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Research Article

Approximation by Szász-Jakimovski-Leviatan-Type Operators via Aid of Appell Polynomials

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The main purpose of the present article is to construct a newly Szász-Jakimovski-Leviatan-type positive linear operators in the Dunkl analogue by the aid of Appell polynomials. In order to investigate the approximation properties of these operators, first we estimate the moments and obtain the basic results. Further, we study the approximation by the use of modulus of continuity in the spaces of the Lipschitz functions, Peetres K-functional, and weighted modulus of continuity. Moreover, we study A-statistical convergence of operators and approximation properties of the bivariate case.

1. Introduction

In 1969, Jakimovski and Leviatan introduced a sequence of positive linear operators $\{L_n\}_{n\geq 1}$ [1], by using Appell polynomials [2] $F(v)e^{vy}=\sum_{s=0}^{\infty}P_s(y)v^s$ and defined as

$$L_n(h;y)=\frac{e^{-ny}}{F(1)}\sum_{s=0}^{\infty}P_s(ny)h\left(\frac{s}{n}\right), \tag{1}$$

where $F(1)\neq 0$, $F(v)=\sum_{s=0}^{\infty}c_s v^s$, and $P_s(y)=\sum_{i=0}^{s}c_i(y^{s-i}/(s-i)!)(s\in\mathbb{N})$. For all $n\in\mathbb{N}$ and $c_i/F(1)\geq 0$, the positive linear operators L_n are defined on $[0,1)$ given by Wood in [3]. If we take $h\in E[0,\infty)$, then an analogue of Szász operators was proved by Jakimovski and Leviatan, where $E[0,\infty)$ denotes the set of functions on $[0,\infty)\}$ such that $|h(y)|\leq ae^{\kappa y}$, where a, κ are positive constants. They established $\lim_{n\to\infty}L_n(h;y)\longrightarrow h(y)$ is uniformly on each compact subset of $[0,1)$ (see [1, 4]). Precisely, for $F(1)=1$ in (1), the well-known classical Szász operators [5] were obtained defined in 1950 such that

$$S_n(h;y)=e^{-ny}\sum_{s=0}^{\infty}\frac{(ny)^s}{s!}h\left(\frac{s}{n}\right). \tag{2}$$

Recently, Szász-Mirakyan operators have been obtained by researchers via the Dunkl generalization in approximation process; for instance, we refer the readers to [6–12]. For more details, related results relevant to the present article in different functional spaces are seen in [13–19] and [20–23]. Sucu [24] introduced Szász-Mirakyan operators by using the new exponential function given in [25] as

$$e_\lambda(y)=\sum_{s=0}^{\infty}\frac{y^s}{\gamma_\lambda(s)}.$$
$$\gamma_\lambda(2p)=\frac{2^{2p}p!\Gamma(p+\lambda+1/2)}{\Gamma(\lambda+1/2)},\quad \gamma_\lambda(2p+1)=\frac{2^{2p+1}p!\Gamma(p+\lambda+3/2)}{\Gamma(\lambda+1/2)}. \tag{3}$$

For $p=0,1,2,\cdots$ a recursion of γ_λ is given as

$$\frac{\gamma_\lambda(p+1)}{(p+1+2\lambda\theta_{p+1})}=\gamma_\lambda(p),\quad \theta_p=\begin{cases}0 & \text{if } p=2s,\ s\in\mathbb{N},\\ 1 & \text{if } p=2s+1,\ s\in\mathbb{N}.\end{cases} \tag{4}$$

These types of generalizations gave rise to exponential function and generalization of Hermite-type polynomials, expressed in the form of the confluent hypergeometric function (see [25]).

2. Construction of Operators and Estimation of Moments

For every $h \in C_\vartheta[0,\infty) = \{h \in C[0,\infty): h(s) = O(s^\vartheta)\}$ as $s \longrightarrow \infty$, and all $y \in [0,\infty)$, $\vartheta > n$, $n \in \mathbb{N}$, $F(1) \neq 0$, $\lambda \geq 0$, we define

$$\mathcal{J}^*_{n,\lambda}(h;y) = \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny) h\left(\frac{s+2\lambda\theta_s}{n}\right). \tag{5}$$

Lemma 1. *For all $y \in [0,\infty)$, $P_s(y) \geq 0$, $\lambda \geq 0$, and $F(1) \neq 0$, if we define*

$$F(\alpha)e_\lambda(\alpha y) = \sum_{s=0}^{\infty} P_s(y)\alpha^s. \tag{6}$$

Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
F(1)e_\lambda(ny) &= \sum_{s=0}^{\infty} P_s(ny),\\
\sum_{s=0}^{\infty} sP_s(ny) &= \left(F'(1) + nyF(1)\right)e_\lambda(ny),\\
\sum_{s=0}^{\infty} s^2P_s(ny) &= \Big(F''(1) + (2ny+1)F'(1)\\
&\quad + ny(ny+1)F(1)\Big)e_\lambda(ny),\\
\sum_{s=0}^{\infty} s^3P_s(ny) &= \Big(F'''(1) + 3(ny+1)F''(1)\\
&\quad + \left(3n^2y^2 + 6ny + 2\right)F'(1)\\
&\quad + ny\left(n^2y^2 + 3ny + 2\right)F(1)\Big)e_\lambda(ny),\\
\sum_{s=0}^{\infty} s^4P_s(ny) &= \Big(F^{\prime v}(1) + (4ny+6)F'''(1)\\
&\quad + \left(6n^2y^2 + 18ny + 11\right)F''(1)\\
&\quad + \left(4n^3y^3 + 18n^2y^2 + 22ny + 6\right)F'(1)\\
&\quad + ny\left(n^3y^3 + 6n^2y^2 + 11ny + 6\right)F(1)\Big)e_\lambda(ny).
\end{aligned} \tag{7}$$

Lemma 2. *Let $\lambda \in [0,\infty)$, $F(1) \neq 0$ and take $\phi_r = s^r$ for $r = 0, 1, 2, 3, 4$.*

Then, for operators $\mathcal{J}^*_{n,\lambda}(\cdot;\cdot)$ by (5), we have the following estimates:

$$\begin{aligned}
\mathcal{J}^*_{n,\lambda}(\phi_0;y) &= 1,\\
\mathcal{J}^*_{n,\lambda}(\phi_1;y) &= y + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right),\\
\mathcal{J}^*_{n,\lambda}(\phi_2;y) &= y^2 + \frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right)y\\
&\quad + \frac{1}{n^2}\left(\frac{F''(1)}{F(1)} + (1+4\lambda)\frac{F'(1)}{F(1)} + 4\lambda^2\right),\\
\mathcal{J}^*_{n,\lambda}(\phi_3;y) &= y^3 + \frac{3}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda + 1\right)y^2\\
&\quad + \frac{1}{n^2}\left(\frac{3F''(1)}{F(1)} + 6(1+2\lambda)\frac{F'(1)}{F(1)} + 2 + 6\lambda\right)y\\
&\quad + \frac{1}{n^3}\left(\frac{3F'''(1)}{F(1)} + 3(1+2\lambda)\frac{F''(1)}{F(1)}\right.\\
&\quad \left. + 2\left(1 + 3\lambda + 6\lambda^2\right)\frac{F'(1)}{F(1)} + 8\lambda^3\right),\\
\mathcal{J}^*_{n,\lambda}(\phi_4;y) &= y^4 + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 8\lambda + 6\right)y^3 + \frac{1}{n^2}\left(\frac{6F''(1)}{F(1)}\right.\\
&\quad \left. + 8(1+3\lambda)\frac{F'(1)}{F(1)} + 11 + 24\lambda + 24\lambda^2\right)y^2\\
&\quad + \frac{1}{n^3}\left(\frac{6F'''(1)}{F(1)} + 8(1+3\lambda)\frac{F''(1)}{F(1)}\right.\\
&\quad + 2\left(11 + 24\lambda + 24\lambda^2\right)\frac{F'(1)}{F(1)}\\
&\quad \left. + 6 + 16\lambda + 24\lambda^2 + 32\lambda^2\right)y\\
&\quad + \frac{1}{n^4}\left(\frac{F^{\prime v}(1)}{F(1)} + 2(3+4\lambda)\frac{F'''(1)}{F(1)}\right.\\
&\quad + \left(11 + 24\lambda + 24\lambda^2\right)\frac{F''(1)}{F(1)}\\
&\quad \left. + \left(6 + 16\lambda + 24\lambda^2 + 32\lambda^3\right)\frac{F'(1)}{F(1)} + 16\lambda^4\right).
\end{aligned} \tag{8}$$

Proof.

(1) Take $h=\phi_0$, then

$$\mathcal{J}^*_{n,\lambda}(\phi_0\,;y)=\frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty}P_s(ny)=1. \tag{9}$$

(2) For $h=\phi_1$

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\phi_1\,;y)&=\frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty}P_s(ny)\left(\frac{s+2\lambda\theta_s}{n}\right)\\&=\frac{1}{nF(1)e_\lambda(ny)}\sum_{s=0}^{\infty}sP_s(ny)\\&\quad+\frac{2\lambda}{nF(1)e_\lambda(ny)}\sum_{s=2k+1}^{\infty}\theta_sP_s(ny)\ \text{for}\ k\\&=0,1,2,3,\cdots=\frac{1}{nF(1)e_\lambda(ny)}\\&\quad\cdot\left(F'(1)+nyF(1)\right)e_\lambda(ny)+\frac{2\lambda}{n}.\end{aligned} \tag{10}$$

(3) For $h=\phi_2$

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\phi_2\,;y)&=\frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty}P_s(ny)\left(\frac{s+2\lambda\theta_s}{n}\right)^2\\&=\frac{1}{n^2F(1)e_\lambda(ny)}\sum_{s=0}^{\infty}s^2P_s(ny)+\frac{4\lambda}{n^2F(1)e_\lambda(ny)}\\&\quad\cdot\sum_{s=2k+1}^{\infty}sP_s(ny)\theta_s\ \text{for}\ k=0,1,2,3,\cdots\\&\quad+\frac{4\lambda^2}{n^2F(1)e_\lambda(ny)}\sum_{s=2k+1}^{\infty}P_s(ny)\theta_s^2\ \text{for}\ k\\&=0,1,2,3,\cdots=\frac{1}{F(1)e_\lambda(ny)}\Big(F''(1)\\&\quad+(2ny+1)F'(1)+ny(ny+1)F(1)\Big)e_\lambda(ny)\\&\quad+\frac{4\lambda^2}{n^2F(1)e_\lambda(ny)}\left(F'(1)+nyF(1)\right)e_\lambda(ny)\\&\quad+\frac{4\lambda^2}{n^2}.\end{aligned} \tag{11}$$

Similarly, we can prove easily (4) and (5).

Lemma 3. *Let $\psi_j=(\phi_1-y)^j$ for $j=1,2,3$, be the central moments, then*

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\psi_1\,;y)&=\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right),\\\mathcal{J}^*_{n,\lambda}(\psi_2\,;y)&=\frac{y}{n}++\frac{1}{n^2}\left(\frac{F''(1)}{F(1)}+(1+4\lambda)\frac{F'(1)}{F(1)}+4\lambda^2\right),\\\mathcal{J}^*_{n,\lambda}(\psi_4\,;y)&=\frac{1}{n}\left(\frac{4F'(1)}{F(1)}+8\lambda\right)y^3+\frac{1}{n^2}\left(-10\frac{F'(1)}{F(1)}+3\right)y^2\\&\quad+\frac{1}{n^3}\left(\frac{-4F''(1)}{F(1)}+2(7+12\lambda)\frac{F'(1)}{F(1)}\right.\\&\quad\left.+2\left(3+8\lambda+12\lambda^2\right)\right)y+\frac{1}{n^4}\left(\frac{F'^{v}(1)}{F(1)}\right.\\&\quad+2(3+4\lambda)\frac{F'''(1)}{F(1)}+\left(11+24\lambda+24\lambda^2\right)\frac{F''(1)}{F(1)}\\&\quad\left.+\left(6+16\lambda+24\lambda^2+32\lambda^3\right)\frac{F'(1)}{F(1)}+16\lambda^4\right)\\&\quad+\frac{1}{n}\left(\frac{-4F'(1)}{F(1)}-8\lambda\right).\end{aligned} \tag{12}$$

3. Global Approximation

In the present section, we follow Gadžiev [11] and recall the weighted spaces of the functions on $[0,\infty)$, as well as additional conditions under which the analogous theorem of P.P. Korovkin holds for such a kind of functions. Take $y\longrightarrow\phi(y)$ be continuous and strictly increasing function with $\sigma(y)=1+\phi^2(y)$ and $\lim_{y\to\infty}\sigma(y)=\infty$. Let $B_\sigma[0,\infty)$ be a set of functions defined on $[0,\infty)$, verifying the results

$$B_\sigma[0,\infty)=\{h(y):|h(y)|\le K_h\sigma(y)\}, \tag{13}$$

where K_h is a constant and depending only on function h and $B_\sigma[0,\infty)$ is space of all continuous as well as bounded functions on $[0,\infty)$. Let the set of all continuous functions on $[0,\infty)$ will be denoted by $C_\sigma[0,\infty)$ and $B_\sigma[0,\infty)\subset C_\sigma[0,\infty)$ equipped with the norm $\|h\|_\sigma=\sup_{y\in[0,\infty)}|h|/\sigma(y)$.

Let us denote

$$C_\sigma^m[0,\infty)=\left\{h\in C_\sigma:\lim_{y\to\infty}\frac{h(y)}{\sigma(y)}=k,k\ \text{is positive constant}\right\}. \tag{14}$$

It is well known that (see [26]) the sequence of linear positive operators $\{L_n\}_{n\ge1}$ maps $C_\sigma[0,\infty)$ into $B_\sigma[0,\infty)$ if and only if

$$|L_n(\sigma\,;y)| \leq C\sigma(y), \tag{15}$$

where C is a positive constant.

Definition 4. For all $h \in C[0,\infty)$, the modulus of continuity for a uniformly continuous function h defined by

$$\omega(h\,;\delta) = \sup_{|s_1-s_2|\leq\delta} |h(s_1) - h(s_2)|, s_1, s_2 \in [0,\infty). \tag{16}$$

For every $\delta > 0$ and uniformly continuous function $h \in C[0,\infty)$, we suppose

$$|h(s_1) - h(s_2)| \leq \left(1 + \frac{|s_1 - s_2|}{\delta^2}\right)\omega(h\,;\delta). \tag{17}$$

Theorem 5. *For all $h \in [0,\infty) \cap \{h : y \geq 0, h(y)/\sigma(y)$ is convergent as $y \longrightarrow \infty\}$, operators $\mathcal{J}^*_{n,\lambda}$ defined in (5) satisfy $\mathcal{J}^*_{n,\lambda} \rightrightarrows h$ on each compact subset of $[0,\infty)$, with $\rightrightarrows$ stands for uniform convergence.*

Proof. From the well-known Korovkin's theorem (see [27]), for all $r = 0, 1, 2$, it is sufficient to see that

$$\mathcal{J}^*_{n,\lambda}(\phi_r\,;y) \longrightarrow y^r. \tag{18}$$

In the view of Lemma 2, it is obvious that $\mathcal{J}^*_{n,\lambda}(\phi_r\,;y) \longrightarrow y^r$ as $n \longrightarrow \infty$, $r = 0, 1, 2$, which completes Theorem 5.

Theorem 6. *Let $\mathcal{J}^*_{n,\lambda} : C^m_\sigma[0,\infty) \longrightarrow B_\sigma[0,\infty)$. Then for every $h \in C^m_\sigma[0,\infty)$, we have*

$$\lim_{n\to\infty} \left\| \mathcal{J}^*_{n,\lambda}(h\,;y) - h \right\|_\sigma = 0. \tag{19}$$

Proof. We prove this theorem by applying Korovkin's theorem so it is sufficient to show that

$$\lim_{n\to\infty} \left\| \mathcal{J}^*_{n,\lambda}\left(\phi_j\,;y\right) - y^j \right\|_\sigma = 0, \text{ for } j = 0, 1, 2. \tag{20}$$

From Lemma 2, we easily see that

$$\left\| \mathcal{J}^*_{n,\lambda}(\phi_o\,;y) - y^0 \right\|_\sigma = \sup_{y\in[0,\infty)} \frac{\left|\mathcal{J}^*_{n,\lambda}(1\,;y) - 1\right|}{\sigma(y)} = 0 \text{ for } j = 0. \tag{21}$$

Similarly, for

$$\left\| \mathcal{J}^*_{n,\lambda}(\phi_1\,;y) - y \right\|_\sigma = \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right) \sup_{y\in[0,\infty)} \frac{1}{\sigma(y)}. \tag{22}$$

which imply that $\|\mathcal{J}^*_{n,\lambda}(\phi_1\,;y) - y\|_\sigma \longrightarrow 0$ as $n \longrightarrow \infty$. For $j = 2$

$$\begin{aligned} &\left\| \mathcal{J}^*_{n,\lambda}(\phi_2\,;y) - y^2 \right\|_\sigma \\ &\quad = \sup_{y\in[0,\infty)} \frac{\left|\mathcal{J}^*_{n,\lambda}(\phi_2\,;y) - y^2\right|}{\sigma(y)} \\ &\quad = \frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right) \sup_{y\in[0,\infty)} \frac{y}{\sigma(y)} \\ &\qquad + \frac{1}{n^2}\left(\frac{F''(1)}{F(1)} + (1 + 4\lambda)\frac{F'(1)}{F(1)} + 4\lambda^2\right) \sup_{y\in[0,\infty)} \frac{1}{\sigma(y)}, \end{aligned} \tag{23}$$

which clearly shows that $\|\mathcal{J}^*_{n,\lambda}(\phi_2\,;y) - y^2\|_\sigma \longrightarrow 0$, whenever $n \longrightarrow \infty$.

Theorem 7. *For all $h \in C_B[0,\infty)$, operators given by (5) satisfy*

$$\left|\mathcal{J}^*_{n,\lambda}(h\,;y) - h(y)\right| \leq 2\omega(h\,;\delta_n(y)), \tag{24}$$

where $\delta_n(y) = \sqrt{\mathcal{J}^*_{n,\lambda}(\psi_2\,;y)}$ and $C_B[0,\infty)$ stand for space of all continuous and bounded functions defined on $[0,\infty)$.

Proof. We prove Theorem 7 by using the well-known Cauchy-Schwarz inequality and modulus of continuity. Thus, we see that

$$\begin{aligned} &\mathcal{J}^*_{n,\lambda}(h\,;y) - h(y) \\ &\quad \leq \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny)\,; \left| h\left(\frac{s + 2\lambda\theta_s}{n}\right) - h(y) \right| \\ &\quad \leq \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny) \left(1 + \frac{1}{\delta}\left|\frac{s + 2\lambda\theta_s}{n} - y\right|\right)\omega(h\,;\delta) \\ &\quad = \left\{1 + \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny) \left|\frac{s + 2\lambda\theta_s}{n} - y\right| \omega(h\,;\delta)\right. \\ &\quad \leq \left\{1 + \frac{1}{\delta}\left(\frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny) \left(\frac{s + 2\lambda\theta_s}{n} - y\right)^2\right)^{\frac{1}{2}} \left(\mathcal{J}^*_{n,\lambda}(\phi_0\,;y)\right)^{\frac{1}{2}}\right\}\omega(h\,;\delta) \\ &\quad = \left(1 + \frac{1}{\delta}\left(\mathcal{J}^*_{n,\lambda}\left((\phi_1 - y)^2\,;y\right)\right)^{\frac{1}{2}}\right)\omega(h\,;\delta). \end{aligned} \tag{25}$$

If we take $\delta = \delta_n = \sqrt{\mathcal{J}^*_{n,\lambda}(\psi_2\,;y)}$, we get the required result asserted by Theorem 7.

4. Some Direct Results of $\mathcal{J}^*_{n,\lambda}$

The present section gives some direct approximation results in the space of K-functional and in the Lipschitz spaces. We suppose the following.

Definition 8. For every $\delta > 0$ and $h \in C[0,\infty)$, we define

$$\begin{aligned} K_2(h\,;\delta) = \inf \Big\{ \Big(&\|h - \psi\|_{C_B[0,\infty)} \\ &+ \delta\left\|\psi''\right\|_{C_B[0,\infty)}\Big) : \psi, \psi' \in \left|C^2_B[0,\infty)\right\}, \end{aligned} \tag{26}$$

where $C_B^2[0,\infty)$ is defined by

$$C_B^k[0,\infty)=\left\{h: h\in C_B[0,\infty), k\in\mathbb{N};\right.\\ \left.\text{such that}\lim_{y\to\infty}\frac{h(y)}{\sigma(y)}=k_h<\infty\right\}. \tag{27}$$

Now, there exists an absolute constant $\mathscr{C}>0$ such that

$$K_2(h;\delta)<\mathscr{C}\left\{\omega_2\left(h;\sqrt{\delta}\right)+\min(1,\delta)\|h\|_{C_B[0,\infty)}\right\}, \tag{28}$$

where $\omega_2(h;\delta)$ is the second-order modulus of continuity given by

$$\omega_2(h;\delta)=\sup_{0<\eta<\delta}\sup_{y\in[0,\infty)}|h(y+2\eta)-2h(y+\eta)+h(y)|. \tag{29}$$

Moreover, the modulus of continuity of order one is

$$\omega(h;\delta)=\sup_{0<\eta<\delta}\sup_{y\in[0,\infty)}|h(y+\eta)-h(y)|. \tag{30}$$

Theorem 9. *Let $h\in C_B^2[0,\infty)$, we define an auxiliary operators $\mathscr{K}_{n,\lambda}^*$ such that*

$$\mathscr{K}_{n,\lambda}^*(h;y)=\mathscr{J}_{n,\lambda}^*(h;y)+h(y)-h\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}. \tag{31}$$

Then, for every $\psi\in C_B^2[0,\infty)$, operators $\mathscr{K}_{n,\lambda}^*$ satisfy

$$\left|\mathscr{K}_{n,\lambda}^*(\psi;y)-\psi(y)\right|\le\{\Theta_n(y)\}\left\|\psi''\right\|, \tag{32}$$

where $\Theta_n(y)=(\delta_n(y))^2+(1/n^2)((F'(1)/F(1))+2\lambda)^2$ and $\delta_n(y)$ are defined in Theorem 7.

Proof. Take $\psi\in C_B^2[0,\infty)$; then, we easily conclude that $\mathscr{K}_{n,\lambda}^*(\phi_0;y)=1$ and

$$\mathscr{K}_{n,\lambda}^*(\phi_1;y)=\mathscr{J}_{n,\lambda}^*(\phi_1;y)+y-\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}=y. \tag{33}$$

We also know easily

$$\left\|\mathscr{J}_{n,\lambda}^*(h;y)\right\|\le\|h\|. \tag{34}$$

Therefore,

$$\left|\mathscr{K}_{n,\lambda}^*(h;y)\right|\le\left|\mathscr{J}_{n,\lambda}^*(h;y)\right|+|h(y)|\\+\left|h\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}\right|\le3\|h\|. \tag{35}$$

From the Taylor series we see

$$\psi(s)=\psi(y)+(s-y)\psi'(y)+\int_y^s(s-\mu)\psi''(\mu)\mathrm{d}\mu. \tag{36}$$

Applying $\mathscr{K}_{n,\lambda}^*$, we have

$$\begin{aligned}\mathscr{K}_{n,\lambda}^*(\psi;y)-\psi(y)&=\psi'(y)\mathscr{K}_{n,\lambda}^*(\phi_1-y;y)+\mathscr{K}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu;y\right)\\&=\mathscr{K}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu;y\right)\\&=\mathscr{J}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu;y\right)\\&\quad-\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu,\end{aligned}$$

$$\begin{aligned}\left|\mathscr{K}_{n,\lambda}^*(\psi;y)-\psi(y)\right|&\le\left|\mathscr{J}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu;y\right)\right|\\&\quad+\left|\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu\right|.\end{aligned} \tag{37}$$

Since we know

$$\begin{aligned}&\left|\int_y^s(s-\mu)\psi''(\mu)\mathrm{d}\mu\right|\le(s-y)^2\left\|\psi''\right\|,\\&\left|\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu\right|\\&\quad\le\left(\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right)^2\left\|\psi''\right\|.\end{aligned} \tag{38}$$

Therefore, we get

$$\left|\mathscr{K}_{n,\lambda}^*(\psi;y)-\psi(y)\right|\le\left\{\mathscr{J}_{n,\lambda}^*\left((\phi_1-y)^2;y\right)+\frac{1}{n^2}\left(\frac{F'(1)}{F(1)}+2\lambda\right)^2\right\}\left\|\psi''\right\|. \tag{39}$$

This gives the complete proof.

Theorem 10. *Let $h \in C_B[0,\infty)$ and any $\psi \in C_B^2[0,\infty)$. Then, there exists a constant $\mathscr{C} > 0$ such that*

$$\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \le \mathscr{C}\left\{\omega_2\left(h;\frac{\sqrt{\Theta_n(y)}}{2}\right) + \min\left(1,\frac{\Theta_n(y)}{4}\right)\|h\|_{C_{B[0,\infty)}}\right\} + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right), \quad (40)$$

where $\Theta_n(y)$ is defined by Theorem 9.

Proof. We prove the result asserted by Theorem 10 in the light of Theorem 9. Therefore, for all $h \in C_B[0,\infty)$ and $\psi \in C_B^2[0,\infty)$, we get

$$\begin{aligned}\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| &= \left|\mathscr{K}_{n,\lambda}^*(h;y) - h(y) + h\left(y + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right) - h(y)\right| \\ &\le \left|\mathscr{K}_{n,\lambda}^*(h-\psi;y)\right| + \left|\mathscr{K}_{n,\lambda}^*(\psi;y) - \psi(y)\right| + |\psi(y) - h(y)| \\ &\quad + \left|h\left(y + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right) - h(y)\right| \\ &\le 4\|h-\psi\| + \Theta_n(y)\|\psi''\| + \omega\left(h;\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right|\right).\end{aligned} \quad (41)$$

Taking infimum over all $\psi \in C_B^2[0,\infty)$ and using (26), we get

$$\begin{aligned}\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| &\le 4K_2\left(h;\frac{\Theta_n(y)}{4}\right) + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right) \\ &\le \mathscr{C}\left\{\omega_2\left(h;\frac{\sqrt{\Theta_n(y)}}{4}\right) + \min\left(1;\frac{\Theta_n(y)}{4}\right)\|h\|_{C_B[0,\infty)}\right\} \\ &\quad + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right).\end{aligned} \quad (42)$$

Here, we obtain some local approximation results of $\mathscr{J}_{n,\lambda}^*$ in the Lipschitz spaces. For all the Lipschitz maximal function $h \in C[0,\infty)$, $0 < \vartheta \le 1$ and $s, y \in [0,\infty)$, we recall that

$$\omega_\vartheta(h;y) = \sup_{s\ne y, s\in[0.\infty)} \frac{|h(s) - h(y)|}{|s-y|^\vartheta}. \quad (43)$$

Theorem 11. *Let $0 < \vartheta \le 1$, then for all $h \in C_B[0,\infty)$, operators $\mathscr{J}_{n,\lambda}^*$ satisfy*

$$\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \le \omega_\vartheta(h;y)(\delta_n(y))^\vartheta, \quad (44)$$

where $\omega_\vartheta(h;y)$ is the Lipschitz maximal function defined by (43) and $\delta_n(y)$ by Theorem 7.

Proof. To prove Theorem 11, we use the well-known Hölder inequality by applying (43)

$$\begin{aligned}&\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \\ &\quad \le \mathscr{J}_{n,\lambda}^*(|h(s) - h(y)|;y) \le \omega_\vartheta(h;y)\left|\mathscr{J}_{n,\lambda}^*\left(|s-y|^\vartheta;y\right)\right. \\ &\quad \le \omega_\vartheta(h;y)\left(\mathscr{J}_{n,\lambda}^*(\phi_0;y)\right)^{\frac{2-\vartheta}{2}}\left(\mathscr{J}_{n,\lambda}^*\left((\phi_1 - y)^2;y\right)\right)^{\frac{\vartheta}{2}} \\ &\quad = \omega_\vartheta(h;y)\left(\mathscr{J}_{n,\lambda}^*(\psi_2;y)\right)^{\frac{\vartheta}{2}}.\end{aligned} \quad (45)$$

The proof is complete.

From [28] for an arbitrary $h \in C_\sigma^m[0,\infty)$, the weighted modulus of continuity is introduced such that

$$\Omega(h;\delta) = \sup_{y\in[0,\infty),|\eta|\le\delta} \frac{|h(y+\eta) - h(y)|}{(1+\eta^2)(1+y^2)}. \quad (46)$$

The two main properties of this modulus of continuity are $\lim_{\delta\to 0}\Omega(h;\delta) = 0$ and

$$|h(s) - h(y)| \le 2\left(1 + \frac{|s-y|}{\delta}\right)\left(1+\delta^2\right)\left(1+y^2\right) \cdot \left(1 + (s-y)^2\right)\Omega(h;\delta), \quad (47)$$

where $s, y \in [0,\infty)$.

Theorem 12. *Let the operators $\mathscr{J}_{n,\lambda}^*$ be defined by (5); then for every $h \in C_\sigma^m[0,\infty)$, there exists a constant $C > 0$ such that*

$$\sup_{y\in\left[0,O\left(\frac{1}{n}\right)\right)} \frac{\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right|}{\sigma(y)} \le C\left(1 + O\left(\frac{1}{n}\right)\right)\Omega\left(h;\sqrt{O\left(\frac{1}{n}\right)}\right), \quad (48)$$

where $\sigma(y) = 1 + y^2$ and $C = (2 + C_1 + 2C_2)$ with $C_1 > 0, C_2 > 0$.

Proof. In light of (46), (47), and Cauchy-Schwarz inequality, we prove this theorem. Thus, we see

$$\begin{aligned}&\left|\mathcal{J}^{*}_{n,\lambda}(h;y)-h(y)\right|\\&\le 2\left(1+\delta^2\right)\left(1+y^2\right)\Omega(h;\delta)\Bigg(1+\mathcal{J}^{*}_{n,\lambda}\left((\phi_1-y)^2;y\right)\\&+\mathcal{J}^{*}_{n,\lambda}\left(\left(1+(\phi_1-y)^2\right)\frac{|\phi_1-y|}{\delta};y\right)\Bigg)\end{aligned} \tag{49}$$

$$\begin{aligned}&\mathcal{J}^{*}_{n,\lambda}\left(\left(1+(\phi_1-y)^2\right)\frac{|\phi_1-y|}{\delta};y\right)\\&\le 1+2\left(\mathcal{J}^{*}_{n,\lambda}\left((\phi_1-y)^4;y\right)\right)^{\frac{1}{2}}\left(\mathcal{J}^{*}_{n,\lambda}\left(\frac{(\phi_1-y)^4}{\delta^2};y\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{50}$$

From Lemma 3, we easily conclude that for any positive C_1 and C_1

$$\mathcal{J}^{*}_{n,\lambda}\left((\phi_1-y)^2;y\right)=O\left(\frac{1}{n}\right)(y+1)^2\le C_1(y+1)^2 \text{ as } n\longrightarrow\infty, \tag{51}$$

$$\mathcal{J}^{*}_{n,\lambda}\left((\phi_1-y)^4;y\right)=O\left(\frac{1}{n}\right)(y+1)^4\le C_2(y+1)^4 \text{ as } n\longrightarrow\infty. \tag{52}$$

Therefore,

$$\left(\mathcal{J}^{*}_{n,\lambda}\left(\frac{(\phi_1-y)^2}{\delta^2};y\right)\right)^{\frac{1}{2}}=\frac{1}{\delta}\sqrt{O\left(\frac{1}{n}\right)}(1+y). \tag{53}$$

$$\left(\mathcal{J}^{*}_{n,\lambda}\left((\phi_1-y)^4;y\right)\right)^{\frac{1}{2}}\le C_2(1+y)^2. \tag{54}$$

Hence, in light of (49), (50), (51), (52), (53) and (54) and choosing $\delta\sqrt{O(1/n)}$, if we take the supremum $y\in[0,O(1/n))$, we get the result.

5. A-Statistical Convergence

Here, we obtain the A-statistical convergence for the operators $\mathcal{J}^{*}_{n,\lambda}$ by (5). From [29], we recall the needed notations and notions for A-statistical convergence. Take $G=(D_{nk})$ be a nonnegative infinite summability matrix. For a given sequence $y=(y_k)$, the A-transform of y is denoted by $Gy:(Gy)_n$ where the series converges for each n and defined by

$$(Gy)_n=\sum_{k=1}^{\infty}y_kD_{nk}. \tag{55}$$

The matrix G is said to be regular if $\lim(Gy)_n=L$ whenever $\lim x=L$ and $y=(y_n)$ are said to be a A-statistically convergent to L, i.e., $st_G-\lim y=L$ if for each $\epsilon>0$, $\lim_n\sum_{k:|y_k-L|\ge\epsilon}D_{nk}=0$. For the recent work on statistical convergence and statistical approximation, we refer to [30–37].

Theorem 13. *Let operators $\mathcal{J}^{*}_{n,\lambda}$ be defined by 1 and a nonnegative regular summability matrix be $G=(D_{nk})$; then, for every $h\in C^m_\sigma[0,\infty)$*

$$st_G-\lim_n\left\|\mathcal{J}^{*}_{n,\lambda}(h;y)-h\right\|_\sigma=0. \tag{56}$$

Proof. It is enough to show that

$$st_G-\lim_n\left\|\mathcal{J}^{*}_{n,\lambda}\left(\phi_j;y\right)-y^j\right\|_\sigma=0,\quad \text{for} j=0,1,2. \tag{57}$$

From Lemma 2, we conclude that

$$\begin{aligned}\left\|\mathcal{J}^{*}_{n,\lambda}(\phi_2;y)-y\right\|_\sigma&=\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|\sup_{y\in[0,\infty)}\frac{1}{\sigma(y)}\\&\le\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|.\end{aligned} \tag{58}$$

which implies that

$$st_G-\lim_n\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|=0. \tag{59}$$

Similarly for $j=2$

$$\begin{aligned}&\left\|\mathcal{J}^{*}_{n,\lambda}(\phi_2;y)-y^2\right\|_\sigma\\&=\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right|\sup_{y\in[0,\infty)}\frac{y}{\sigma(y)}\\&+\left|\frac{1}{n^2}\left(\frac{F''(1)}{F(1)}+(1+4\lambda)\frac{F'(1)}{F(1)}+4\lambda\right)\right|\sup_{y\in[0,\infty)}\frac{1}{\sigma(y)}\\&\le\frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right|+\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|.\end{aligned} \tag{60}$$

which shows that

$$st_G - \lim_n \frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right)\right| = st_G - \lim_n \left|\frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right)\right| = 0. \quad (61)$$

For a given $\varepsilon > 0$, we define the sets such that

$$U_1 := \left\{n : \left\|\mathcal{I}^*_{n,\lambda}(\phi_2 ; y) - y^2\right\| \geq \epsilon\right\},$$

$$U_2 := \left\{n : \frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right)\right| \geq \frac{\epsilon}{2}\right\},$$

$$U_3 := \left\{n : \left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 2\lambda\right)\right| \geq \frac{\epsilon}{2}\right\}. \quad (62)$$

Therefore, we conclude that $U_1 \subseteq U_2 \cup U_3$, and $\sum_{k_1 \in U_1} D_{nk_1} \leq \sum_{k_1 \in U_2} D_{nk_1} + \sum_{k_1 \in U_3} D_{nk_1}$. Hence, (61) implies that

$$st_G - \lim_n \left\|\mathcal{I}^*_{n,\lambda}(\phi_2 ; y) - y^2\right\|_\sigma = 0. \quad (63)$$

This is denumerable to complete the proof.

6. Bivariate Operators and Their Moments Estimation

Let $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$ with $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}^2_+ = \{(y, y_1): 0 \leq y < \infty, 0 \leq y_1 < \infty\}$. Suppose $C(\mathbb{R}^2_+)$ is a set of all continuous functions on $\mathbb{R}^2_+$, endowed with the norm given by $\|g\|_{C(\mathbb{R}^2_+)} = \sup_{(y,y_1)\in\mathbb{R}^2_+} |g(y, y_1)|$. Then, for all $g \in C(\mathbb{R}^2_+)$ and $n, m \in \mathbb{N}$, we define

$$\mathcal{I}^*_{n,m}(g ; y, y_1) = \frac{1}{F(1)e_\lambda(ny)G(1)e_\eta(my_1)} \cdot \sum_{s,t=0}^{\infty} P_s(ny)P_t(my_1)g\left(\frac{s + 2\lambda\theta_s}{n}, \frac{t + 2\eta\theta_t}{m}\right), \quad (64)$$

where $\lambda, \eta \geq 0$ and $F(1), G(1) \neq 0$.

For $i, j = \{0, 1, 2, 3, 4\}$, if we take $s = s + 2\lambda\theta_s/n$, $t = t + 2\eta\theta_t/m$, and consider central moments as

$$g\left(s^i, t^j\right) = (s - y)^i (t - y_1)^j. \quad (65)$$

then from Lemma 2 and Lemma 3, we easily conclude Lemma 14 as follows:

Lemma 14. *For all $y, y_1 \in \mathbb{R}^2_+$ and sufficiently large $n, m \in \mathbb{N}$, we have*

$$\begin{aligned}
\mathcal{I}^*_{n,m}\left((s - y)^2 ; y, y_1\right) &= O\left(\frac{1}{n}\right)(y + 1)^2 \\
&\leq M_1(y + 1)^2 \text{ as } n, m \longrightarrow \infty, \\
\mathcal{I}^*_{n,m}\left((t - y_1)^2 ; y, y_1\right) &= O\left(\frac{1}{m}\right)(y_1 + 1)^2 \\
&\leq M_2(y_1 + 1)^2 \text{ as } n, m \longrightarrow \infty, \\
\mathcal{I}^*_{n,m}\left((s - y)^4 ; y, y_1\right) &= O\left(\frac{1}{n}\right)(y + 1)^4 \\
&\leq M_3(y + 1)^4 \text{ as } n, m \longrightarrow \infty, \\
\mathcal{I}^*_{n,m}\left((t - y_1)^4 ; y, y_1\right) &= O\left(\frac{1}{m}\right)(y_1 + 1)^4 \\
&\leq M_4(y_1 + 1)^4 \text{ as } n, m \longrightarrow \infty.
\end{aligned} \quad (66)$$

Lemma 15. *If we let*

$$\mathcal{K}^*_{n,\lambda}(g ; y, y_1) = \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny)g\left(\frac{s + 2\lambda\theta_s}{n}, y_1\right),$$

$$\mathcal{L}^*_{m,\eta}(g ; y, y_1) = \frac{1}{G(1)e_\eta(my_1)} \sum_{t=0}^{\infty} P_t(my_1)g\left(y, \frac{t + 2\eta\theta_t}{m}\right). \quad (67)$$

Then, it follows that

$$\mathcal{I}^*_{n,m}(g ; y, y_1) = \mathcal{K}^*_{n,\lambda}\left(\mathcal{L}^*_{m,\eta}(g ; y, y_1)\right) = \mathcal{L}^*_{m,\eta}\left(\mathcal{K}^*_{n,\lambda}(g ; y, y_1)\right). \quad (68)$$

Proof. We easily see that

$$\begin{aligned}
&\mathcal{K}^*_{n,\lambda}\left(\mathcal{L}^*_{m,\eta}(g ; y, y_1)\right) \\
&= \mathcal{K}^*_{n,\lambda}\left(\frac{1}{G(1)e_\eta(my_1)} \sum_{t=0}^{\infty} P_t(my_1)g\left(y, \frac{t + 2\eta\theta_t}{m}\right)\right) \\
&= \frac{1}{G(1)e_\eta(my_1)} \sum_{t=0}^{\infty} \mathcal{K}^*_{n,\lambda}\left(g\left(y, \frac{t + 2\eta\theta_t}{m}\right)\right)P_t(my_1) \\
&= \frac{1}{F(1)e_\lambda(ny_1)G(1)e_\eta(my_1)} \sum_{s,t=0}^{\infty} P_s(ny)(P_t)(my_1)g \\
&\cdot \left(\frac{s + 2\lambda\theta_s}{n}, \frac{t + 2\eta\theta_t}{m}\right) = \mathcal{I}^*_{n,n}(g ; y, y_1).
\end{aligned} \quad (69)$$

Similarly, we can see $\mathscr{L}^*_{m,\eta}(\mathscr{K}^*_{n,\lambda}(g;y,y_1)) = \mathscr{L}^*_{m,\eta}(g;y,y_1)$.

Let the weighted function ρ be $\rho(y,y_1) = 1 + y^2 + y_1^2$. Take $B_\rho(\mathbb{R}^2_+) = \{g : |g(y,y_1)| \le M_g\rho(y,y_1) \quad M_g > 0\}$. We denote the set of k-times continuously differentiable functions on $\mathbb{R}^2_+ = \{(y,y_1) \in \mathbb{R}^2 : y, y_1 \in [0,\infty)\}$ by $C^{(k)}(\mathbb{R}^2_+)$. We also denote the class of functions such that

$$C_\rho(\mathbb{R}^2_+) = \{g : g \in B_\rho \cap C_\rho(\mathbb{R}^2_+)\},$$

$$C^k_\rho(\mathbb{R}^2_+) = \left\{g : g \in C_\rho(\mathbb{R}^2_+);\ \text{such that} \lim_{(y,y_1)\to\infty} \frac{g(y,y_1)}{\rho(y,y_1)} = k_g < \infty\right\},$$

$$C^0_\rho(\mathbb{R}^2_+) = \left\{g : g \in C^k_\rho(\mathbb{R}^2_+);\ \text{such that} \lim_{(y,y_1)\to\infty} \frac{g(y,y_1)}{\rho(y,y_1)} = 0\right\}. \tag{70}$$

Let the norm on B_ρ be defined as $\|g\|_\rho = \sup_{(y,y_1)\in\mathbb{R}^2_+}(|g(y,y_1)|/\rho(y,y_1))$.

For all $g \in C^0_\rho(\mathbb{R}^2_+)$ and $\delta_1, \delta_2 > 0$, the weighted modulus of continuity is given as

$$\omega_\rho(g;\delta_1,\delta_2) = \sup_{(y,y_1)\in[0,\infty)} \sup_{0\le|\alpha|\le\delta_1, 0\le|\beta|\le\delta_2} \cdot \frac{|g(y+\alpha, y_1+\beta) - g(y,y_1)|}{\rho(y,y_1)\rho(\alpha,\beta)}, \tag{71}$$

and for any $r_1, r_2 > 0$ satisfying the inequality

$$\omega_\rho(g;r_1\delta_1, r_2\delta_2) \le 4(1+r_1)(1+r_2)(1+\delta_1^2) \cdot (1+\delta_2^2)\omega_\rho(g;\delta_1,\delta_2), \tag{72}$$

it also follows that

$$\begin{aligned}|g(s,t) - g(y,y_1)| &\le \rho(y,y_1)\rho(|s-y|, |t-y_1|)\omega_\rho(g;|s-y|,|t-y_1|)\\ &\le (1+y^2+y_1^2)(1+(s-y)^2) \cdot (1+(t-y_1)^2)\omega_\rho(g;|s-y|,|t-y_1|).\end{aligned} \tag{73}$$

Theorem 16. *For all $g \in C^0_\rho(\mathbb{R}^2_+)$ and sufficiently large $n, m \in \mathbb{N}$*

$$\begin{aligned}&\frac{|\mathscr{I}^*_{n,m}(g;y,y_1) - g(y,y_1)|}{(1+y^2+y_1^2)}\\ &\le \xi_{y,y_1}(1+O(n^{-1}))(1+O(m^{-1}))\omega_\rho \cdot \left(g; O\left(n^{-\frac{1}{2}}\right), O\left(m^{-\frac{1}{2}}\right)\right),\end{aligned} \tag{74}$$

where $\xi_{y,y_1} = (1 + (y+1) + M_1(y+1)^2 + \sqrt{M_3}(y+1)^3)(1 + (y_1+1) + M_2(y_1+1)^2 + \sqrt{M_4}(y_1+1)^3)$, and M_1, M_2, M_3, $M_4 > 0$.

Proof. In view of the above explanation for all $\delta_n, \delta_m > 0$, we see that

$$\begin{aligned}&|g(s,t) - g(y,y_1)|\\ &\le 4(1+y^2+y_1^2)(1+(s-y)^2)(1+(t-y_1)^2) \times \left(1 + \frac{|s-y|}{\delta_n}\right)\left(1 + \frac{|t-y_1|}{\delta_m}\right) \cdot (1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m)\\ &= 4(1+y^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2) \times \left(1 + \frac{|s-y|}{\delta_n} + (s-y)^2 + \frac{|s-y|}{\delta_n} + (s-y)^2\right)\\ &\quad \times \left(1 + \frac{|t-y_1|}{\delta_m} + (t-y_1)^2 + \frac{|t-y_1|}{\delta_m}(t-y_1)^2\right)\omega_\rho \cdot (g;\delta_n,\delta_m).\end{aligned} \tag{75}$$

On applying the operators $\mathscr{I}^*_{n,m}$, we get

$$\begin{aligned}&|\mathscr{I}^*_{n,m}(g;y,y_1) - g(y,y_1)|\\ &\le \mathscr{I}^*_{n,m}(|g(s,t) - g(y,y_1)|;y,y_1)4(1+y^2+y_1^2) \times \mathscr{I}^*_{n,m}\left(1 + \frac{|s-y|}{\delta_n} + (s-y)^2 + \frac{|s-y|}{\delta_n}(s-y)^2;y,y_1\right)\\ &\quad \times \mathscr{I}^*_{n,m}\left(1 + \frac{|t-y_1|}{\delta_m} + (t-y_1)^2 + \frac{|t-y_1|}{\delta_m} \cdot (t-y_1)^2;y,y_1\right) \times (1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m)\\ &= 4(1+y^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m) \times \left(1 + \frac{1}{\delta_n}\mathscr{I}^*_{n,m}(|s-y|;y,y_1) + \mathscr{I}^*_{n,m}((s-y)^2;y,y_1) + \frac{1}{\delta_n}\mathscr{I}^*_{n,m}(|s-y|(s-y)^2;y,y_1)\right)\\ &\quad \times \left(1 + \frac{1}{\delta_m}\mathscr{I}^*_{n,m}(|t-y_1|;y,y_1)\right)\mathscr{I}^*_{n,m}((t-y_1)^2;y,y_1) + \frac{1}{\delta_m}\mathscr{I}^*_{n,m}(|t-y_1|(t-y_1)^2;y,y_1).\end{aligned} \tag{76}$$

From Cauchy-Schwarz inequality, we see

$$\begin{aligned}&\left|\mathcal{I}^*_{n,m}(g;y,y_1)-g(y,y_1)\right|\\&\leq 4\left(1+y^2+y_1^2\right)\left(1+\delta_n^2\right)\left(1+\delta_m^2\right)\omega_\rho(g;\delta_n,\delta_m)\\&\quad\times\left[1+\frac{1}{\delta_n}\sqrt{\mathcal{I}^*_{n,m}\left((s-y)^2;y,y_1\right)}+\mathcal{I}^*_{n,m}\left((s-y)^2;y,y_1\right)\right.\\&\quad\left.+\frac{1}{\delta_n}\sqrt{\mathcal{I}^*_{n,m}\left((s-y)^2;y,y_1\right)}\sqrt{\mathcal{I}^*_{n,m}\left((s-y)^4;y,y_1\right)}\right]\\&\quad\times\left[1+\frac{1}{\delta_m}\sqrt{\mathcal{I}^*_{n,m}\left((t-y_1)^2;y,y_1\right)}\right.\\&\quad+\mathcal{I}^*_{n,m}\left((t-y_1)^2;y,y_1\right)+1\\&\quad+\frac{1}{\delta_m}\sqrt{\mathcal{I}^*_{n,m}\left((t-y_1)^2;y,y_1\right)}\\&\quad\left.+\sqrt{\mathcal{I}^*_{n,m}\left((t-y_1)^4;y,y_1\right)}\right].\end{aligned} \tag{77}$$

From Lemma 14 we get

$$\begin{aligned}&\left|\mathcal{I}^*_{n,m}(g;y,y_1)-g(y,y_1)\right|\\&\leq 4\left(1+y^2+y_1^2\right)\left(1+\delta_n^2\right)\left(1+\delta_m^2\right)\omega_\rho(g;\delta_n,\delta_m)\\&\quad\times\left[1+\frac{1}{\delta_n}\sqrt{O\left(\frac{1}{n}\right)}(y+1)+M_1(y+1)^2\right.\\&\quad\left.+\frac{1}{\delta_n}\sqrt{O\left(\frac{1}{n}\right)}\sqrt{M_3}(y+1)^3\right]\\&\quad\times\left[1+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)}(y_1+1)+M_2(y_1+1)^2\right.\\&\quad\left.+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)}\sqrt{M_4}(y_1+1)^3\right].\end{aligned} \tag{78}$$

By choosing $\delta_n=O(n^{-1/2})$ and $\delta_m=O(m^{-1/2})$, we arrived to our desired results.

7. Conclusion

The motivation of this present article is to provide the generalized error estimation of convergence rather than the classical Dunkl-Szász-Mirakyan operators. Here, we have defined Szász-Jakimovski-Leviatan operators by using the Appel polynomials with the aid of a new parameter $\lambda\in[0,\infty)$. These types of approximation are able to give the generalized results and error estimation in comparison to earlier study demonstrations. We have obtained the approximations via the well-known weighted Korovkin's spaces and investigated approximations in Peetre's K-functional and Lipschitz spaces with the aid of modulus of continuity. Further, we have also obtained the approximation in A-statistical convergence. Lastly, we have studied the approximation properties for the bivariate case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors are grateful to this manuscript and declare that have no competing interest.

Authors' Contributions

All contents of this research article are checked and agreed to the integrity and accuracy of this manuscript.

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