_{p,r}, \tag{41}$$

for any $\varepsilon > 0, r \in (0, \infty]$.

We prove the following two propositions used for the proof of the Theorem 13.

Proposition 16. *Let $0 < p < \infty$, $0 < q, r \le \infty$, $s \in \mathbb{R}$, and $0 \le \alpha < 1$. Then we have*

(1) $M^{s,\alpha}_{p,q} \subset F_{p,r} \Rightarrow \ell_q^{(n(1-\alpha))/(q)+n\alpha(1-(1/p))+s,1} \subset \ell_p^{(n(1-(1/p))),1}$;
(2) $F_{p,r} \subset M^{s,\alpha}_{p,q} \Rightarrow \ell_p^{(n(1-(1/p))),1} \subset \ell_q^{(n(1-\alpha))/(q)+n\alpha(1-(1/p))+s,1}$.

Proof. Let g be a nonzero Schwartz function whose Fourier transform has compact support in $\{\xi : 3/4 \le |\xi| \le 4/3\}$, satisfying $g(\xi) = 1$ on $\{\xi : 7/8 \le |\xi| \le 8/7\}$. Set $\widehat{g_j}(\xi) := \widehat{g}(\xi/2^j)$. By the definition of Δ_j, we have $\Delta_j g_j = g_j$ for $j \ge 0$, and $\Delta_l g_j = 0$ if $l \ne j$. Denote

$$\tilde{A}_j = \left\{k \in \mathbb{Z}^n : \square_k^\alpha g_j = \mathcal{F}^{-1}\eta_k^\alpha\right\}, \quad A_j = \left\{k \in \mathbb{Z}^n : \square_k^\alpha g_j \ne 0\right\}, \tag{42}$$

we have $|A_j| \sim \left|\tilde{A}_j\right| \sim 2^{jn(1-\alpha)}$ for $j \ge N$, where N is a sufficiently large number. We define

$$G_N = \sum_{j\ge N} b_j g_j^N; \quad g_j^N(x) = g_j(x - jN), \tag{43}$$

for a truncated (only finite nonzero items) nonnegative sequence $\left\{b_j\right\}_{j=0}^\infty$.

We first prove that the inclusion $M_{p,q}^{s,\alpha} \subset F_{p,r}$ implies $\ell_q^{(n(1-\alpha)/q)+n\alpha(1-(1/p))+s,1} \subset \ell_p^{n(1-(1/p)),1}$. By the definition of α-modulation space, we obtain that

$$\begin{aligned}
\|G_N\|_{M_{p,q}^{s,\alpha}} &= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq/1-\alpha}\left\|\square_k^\alpha G_N\right\|_{L_p}^q\right)^{1/q} \\
&\lesssim \left(\sum_{j\ge N}\sum_{k\in A_j} \langle k\rangle^{sq/1-\alpha} b_j^q \left\|\square_k^\alpha g_j^N\right\|_{L_p}^q\right)^{1/q} \\
&\lesssim \left(\sum_{j\ge N}\sum_{k\in A_j} b_j^q \langle k\rangle^{sq/1-\alpha}\left\|\mathcal{F}^{-1}\eta_k^\alpha\right\|_{L_p}^q\right)^{1/q} \\
&\sim \left(\sum_{j\ge N}\sum_{k\in A_j} b_j^q \langle k\rangle^{sq/1-\alpha}\langle k\rangle^{(n\alpha q/1-\alpha)(1-(1/p))}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} \left|A_j\right| b_j^q \langle k\rangle^{q/1-\alpha[s+n\alpha(1-(1/p))]}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} b_j^q 2^{jn(1-\alpha)} . 2^{jq[s+n\alpha(1-(1/p))]}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} b_j^q 2^{jq[s+n\alpha(1-(1/p))+(n(1-\alpha)/q)]}\right)^{1/q} \\
&\sim \left\|b_j\right\|_{\ell_q^{s+n\alpha(1-(1/p))+(n(1-\alpha)/q),1}}.
\end{aligned} \tag{44}$$

On the other hand, we turn to the estimate of $\left\|G_N\right\|_{F_{p,r}}$, using the orthogonality of $\left\{g_j^N\right\}$ as $N \to \infty$, we obtain

$$\begin{aligned}
\|G_N\|_{F_{p,r}} &= \left\|\left(\sum_{j\in\mathbb{N}_0} \left|\Delta_j G_N\right|^r\right)^{1/r}\right\|_{L_p} \\
&= \left\|\left(\sum_{j=N}^\infty \left|b_j g_j^N\right|^r\right)^{1/r}\right\|_{L_p} \xrightarrow{N\to\infty} \left(\int_{\mathbb{R}^n} \sum_{j=N}^\infty \left|b_j g_j\right|^p dx\right)^{1/p} \\
&\simeq \left(\sum_{j=N}^\infty b_j^p 2^{jn(1-(1/p))p}\right)^{1/p} = \left\|\left\{b_j\right\}_{j\ge N}\right\|_{\ell_p^{n(1-(1/p)),1}}.
\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}
&\|G_N\|_{M_{p,q}^{s,\alpha}} \lesssim \left\|b_j\right\|_{\ell_q^{s+n\alpha(1-(1/p))+(n(1-\alpha)/q),1}}, \\
&\lim_{N\to\infty} \|G_N\|_{F_{p,r}} \sim \left\|b_j\right\|_{\ell_p^{n(1-(1/p)),1}}.
\end{aligned} \tag{46}$$

Thus, if $M_{p,q}^{s,\alpha} \subset F_{p,r}$, we obtain the desired inclusion

$$\ell_q^{n(1-\alpha)/q+n\alpha(1-(1/p))+s,1} \subset \ell_p^{n(1-(1/p)),1}. \tag{47}$$

Next we prove that the inclusion $F_{p,r} \subset M_{p,q}^{s,\alpha}$ implies $\ell pn(1-(1/p)), 1 \subset \ell q(n(1-\alpha)/q) + n\alpha(1-(1/p)) + s, 1$. By the definition of α-modulation space, we obtain that

$$\begin{aligned}
\|G_N\|_{M_{p,q}^{s,\alpha}} &= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq/1-\alpha}\left\|\square_k^\alpha g_N\right\|_{L_p}^q\right)^{1/q} \\
&\ge \left(\sum_{j\ge N}\sum_{k\in\tilde{A}_j} \langle k\rangle^{sq/1-\alpha} b_j^q\left\|\square_k^\alpha g_j^N\right\|_{L_p}^q\right)^{1/q} \\
&= \left(\sum_{j\ge N}\sum_{k\in\tilde{A}_j} b_j^q\langle k\rangle^{sq/1-\alpha}\left\|\mathcal{F}^{-1}\eta_k^\alpha\right\|_{L_p}^q\right)^{1/q} \\
&\sim \left(\sum_{j\ge N}\sum_{k\in\tilde{A}_j} b_j^q\langle k\rangle^{sq/1-\alpha}\langle k\rangle^{(n\alpha q/1-\alpha)(1-(1/p))}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} \left|\tilde{A}_j\right| b_j^q \langle k\rangle^{(q/1-\alpha)[s+n\alpha(1-(1/p))]}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} b_j^q 2^{jn(1-\alpha)} . 2^{jq[s+n\alpha(1-(1/p))]}\right)^{1/q} \\
&\sim \left(\sum_{j\ge N} b_j^q 2^{jq[s+n\alpha(1-(1/p))+(n(1-\alpha)/q)]}\right)^{1/q} \\
&\sim \left\|b_j\right\|_{\ell_q^{s+n\alpha(1-(1/p))+(n(1-\alpha)/q),1}}.
\end{aligned} \tag{48}$$

Hence,

$$\|G_N\|_{M_{p,q}^{s,\alpha}} \gtrsim \left\|b_j\right\|_{\ell_q^{s+n\alpha(1-(1/p))+(n(1-\alpha)/q),1}}. \tag{49}$$

On the other hand, by the same argument of the previous proof, we deduce that

$$\lim_{N\to\infty} \|G_N\|_{F_{p,r}} \sim \left\|b_j\right\|_{\ell_p^{n(1-(1/p)),1}}. \tag{50}$$

Thus, if $F_{p,r} \subset M_{p,q}^{s,\alpha}$ holds, we obtain the desired inclusion

$$\ell_p^{n(1-(1/p)),1} \subset \ell_q^{(n(1-\alpha)/q)+n\alpha(1-(1/p))+s,1}. \tag{51}$$

□

Proposition 17. *Let $0 < p \le 1$. We have the following inclusion relation:*

$$F_{p,\infty}^{n(1-\alpha)((2/p)-1)} \subset M_{p,q}^{0,\alpha}. \tag{52}$$

Proof. We first verify

$$\|a\|_{M^{0,\alpha}_{p,q}} \lesssim 1, \tag{53}$$

for any $n(1-\alpha)((2/p)-1)$-atom a. Tack a to be an $n(1-\alpha)((2/p)-1)$-atom as in Definition 7 (with $s=n(1-\alpha)((2/p)-1)$). Observing that $K \geq [n(1-\alpha)((2/p)-1)]+1 \geq [n(1-\alpha)((1/p)-(1/2))]+1$, we have

$$\left|\partial^\delta a\right| \leq 1, \tag{54}$$

for $|\delta| \leq [n(1-\alpha)((1/p)-(1/2))]+1$. By the Bernstein multiplier theorem, we have the following estimate of a:

$$\|a\|_{M^{0,\alpha}_{p,q}} \sim \left\|\mathcal{F}^{-1}a\right\|_{L_p} \lesssim \sum_{|\delta|\leq[n(1-\alpha)((1/p)-(1/2))]+1} \left\|\partial^\delta a\right\|_{L_2} \lesssim 1. \tag{55}$$

Next, we turn to the estimate of an (s,p,∞)-atom for $F^{n(1-\alpha)((2/p)-1)}_{p,\infty}$. By Definition 7, an (s,p,∞)-atom g can be represented by

$$g = \sum_{(\mu,l)<(\nu,k)} d_{\mu l} a_{\mu l}(x) \left(\text{convergence in } F^{n(1-\alpha)((2/p)-1)}_{p,\infty}\right), \tag{56}$$

for some $k \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, where $a_{\mu l}(x)$ are $(Q_{\mu l}, s, p, \infty)$-atoms and $d_{\mu l}$ are complex numbers with

$$\sup_{(\mu,l)<(\nu,k)}\left|d_{\mu l}\right| \leq \left|Q_{\nu k}\right|^{-(1/p)}, \tag{57}$$

for a fixed $\tau \leq \nu$, we denote

$$g_\tau = \sum_{(\tau,l)<(\nu,k)} d_{\tau l} a_{\tau l}(x). \tag{58}$$

Then, g can be represented by

$$g = \sum_{\tau\leq\nu} g_\tau \left(\text{convergence in } F^{n(1-\alpha)((2/p)-1)}_{p,\infty}\right). \tag{59}$$

We now concentrate on the estimate of g_τ. By Definition 7, we have.

$$\left|\partial^\delta a_{\tau l}\right| \leq \left|Q_{\tau l}\right|^{(1-\alpha)((2/p)-1)-(\delta/n)} = \left|2^{-\tau n}\right|^{(1-\alpha)((2/p)-1)-(\delta/n)}, \tag{60}$$

for $|\delta| \leq [n(1-\alpha)((1/p)-(1/2))]+1 \leq K$. Recalling $\operatorname{supp} a_{\tau l} \subset 5Q_{\tau l}$, we use (60) and the almost orthogonality of $a_{\tau l}$ to deduce that

$$\begin{aligned}\left|\partial^\delta g_\tau\right| &= \left|\sum_{(\tau,l)<(\nu,k)} d_{\tau l}\partial^\delta a_{\tau l}(x)\right| \\ &\lesssim \sup_{(\tau,l)<(\nu,k)}\left|d_{\mu l}\right|\left|2^{-\tau n}\right|^{(1-\alpha)((2/p)-1)-(\delta/n)} \\ &\lesssim \left|Q_{\nu k}\right|^{-(1/p)}\left|2^{-\tau n}\right|^{(1-\alpha)((2/p)-1)-(\delta/n)},\end{aligned} \tag{61}$$

for all $|\delta| \leq [n(1-\alpha)((1/p)-(1/2))]+1$. By the Bernstein multiplier theorem, we deduce that

$$\begin{aligned}\left\|g_\tau\right\|_{M^{0,\alpha}_{p,q}} &\sim \left\|\mathcal{F}^{-1}g_\tau\right\|_{L_p} \\ &\lesssim \sum_{|\delta|\leq[n(1-\alpha)((1/p)-(1/2))]+1} \left\|\partial^\delta g_\tau\right\|_{L_2} \\ &\lesssim \sum_{|\delta|\leq[n(1-\alpha)((1/p)-(1/2))]+1} \left|Q_{\nu k}\right|^{(1/2)-(1/p)}\left|2^{-\tau n}\right|^{(1-\alpha)((2/p)-1)-(\delta/n)}.\end{aligned} \tag{62}$$

By a dilation argument, we have

$$\left\|g_\tau\right\|_{M^{0,\alpha}_{p,q}} \sim \left\|\mathcal{F}^{-1}g_\tau\right\|_{L_p} \lesssim \left|Q_{\nu k}\right|^{(1/2)-(1/p)}\left|2^{-\tau n}\right|^{(1-\alpha)((1/p)-(1/2))}. \tag{63}$$

Thus,

$$\begin{aligned}\|g\|^p_{M^{0,\alpha}_{p,q}} &= \left\|\sum_{\tau\leq\nu} g_\tau\right\|^p_{M^{0,\alpha}_{p,q}} \leq \sum_{\tau\leq\nu}\left\|g_\tau\right\|_{M^{0,\alpha}_{p,q}} \\ &\leq \left|Q_{\nu k}\right|^{(p/2)-1}\sum_{\tau\leq\nu}\left|2^{-\tau n}\right|^{(1-\alpha)(1-(p/2))} \\ &\lesssim \left|Q_{\nu k}\right|^{(p/2)-1}.\left|2^{-\nu n}\right|^{(1-\alpha)(1-(p/2))} \sim 1.\end{aligned} \tag{64}$$

By Lemma 8 we have

$$\begin{aligned}\|f\|_{M^{0,\alpha}_{p,q}} &= \left\|\sum_{j=1}^{\infty}\left(\mu_j a_j + \lambda_j g_j\right)\right\|_{M^{0,\alpha}_{p,q}} \\ &\leq \left(\sum_{j=1}^{\infty}\left(\left|\mu_j\right|^p\left\|a_j\right\|_{M^{0,\alpha}_{p,q}} + \left|\lambda_j\right|^p\left\|g_j\right\|_{M^{0,\alpha}_{p,q}}\right)\right)^{(1/p)} \\ &\lesssim \left(\sum_{j=1}^{\infty}\left|\mu_j\right|^p + \left|\lambda_j\right|^p\right)^{(1/p)} \lesssim \|f\|_{F^{n(1-\alpha)((2/p)-1)}_{p,\infty}},\end{aligned} \tag{65}$$

which is the desired conclusion. □

Proof of Theorem 13. We divide this proof into two parts.

Sufficiency. For $p \geq q$, by Lemma 9, we obtain $B_{p,\infty} \subset M^{n(1-\alpha)(1-(1/p)),\alpha}_{p,\infty}$. Using $F_{p,\infty} \subset B_{p,\infty}$, we deduce that

$$F_{p,\infty} \subset M^{n(1-\alpha)(1-(1/p)),\alpha}_{p,\infty}. \tag{66}$$

In addition, we have $F^{n(1-\alpha)((2/p)-1)}_{p,\infty} \subset M^{0,\alpha}_{p,p}$ by Proposition 17. By potential lifting, we obtain

$$F_{p,\infty} \subset M^{n(1-\alpha)(1-(2/p)),\alpha}_{p,p}. \tag{67}$$

Thus, the desired conclusion can be deduced by a standard interpolation argument between (66) and (67).

For $p < q$, recalling $F_{p,\infty} \subset M^{n(1-\alpha)(1-(2/p)),\alpha}_{p,p}$ obtained in Proposition 17, we deduce that

$$F_{p,r} \subset F_{p,\infty} \subset M^{n(1-\alpha)(1-(2/p)),\alpha}_{p,p} \subset M^{n(1-\alpha)(1-(1/p)-(1/q))-\varepsilon,\alpha}_{p,q}, \tag{68}$$

for any $\varepsilon > 0, r \in (0,\infty]$.

Necessity. We use Proposition 16 to deduce inclusion relation $\ell_p^{n(1-(1/p)),1} \subset \ell_q^{((n(1-\alpha))/q)+n\alpha(1-(1/p))+s,1}$. Then, Lemma 4 yields that $s \le -n(1-\alpha)((1/p)+(1/q)-1)$ for $p \le q$, while the inequality is strict for $p > q$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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