Hindawi
Journal of Function Spaces
Volume 2020, Article ID 9841640, 14 pages
https://doi.org/10.1155/2020/9841640

Research Article

Sign Changing Solutions for Coupled Critical Elliptic Equations

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Received 1 December 2019; Accepted 3 January 2020; Published 11 February 2020

Academic Editor: Guozhen Lu

In this paper, we consider the coupled elliptic system with a Sobolev critical exponent. We show the existence of a sign changing solution for problem $(\mathscr{P})$ for the coupling parameter $-\sqrt{\mu_1\mu_2} < \beta < 0$. We also construct multiple sign changing solutions for the symmetric case.

1. Introduction

In this paper, we consider the following coupled elliptic system with a Sobolev critical exponent:

$$(\mathscr{P})\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \nu_1 |u_1|^{p_1-2} u_1 + \mu_1 |u_1|^{2^*-2} u_1 \\ \quad + \beta |u_1|^{(2^*/2)-2} u_1 |u_2|^{2^*/2}, \quad \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \nu_2 |u_2|^{p_2-2} u_2 + \mu_2 |u_2|^{2^*-2} u_2 \\ \quad + \beta |u_1|^{2^*/2} |u_2|^{(2^*/2)-2} u_2, \quad \text{in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 6$, $2 < p_j < 2^*$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, $\nu_j, \mu_j > 0$ for $j = 1, 2$, and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

In recent years, the following coupled elliptic system has attracted much interest:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1 = u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

$\Omega \subset \mathbb{R}^N$ is a bounded smooth domain or $\Omega = \mathbb{R}^N$, $N \geq 3$. System (2) arises when we consider the standing wave solutions of the two coupled Gross–Pitaevskii equations:

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, & x \in \Omega,\ t > 0, \\ -i\dfrac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, & x \in \Omega,\ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, & j = 1, 2, \\ \Phi_j(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ & j = 1, 2. \end{cases} \tag{3}$$

The system above has applications in many physical problems (see [1, 2]). It also arises in the Hartree–Fock theory for a double condensate, which is a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [3] and the references therein). Considering the solitary wave solutions of system (3), we set $\Phi_j(x,t) = e^{i\lambda_j t} u_j(x)$ for $j = 1, 2$. Then, it is reduced to system (2). For the subcritical case, i.e., $N = 3$, the existence of least energy and other finite energy solutions as well as the existence and multiplicity of positive and sign-changing solutions are studied in [2, 4–23] and the references therein. For the critical case, i.e., $N = 4\,(2^* := ((2N)/(N-2)) = 4)$, the existence of a positive least energy solution is proved when β is negative, positive small, and positive large in [24]. For the higher dimension $N \geq 5$, the authors in [25, 26] also consider the following critical case:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^{2^*-1} + \beta u_1^{(2^*/2)-1} u_2^{2^*/2}, & \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^{2^*-1} + \beta u_1^{(2^*/2)} u_2^{(2^*/2)-1}, & \text{in } \Omega, \\ u_1, u_2 \geq 0 \text{ in } \Omega, u_1 = u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{4}$$

Note that when $N = 4$, system (4) is the same as system (2). But interestingly, the authors in [25] find different results for the higher dimension case $N \geq 5$ from that of $N = 4$. The authors in [27] also consider the even case:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{2^*-2} u_1 + \beta |u_1|^{(2^*/2)-2} u_1 |u_2|^{2^*/2}, & \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{2^*-2} u_2 + \beta |u_1|^{2^*/2} |u_2|^{(2^*/2)-2} u_2, & \text{in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{5}$$

They show that when $N \geq 6$, system (5) has a sign changing solution for any $\beta < 0$.

Recall that when $\beta = 0$, system (4) becomes the famous Brezis–Nirenberg problem:

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^{2^*-1}, & \text{in } \Omega, \\ u_j \geq 0 \text{ in } \Omega,\ u_j = 0 \text{ on } \partial\Omega, & j = 1, 2, \end{cases} \tag{6}$$

which are two different independent scalar equations. In [28], the authors show that when $N \geq 4, -\lambda_1(\Omega) < \lambda_j < 0$, and $\mu_j = 1$, equation (6) has a ground state solution. For the even case,

$$\begin{cases} -\Delta u + \lambda_j u = \mu_j |u|^{2^*-2} u, & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, & j = 1, 2. \end{cases} \tag{7}$$

The authors in [29] show that when $N \geq 6, -\lambda_1(\Omega) < \lambda_j < 0$, and $\mu_j > 0$, equation (7) has sign changing solutions for $j = 1, 2$. In [30, 31], when $N \geq 4, \lambda_j < 0$, and $\mu_j > 0$, the authors obtained nontrivial solutions of (7) for $j = 1, 2$.

In fact, in the pioneering paper [28] of Brezis and Nirenberg, they also study a more general equation including the following classical case:

$$\begin{cases} -\Delta u + \lambda_j u = \nu_j u^{p_j-1} + \mu_j u^{2^*-1}, & \text{in} \Omega, \\ u \geq 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, & j = 1, 2, \end{cases} \tag{8}$$

where $2 < p_j < 2^*$ and $\nu_j > 0$. From [28], we know that when $N \geq 4, \nu_j, \mu_j > 0$, and $\lambda_j \in (-\lambda_1(\Omega), 0)$, then problem (8) has positive ground state solutions $v_1, v_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ with the energy:

$$\begin{aligned} 0 < B_j &:= \frac{1}{2}\int_\Omega \left(\left|\nabla v_j\right|^2 + \lambda_j v_j^2\right) - \int_\Omega \frac{\nu_j}{p_j} v_j^p - \frac{1}{2^*}\int_\Omega \mu_j v_j^{2^*} \\ &< \frac{1}{N}\mu_j^{-((N-2)/2)} S^{N/2}, \quad j = 1, 2, \end{aligned} \tag{9}$$

where S is the sharp imbedding constant from $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$.

Based on these, firstly, we try to show similar results in [29–31] for the even case of the general equation (8), i.e.,

$$\begin{cases} -\Delta u + \lambda_j u = \nu_j |u|^{p_j-2} u + \mu_j |u|^{2^*-2} u, & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, & j = 1, 2. \end{cases} \tag{10}$$

Then, naturally and interestingly, we guess that the more general system corresponding to system (5) also has similar result of the existence of sign changing solutions. That is, we consider the more general critical elliptic system as follows:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \nu_1 |u_1|^{p_1-2} u_1 + \mu_1 |u_1|^{2^*-2} u_1 \\ \quad + \beta |u_1|^{(2^*/2)-2} u_1 |u_2|^{2^*/2}, & \text{in } \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \nu_2 |u_2|^{p_2-2} u_2 + \mu_2 |u_2|^{2^*-2} u_2 \\ \quad + \beta |u_1|^{2^*/2} |u_2|^{(2^*/2)-2} u_2, & \text{in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega\ . \end{cases} \tag{11}$$

Precisely, we get the following results.

Theorem 1. *Assume that $N \geq 6, 2 < p_j < 2^*, \lambda_j \in (-\lambda_1(\Omega), 0)$, and $\nu_j, \mu_j > 0$, for $j = 1, 2$; then, (10) has sign changing solutions for $j = 1, 2$ and the energy is the least one among all sign changing solutions.*

Then, we can consider the case $\lambda_1 = \lambda_2 = \lambda$ and $p_1 = p_2 = p$. The following equation is one case of (10):

$$\begin{cases} -\Delta u + \lambda u = \nu |u|^{p-2} u + |u|^{2^*-2} u, \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{12}$$

where $\nu > 0$. Let v and w be the sign changing solutions of (12) when $N \geq 6, \lambda \in (-\lambda_1(\Omega), 0)$, and $N \geq 4, \lambda \leq -\lambda_1(\Omega)$, respectively (the latter case really exists and we shall show it later). By Lemma 1 in [25], the system below has a positive solution:

$$\begin{cases} \mu_1 a_1^{2^*-2} + \beta a_1^{(2^*/2)-2} a_2^{2^*/2} = 1, \\ \mu_2 a_2^{2^*-2} + \beta a_1^{2^*/2} a_2^{(2^*/2)-2} = 1. \end{cases} \tag{13}$$

Then, we can construct solutions of (11) by using v and w as in [2, 24, 25].

Theorem 2. *Let (a_1, a_2) be a solution of (13). Assume that $\nu_1 = \nu/a_1^{p-2}$ and $\nu_2 = \nu/a_2^{p-2}, \mu_1, \mu_2, \beta > 0$ and one of the following happens:*

(1) $N \geq 6, -\lambda_1(\Omega) < \lambda_1 = \lambda_2 < 0$,

(2) $N \geq 4, \lambda_1 = \lambda_2 \leq -\lambda_1(\Omega)$,

Then $(a_1 v, a_2 v)$ and $(a_1 w, a_2 w)$ are sign changing solutions of system (11).

We also get a least energy semi-sign changing solution for system (11).

Theorem 3. *Assume that $N \geq 6, 2 < p_j < 2^*, -\lambda_1(\Omega) < \lambda_j < 0$, $\nu_j, \mu_j > 0$, and $-\sqrt{\mu_1\mu_2} < \beta < 0$. Then, system (11) has a semi-sign changing solution with one component changing sign and the other one positive, and the energy is the least one among all these kinds of solutions.*

Next, we give the proof of Theorem 1 in Section 2. And we shall show Theorems 2 and 3 in Sections 3 and 4, respectively.

2. Proof of Theorem 1

In this section, we consider scalar case (10). Fixing $j = 1$ or 2, we show the existence result. The proof is similar to that in [29]. Considering we need this result in the following two Theorems 2 and 3, we verify it also for the completeness of the current paper. The working space and some notations shall be given firstly. We assume that all the integrations below are taken over Ω if without special specification. Since $\lambda_j > -\lambda_1(\Omega)$, we can define the equivalent inner product in $H_0^1(\Omega)$ by

$$(u, v)_j = \int \langle \nabla u, \nabla v \rangle + \lambda_j uv, \tag{14}$$

which gives rise to a norm denoted by $\|\cdot\|_j$. We also use $|u|_q = (\int_\Omega |u|^q \mathrm{d}x)^{1/q} (1 \le q < \infty)$ and $\|u\|_0 = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{1/2}$ for convenience. Then, the energy functional of equation (10) is

$$I_j(u) = \frac{1}{2}\|u\|_j^2 - \frac{1}{p_j}\int \nu_j |u_j|^{p_j} - \frac{1}{2^*}\int \mu_j |u_j|^{2^*}. \tag{15}$$

Recall v_j is a ground state solution of (8), that is,

$$B_j := I_j(v_j) = \inf\left\{I_j(u) \colon u \in H_0^1(\Omega)\backslash\{0\},\ I_j'(u)u = 0\right\}. \tag{16}$$

For the sign changing case, we define the manifold as follows:

$$\mathcal{S}_j := \left\{u \in H_0^1(\Omega) \colon u^\pm \equiv 0, I_j'(u^\pm)u^\pm = 0\right\}. \tag{17}$$

where $u^\pm := \pm\max\{u, 0\}$ and the condition in the definition of (17) is that

$$\|u^\pm\|_j^2 = \nu_j \int |u^\pm|^{p_j} + \mu_j \int |u^\pm|^{2^*}. \tag{18}$$

It is easy to check that $\mathcal{S}_j \ne \varnothing$. Then, we define

$$\mathcal{B}_j := \inf_{u \in \mathcal{S}_j} I_j(u). \tag{19}$$

We need a conclusion in [32].

Lemma 1 (see [32]). *Consider a rectangle* $\mathbf{R} = \Pi_1^s [a_i, b_i] \subset \mathbb{R}^s$ *and a continuous function* $\mathbf{\Phi} = (\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_s)\colon \mathbf{R} \longrightarrow \mathbb{R}^s$. *If* $\mathbf{\Phi}_i|_{x_i = a_i} > 0 > \mathbf{\Phi}_i|_{x_i = b_i}$ *hold for all* $i = 1, \ldots, s$, *then* Φ *has a zero inside* $\mathbf{R}$.

Similarly as in [29], we set

$$E_j(u) := \begin{cases} 0, & u = 0, \\ \dfrac{\nu_j \int |u|^{p_j} + \mu_j \int |u|^{2^*}}{\|u\|_j^2}, & u \ne 0. \end{cases} \tag{20}$$

Then, (18) is equivalent to $E_j(u^+) = E_j(u^-) = 1$. Define

$$\begin{aligned} \mathcal{H}_j := \Big\{ & h \in C\big([0,1] \times [0,1], H_0^1(\Omega)\big) \colon\ h(t, 0) = 0, \\ & h(0, s) \ge 0, h(1, s) \le 0, I_j(h(t, 1)) \le 0, \\ & E_j(h(s, 1)) \ge 2, \quad \forall t, s \in [0, 1]\Big\}. \end{aligned} \tag{21}$$

Since $\mu_j > 0$, it is easy to see that $\mathcal{H}_j \ne \varnothing$. Then, we have the following lemma.

Lemma 2. $\mathcal{B}_j = \inf_{h \in \mathcal{H}_j} \sup_{u \in h([0,1]\times[0,1])} I_j(u)$.

Proof. For any $u \in \mathcal{S}_j$, $t, s \ge 0$, we have

$$\begin{aligned} I_j(tu^+ - su^-) &= \left[\frac{1}{2}t^2\|u^+\|_j^2 - \frac{\nu_j}{p_j}t^{p_j}\int |u^+|^{p_j} - \frac{\mu_j}{2^*}t^{2^*}\int |u^+|^{2^*}\right] \\ &\quad + \left[\frac{1}{2}s^2\|u^-\|_j^2 - \frac{\nu_j}{p_j}s^{p_j}\int |u^-|^{p_j} - \frac{\mu_j}{2^*}s^{2^*}\int |u^-|^{2^*}\right] \\ &:= f_+(t) + f_-(s). \end{aligned} \tag{22}$$

Then,

$$\sup_{t,s \ge 0} I_j(tu^+ - su^-) = f_+(1) + f_-(1) = I_j(u). \tag{23}$$

Denote the map $h \in \mathcal{H}_j$ satisfying

$$h([0,1] \times [0,1]) \subset \{tu^+ - su^- \colon t, s \ge 0\}. \tag{24}$$

By (23), we have

$$\mathcal{B}_j \ge \inf_{h \in \mathcal{H}_j} \sup_{u \in h([0,1]\times[0,1])} I_j(u). \tag{25}$$

On the other hand, for any $h \in \mathcal{H}_j$, we have

$$\begin{aligned} E_j(h(\mathbf{t})^+) - E_j(h(\mathbf{t})^-) &\begin{cases} \ge 0, & \mathbf{t} \in \{(0, s) \colon s \in [0, 1]\}, \\ \le 0, & \mathbf{t} \in \{(1, s) \colon s \in [0, 1]\}, \end{cases} \\ E_j(h(\mathbf{t})^+) + E_j(h(\mathbf{t})^-) - 2 &\begin{cases} < 0, & \mathbf{t} \in \{(t, 0) \colon t \in [0, 1]\}, \\ \ge 0, & \mathbf{t} \in \{(t, 1) \colon t \in [0, 1]\}. \end{cases} \end{aligned} \tag{26}$$

Then, by Lemma 1, we have that there exists $t_0 \in [0,1] \times [0,1]$ such that $E_j(h(\mathbf{t}_0)^+) = E_j(h(\mathbf{t}_0)^-) = 1$, i.e., $u_0 := h(\mathbf{t}_0) \in \mathcal{S}_j$. Consequently,

$$\mathcal{B}_j \le \inf_{h \in \mathcal{H}_j} \sup_{u \in h([0,1]\times[0,1])} I_j(u). \tag{27}$$

This completes the proof.

Now, we have an upper estimate for $\mathcal{B}_j$. □

Lemma 3. $\mathcal{B}_j < c_j^* := B_j + (1/N)\mu_j^{-((N-2)/2)} S^{N/2}$.

Proof. Since $v_j \in C(\bar{\Omega})$ and $v_j = 0$ on $\partial\Omega$, there exists $B(x_0, 2r_0) := \{x \colon |x - x_0| \le 2r\} \subset \Omega$ for some x_0 near $\partial\Omega$ and $r > 0$ small to be decided later. Take a cutoff function $\phi \in C_0^1(B(x_0, 2r))$ with $0 \le \phi \le 1$ and $\phi(x) \equiv 1$ for $|x - x_0| \le r$. Let

$$U_{\varepsilon,x_0}(x) := \frac{\left[N(N-2)\varepsilon^2\right]^{(N-2)/4}}{\left[\varepsilon^2 + \left|x - x_0\right|^2\right]^{(N-2)/2}}, \quad x \in \mathbb{R}^N. \tag{28}$$

Then (see [33, 34]), U_{ε,x_0} is the solution of the following equation:

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}\left(\mathbb{R}^N\right). \tag{29}$$

Set $u_\varepsilon := \phi U_{\varepsilon,x_0}$; then (see [35]),

$$\begin{aligned}
&\int \left|\nabla u_\varepsilon\right|^2 = S^{N/2} + O\left(\varepsilon^{N-2}\right),\\
&\int \left|u_\varepsilon\right|^{2^*} = S^{N/2} + O\left(\varepsilon^{N}\right),\\
&\int \left|u_\varepsilon\right|^2 \geq C\varepsilon^2 + O\left(\varepsilon^{N-2}\right),\\
&\int \left|u_\varepsilon\right| \leq C\varepsilon^{(N-2/2)},\\
&\int \left|u_\varepsilon\right|^{p_j-1} \leq C\varepsilon^{(N-2)\left(3-p_j\right)+4/2},\\
&\int \left|u_\varepsilon\right|^{2^*-1} \leq C\varepsilon^{(N-2)/2}.
\end{aligned} \tag{30}$$

Now, we show that

$$\sup_{t,s\in\mathbb{R}} I_j\left(tv_j + su_\varepsilon\right) < c_j^*. \tag{31}$$

In fact, by (30) and $((N-2)(3-p_j)+4)/(2) > (N-2)/(2)$ since $p_j < 2^*$, we have

$$\begin{aligned}
I_j\left(tv_j + su_\varepsilon\right) \leq{}& \left[\frac{1}{2}t^2\left\|v_j\right\|_j^2 - \frac{\nu_j}{p_j}t^{p_j}\int \left|v_j\right|^{p_j} - \frac{\mu_j}{2^*}t^{2^*}\int \left|v_j\right|^{2^*}\right]\\
&+ \left[\frac{1}{2}s^2\left\|u_\varepsilon\right\|_j^2 - \frac{\mu_j}{2^*}s^{2^*}\int \left|u_\varepsilon\right|^{2^*}\right]\\
&+ C_1\left[|ts|\left|\Delta v_j\right|_{L^\infty\left(B\left(x_0,2r\right)\right)}\left|u_\varepsilon\right|_1\right.\\
&\left.+ |ts|\left|v_j\right|_{L^\infty\left(B\left(x_0,2r\right)\right)}\left|u_\varepsilon\right|_1\right]\\
&+ C_2\left[\left|v_j\right|_{L^\infty\left(B\left(x_0,2r\right)\right)}\left(\int \left|su_\varepsilon\right|^{p_j-1} + \int \left|su_\varepsilon\right|^{2^*-1}\right)\right]\\
\leq{}& B_j + \frac{1}{N}\mu_j^{-((N-2))/2}S^{N/2}\\
&+ C_3\left|v_j\right|_{L^\infty\left(B\left(x_0,2r\right)\right)}\varepsilon^{(N-2)/2} - C_4\varepsilon^2\\
<{}& B_j + \frac{1}{N}\mu^{-((N-2)/2)}S^{N/2},\\
&\text{for } r > 0 \text{ small enough.}
\end{aligned} \tag{32}$$

Then by Lemma 2, the conclusion follows. □

Set

$$\mathcal{G} := \left\{u \in H_0^1(\Omega): \left|E_j\left(u^\pm\right) - 1\right| < \frac{1}{2}\right\}. \tag{33}$$

Then, we show that I_j satisfies a local (PS) condition in the following sense. □

Lemma 4. *Under the assumptions in Theorem 1, for any $c < c^*$, if $\{u_n\} \subset \mathcal{G}$ satisfies that*

$$\begin{aligned}
I_j\left(u_n\right) &\longrightarrow c,\\
I'\left(u_n\right) &\longrightarrow 0,\\
\text{as } n &\longrightarrow \infty,
\end{aligned} \tag{34}$$

then we have that $\{u_n\}$ has a convergent subsequence in $H_0^1(\Omega)$.

Proof. It is easy to check that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, so there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ up to a subsequence and $u_n \longrightarrow u$ strongly in $L^q(\Omega)$ for $1 \leq q < 2^*$. Then, $I_j'(u) = 0$ and $u_n^\pm \rightharpoonup u^\pm$ weakly in $H_0^1(\Omega)$. By the Sobolev inequality, we have that there exist positive constants C_0 and C_1 independent of n such that $\|u_n^\pm\|_j \geq C_0 > 0$ and then

$$\begin{aligned}
I_j\left(u_n\right) &= I_j\left(u_n\right) - \frac{1}{p_j}I_j'\left(u_n\right)u_n + o(1)\\
&= \left(\frac{1}{2} - \frac{1}{p_j}\right)\left\|u_n\right\|_j^2 + \left(\frac{1}{p_j} - \frac{1}{2^*}\right)\mu_j\int \left|u_n\right|^{2^*} \geq C_1 > 0.
\end{aligned} \tag{35}$$

Since

$$\begin{aligned}
o(1) &= \left(I_j'(u_n) - I_j'(u)\right)\left(u_n^\pm - u^\pm\right)\\
&= \left\|u_n^\pm - u^\pm\right\|_0^2 - \mu_j\int \left|u_n^\pm - u^\pm\right|^{2^*} + o(1)\\
&\geq \left\|u_n^\pm - u^\pm\right\|_0^2\left(1 - S^{(N/(N-2))}\mu_j\left\|u_n^\pm - u^\pm\right\|_j^{2^*-2}\right) + o(1),
\end{aligned} \tag{36}$$

we have either $u_n^\pm \longrightarrow u^\pm$ strongly in $H_0^1(\Omega)$ or $\|u_n^\pm - u^\pm\|_0^2 \geq \mu_j^{(N-2)/2}S^{N/2}$. Considering the latter case, we have

$$\begin{aligned}
I_j\left(u_n^\pm\right) &= I_j\left(u_n^\pm - u^\pm\right) + I_j\left(u^\pm\right) + o(1)\\
&= \frac{1}{N}\left\|u_n^\pm - u^\pm\right\|_0^2 + I_j\left(u^\pm\right) + o(1)\\
&\geq \frac{1}{N}\mu_j^{-((N-2)/2)}S^{N/2} + o(1).
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
c_j^* &> \lim_{n\longrightarrow\infty} I_j\left(u_n\right) = \lim_{n\longrightarrow\infty}\left[I_j\left(u_n^+\right) + I_j\left(u_n^-\right)\right]\\
&\geq B_j + \frac{1}{N}\mu_J^{-((N-2)/2)}S^{N/2} = c_j^*,
\end{aligned} \tag{38}$$

is a contradiction. That is, we must have $u_n \longrightarrow u$ strongly in $H_0^1(\Omega)$ and $u^\pm \neq 0$.

Now we are ready to show Theorem 1 based on a deformation lemma (see [36–38] for instance). □

Proof of Theorem 1. Obviously, I_j is coercive on $\mathcal{S}_j$ and then there exists a minimizing sequence $\{u_n\} \subset \mathcal{S}_j$. Let $h_n \in \mathcal{H}_j$ such that

$$h_n([0,1]\times[0,1]) \subset \{tu_n^+ - su_n^-\colon t,s\geq 0\}. \tag{39}$$

Then, by Lemma 2, we have

$$\lim_{n\longrightarrow\infty} \sup_{u\in h_n([0,1]\times[0,1])} I_j(u) = \lim_{n\longrightarrow\infty} I_j(u_n) = \mathscr{B}_j. \tag{40}$$

By the well-known deformation lemma, we have that there exists a sequence $\{\hat{u}_n\} \subset H_0^1(\Omega)$ such that

$$\operatorname{dist}\left(\hat{u}_n, h_n([0,1]\times[0,1])\right) \longrightarrow 0, \tag{41}$$

$$I_j\left(\hat{u}_n\right) \longrightarrow \mathscr{B}_j, I_j'\left(\hat{u}_n\right) \longrightarrow 0, \text{as } n \longrightarrow \infty. \tag{42}$$

Obviously, $\{\hat{u}_n\}$ is bounded and there exists $\hat{u} \in H_0^1(\Omega)$ such that $\hat{u}_n \rightharpoonup \hat{u}$ weakly in $H_0^1(\Omega)$; then, $I_j'(\hat{u}) = 0$ by the definition of weak convergence.

Now, we claim that $\{\hat{u}_n\} \subset \mathscr{G}$. In fact, by (41), there exists a sequence $\{\tilde{u}_n\}$ with

$$\begin{gathered}\tilde{u}_n = t_n u_n^+ - s_n u_n^- \in h_n([0,1]\times[0,1]),\\ \operatorname{dist}\left(\hat{u}_n, \tilde{u}_n\right) \longrightarrow 0.\end{gathered} \tag{43}$$

Then, for n large enough depending on $\xi > 0$,

$$I_j\left(\tilde{u}_n^+\right) = I_j\left(t_n u_n^+\right) > I_j\left(u_n^+\right) - \xi \geq B_j - \xi, \tag{44}$$

$$I_j\left(\tilde{u}_n^-\right) = I_j\left(s_n u_n^-\right) > I_j\left(u_n^-\right) - \xi \geq B_j - \xi. \tag{45}$$

(44), (45), and (43) implies that $\hat{u}_n^\pm \neq 0$. Since $I_j'(\hat{u}_n)\hat{u}_n^\pm = o(1)$, the claim comes true.

Therefore, by Lemmas 3 and 4, we have that $\hat{u}_n \longrightarrow \hat{u}$ strongly in $H_0^1(\Omega)$ with $\hat{u}^\pm \neq 0$, i.e., $\hat{u}$ is a sign changing solution of equation (10) and the energy $I_j(\hat{u})$ is the least one among all sign changing solutions of (10). □

3. Proof of Theorem 2

Firstly, inspired by [30, 31], we show the following theorem about the scalar equation by the minimax method (see [39]), and we would like to just give the sketch for the completeness of the current paper. We also use the same notations as in Section 2 whenever no confusion arises.

Denote $\varphi_1, \varphi_2, \varphi_3, \ldots$ as the L^2 normalized eigenfunctions of $-\Delta$, corresponding to positive eigenvalues $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots \uparrow +\infty$, counted with their multiplicity. We set $H_k := \langle \varphi_1, \varphi_2, \ldots, \varphi_k \rangle$ as the k dimensional space in $H_0^1(\Omega)$. For the fixed $\lambda_j < 0$, define

$$\lambda_j^+ := \min\left\{\lambda_k(\Omega)\colon -\lambda_j < \lambda_k(\Omega)\right\}. \tag{46}$$

We may suppose that $\lambda_j^+ = \lambda_{k_0}(\Omega)$ for some $k_0 \in \mathbb{N}$. Especially, we set $\lambda_0(\Omega) \equiv 0$.

Theorem 4. *Assume that $N \geq 4, 2 < p_j < 2^*, \lambda_j < 0$, and ν_j, $\mu_j > 0$, for $j = 1, 2$, then equation (10) has a nontrivial solution. Moreover, when $\lambda_j \leq -\lambda_1(\Omega)$, the solution changes sign.*

Proof. It is easy to check that I_j satisfies the following geometry.

(a) For $u \in (H_{k_0-1})^\perp$, there exist $r_0, \alpha_0 > 0$ such that $I_j(u) > \alpha_0$ with $\|u\|_0 = r_0$ and $I_j(u) > 0$ for $0 < \|u\|_0 < r_0$

(b) For any finite dimensional subspace $\tilde{H} \subset H_0^1(\Omega)$, it holds that $\hat{I}_j^0 \cap \tilde{H}$ is bounded in $H_0^1(\Omega)$, where $\hat{I}_j^0 := \left\{u \in H_0^1(\Omega)\colon I_j(u) \geq 0\right\}$

As in [36], we define the minimax value:

$$\hat{c}_j := \sup_{h\in\Lambda} \inf_{u\in\partial B_1 \cap (H_{k_0-1})^\perp} I_j(h(u)), \tag{47}$$

where

$$\begin{aligned}\Lambda := \Big\{&h \in C\left(H_0^1(\Omega), H_0^1(\Omega)\right)\colon h(\operatorname{Int}B_1) \subset \hat{I}_j^0 \cup B_{r_0},\\ &h \text{ is an odd homeomorphism of } H_0^1(\Omega) \text{ onto } H_0^1(\Omega)\Big\}.\end{aligned} \tag{48}$$

Obviously, $h = r_0\mathbf{id} \in \Lambda$. We also set another minimax value:

$$\hat{b}_j := \inf_{K\in\hat{\Lambda}} \sup_{u\in K} I_j(u), \tag{49}$$

where

$$\begin{aligned}\hat{\Lambda} := \Big\{&K \subset H_0^1(\Omega)\colon K \text{ is compact, symmetric, and}\\ &\gamma\left(K \cap h(\partial B_1)\right) \geq k_0, \text{for all } h \in \Lambda\Big\},\end{aligned} \tag{50}$$

and γ is the Krasnoselskii genus. It can be verified that $\hat{\Lambda} \neq \varnothing$. By properties (a) and (b), we can show that

$$0 < \alpha_0 \leq \hat{c}_j \leq \hat{b}_j < \hat{c}_j^* := \frac{1}{N}\mu_j^{-((N-2)/2)} S^{N/2}, \tag{51}$$

and I_j satisfies the $(\mathrm{PS})_{\hat{c}_j}$ condition. Also by a deformation lemma, there exists a $(\mathrm{PS})_{\hat{c}_j}$ sequence $\{u_n\}$ of I_j and then $u_n \longrightarrow u$ in $H_0^1(\Omega)$. That is, u is a nontrivial solution of equation (10).

Multiply the equations in (10) with the first eigenfunction φ_1 and integrate over Ω; we have

$$\left(\lambda_1(\Omega) + \lambda_j\right)\int_\Omega u\varphi_1 = \int_\Omega \left(\nu_j |u|^{p_j-2} u\varphi_1 + \mu_j |u|^{2^*-2} u_2 \varphi_1\right). \tag{52}$$

Then, it is easy to see that when $\lambda_j \leq -\lambda_1(\Omega), \nu_j, \mu_j > 0$ any solution of (10) must change sign. □

Proof of Theorem 2. The proof is by direct computation. Recall that v and w are sign changing solutions of (12) being well defined. Since (a_1, a_2) satisfy (13), for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}&\nu_1 \int |a_1 v|^{p-2} a_1 v\varphi + \mu_1 \int |a_1 v|^{2^*-2} a_1 v\varphi\\ &\quad + \beta \int |a_1 v|^{(2^*/2)-2} a_1 v |a_2 v|^{2^*/2} \varphi\\ &= a_1\left(\nu \int |v|^{p-2} v\phi + \int |v|^{2^*-2} v\right) = (a_1 v, \varphi)_1,\end{aligned} \tag{53}$$

That is, $(a_1 v, a_2 v)$ satisfy the first equation in system (11). Similarly, we can verify that Theorem 2 holds. □

4. Proof of Theorem 3

Inspired by [27, 29], we now consider the general case of system (11), i.e., we do not assume that $\lambda_1 = \lambda_2$. We always assume that $\beta < 0$ and $\nu_j, \mu_j > 0$ for $j = 1, 2$. Firstly, we introduce the product space $\mathbf{H} := H_0^1(\Omega) \times H_0^1(\Omega)$ as the working space. Define the inner product as

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{2} \left(u_j, v_j\right)_j, \quad \text{for } \mathbf{u} = (u_1, u_2),\ \mathbf{v} = (v_1, v_2) \in \mathbf{H}, \tag{54}$$

which gives rise to a norm on $\mathbf{H}$ denoted by $\|\cdot\|$. Recall that solutions of (11) are critical points of the following energy functional:

$$I(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 - \int \sum_{j=1}^{2} \frac{\nu_j}{p}|u_j|^{p_j} - \frac{1}{2^*}\int \sum_{j=1}^{2}\left(\mu_j |u_j|^{2^*} + 2\beta |u_1|^{2^*/2}|u_2|^{2^*/2}\right). \tag{55}$$

For $a, b \in \mathbb{R}$ with $a \le b$, we denote

$$\begin{aligned} I^b &:= \{\mathbf{u} \in \mathbf{H}\colon I(\mathbf{u}) \le b\}, \\ \hat{I}^a &:= \{\mathbf{u} \in \mathbf{H}\colon I(\mathbf{u}) \ge a\}, \\ I^{-1}([a,b]) &:= I^b \cap \hat{I}^a. \end{aligned} \tag{56}$$

Similar to $\mathcal{S}_j$ related to the scalar equation, we define

$$\begin{aligned} \mathcal{S} := \{&(u_1, u_2) \in \mathbf{H}\colon u_1^{\pm}, u_2 \equiv 0, I'(u_1^{\pm}, u_2)(u_1^{\pm}, 0) \\ &= I'(u_1, u_2)(0, u_2) = 0\}. \end{aligned} \tag{57}$$

That is,

$$\begin{aligned} \|u_1^{\pm}\|_1^2 &= \int \left(\nu_1 |u_1^{\pm}|^{p_1} + \mu_1 |u_1^{\pm}|^{2^*} + \beta |u_1^{\pm}|^{2^*/2}|u_2|^{2^*/2}\right), \\ \|u_2\|_2^2 &= \int \left(\nu_2 |u_2|^{p_2} + \mu_2 |u_1|^{2^*} + \beta |u_1|^{2^*/2}|u_2|^{2^*/2}\right). \end{aligned} \tag{58}$$

Let $U_1, U_2 \in C_0^\infty(\Omega)$ with $U_1^{\pm}, U_2 \equiv 0$ and $\operatorname{supp} U_1 \cap \operatorname{supp} U_2 = \varnothing$, it is easy to see that there exist $t_1, t_2, s > 0$ such that $(t_1 U_1^+ - t_2 U_2^-, sU_2) \in \mathcal{S}$. Thus, $\mathcal{S} \ne \varnothing$. By Sobolev inequality, it is easy to see that for any $\mathbf{u} \in \mathcal{S}$, there exists a constant $C_0 > 0$ such that

$$\|u_1^{\pm}\|_1, \|u_2\|_2, |u_1^{\pm}|_{2^*}, |u_2|_{2^*} \ge C_0 > 0. \tag{59}$$

Set

$$3\varepsilon_0 := \min\left\{\left(\frac{1}{2} - \frac{1}{p_1}\right)C_0^2, \left(\frac{1}{2} - \frac{1}{p_1}\right)C_0^2\right\}. \tag{60}$$

Then, we define

$$\mathcal{B} := \inf_{u \in \mathcal{S}} I(\mathbf{u}). \tag{61}$$

We firstly have a lower bound for $\mathcal{B}$.

Lemma 5. *Assume that $-\sqrt{\mu_1\mu_2} \le \beta < 0$; then I is coercive on $\mathcal{S}$ and $\mathcal{B} > 0$.*

Proof. We may assume that $p_1 \ge p_2$. For $\mathbf{u} \in \mathcal{S}$, note that for $\beta \ge -\sqrt{\mu_1\mu_2}$, it holds that

$$\int \sum_j \mu_j |u|_j^{2^*} + 2\beta |u_1|^{2^*/2}|u_2|^{2^*/2} \ge 0. \tag{62}$$

Then, we have

$$\begin{aligned} I(\mathbf{u}) &= I(\mathbf{u}) - \frac{1}{p_2} I'(\mathbf{u})(u_1, u_2) \\ &= \left(\frac{1}{2} - \frac{1}{p_2}\right)\left(\|\mathbf{u}^2\|\right) + \left(\frac{1}{p_2} - \frac{1}{p_1}\right)\int \nu_1 |u_1|^{p_1} \\ &\quad + \left(\frac{1}{p_2} - \frac{1}{2^*}\right)\int \left(\sum_j \mu_j |u_j|_j^{2^*} + 2\beta |u_1|^{2^*/2}\|u_2\|^{2^*/2}\right) \\ &\ge \left(\frac{1}{2} - \frac{1}{p_2}\right)\|\mathbf{u}\|^2. \end{aligned} \tag{63}$$

(63) implies that I is coercive on $\mathcal{S}$. Then, by (59), we have $\mathcal{B} > 0$. □

We shall give an upper bound for $\mathcal{B}$ later by defining another manifold and the infimum on it. The idea is similar to that in [27], but since the corresponding equation (10) is different and more complex as well as system (11), some new tricks should be used in the current paper, so we shall give the details of the proofs. Let

$$\begin{aligned} \mathcal{M} := \{&(u_1, u_2) \in \mathbf{H}\colon u_1, u_2 \equiv 0, I'(u_1, u_2)(u_1, 0) \\ &= I'(u_1, u_2)(0, u_2) = 0\}. \end{aligned} \tag{64}$$

Obviously, $\mathcal{S} \subset \mathcal{M}$. Then, we define

$$B := \inf_{u \in \mathcal{M}} I(\mathbf{u}). \tag{65}$$

Before the estimate for the bound of $\mathcal{B}$, we need some preliminaries. □

Lemma 6. *Assume that $-\sqrt{\mu_1\mu_2} \le \beta < 0$; then, for any $\mathbf{u} \in \mathcal{S}$, $\sup_{t_1,t_2,s \ge 0} I(t_1 u_1^+ - t_2 u_1^-, su_2)$ is attained only by $t_1 = t_2 = s = 1$. Similarly, for any $\mathbf{u} \in \mathcal{M}$, $\sup_{t,s>0} I(tu_1, su_2)$ is attained only by $t = s = 1$.*

Proof. For $\mathbf{u} \in \mathcal{S}, t_1, t_2, s \ge 0$; set

$$\begin{aligned} F(t_1, t_2, s) &:= I(t_1 u_1^+ - t_2 u_1^-, su_2) \\ &= \frac{1}{2}\left(t_1^2\|u^+\|_1^2 + t_2^2\|u^-\|_1^2 + s^2\|u_2\|_2^2\right) \\ &\quad - \left(\int \frac{\nu_1}{p_1} t_1^{p_1}|u_1^+|^p + \frac{\nu_1}{p_1} t_2^{p_1}|u_1^-|^{p_1} + \frac{\nu_1}{p_2} s^{p_2}|u_2|^{p_2}\right) \\ &\quad - \frac{1}{2^*}\int \left(\mu_1 t_1^{2^*}\|u_1^+\|^{2^*} + \mu_1 t_2^{2^*}|u_1^-|^{2^*} + \mu_2 s^{2^*}|u_2|^{2^*}\right) \\ &\quad - \frac{2\beta}{2^*} s^{2^*/2}\int \left(t_1^{2^*/2}|u_1^+|^{2^*/2} + t_2^{2^*/2}|u_1^-|^{2^*/2}\right)|u_2|^{2^*/2}. \end{aligned} \tag{66}$$

Then, $F_{t_1}|_{(1,1,1)} = F_{t_2}|_{(1,1,1)} = F_s|_{1,1,1} = 0$ and

$$\begin{aligned}
F_{t_1t_1}\big|_{(1,1,1)} &= \|u_1^+\|_1^2 - (p_1-1)\int \nu_1|u_1^+|^{p_1} - (2^*-1)\int \mu_1|u_1^+|^{2^*} \\
&\quad -\left(\frac{2^*}{2}-1\right)\beta\int |u_1^+|^{2^*/2}|u_2|^{2^*/2}, \\
F_{t_2t_2}\big|_{(1,1,1)} &= \|u_1^-\|_1^2 - (p_1-1)\int \nu_1|u_1^-|^{p_1} - (2^*-1)\int \mu_1|u_1^-|^{2^*} \\
&\quad -\left(\frac{2^*}{2}-1\right)\beta\int |u_1^-|^{2^*/2}|u_2|^{2^*/2}, \\
F_{ss}\big|_{(1,1,1)} &= \|u_2\|_2^2 - (p_2-1)\int \nu_2|u_2|^{p_2} - (2^*-1)\int \mu_2|u_2|^{2^*} \\
&\quad -\left(\frac{2^*}{2}-1\right)\beta\int |u_1|^{2^*/2}|u_2|^{2^*/2}, \\
F_{t_1t_2}\big|_{(1,1,1)} &= F_{t_2t_1}\big|_{(1,1,1)} = 0, \\
F_{t_1s}\big|_{(1,1,1)} &= F_{st_1}\big|_{(1,1,1)} = -\frac{2^*}{2}\beta\int |u_1^+|^{2^*/2}|u_2|^{2^*/2}, \\
F_{t_2s}\big|_{(1,1,1)} &= F_{st_2}\big|_{(1,1,1)} = -\frac{2^*}{2}\beta\int |u_1^-|^{2^*/2}|u_2|^{2^*/2}.
\end{aligned} \tag{67}$$

Set

$$H_F := \begin{pmatrix} F_{t_1t_1} & F_{t_1t_2} & F_{t_1s} \\ F_{t_2t_1} & F_{t_2t_2} & F_{t_2s} \\ F_{st_1} & F_{st_2} & F_{ss} \end{pmatrix}_{(1,1,1)}. \tag{68}$$

Denote

$$\begin{aligned}
A_1^{\pm} &= \int \nu_1|u_1^{\pm}|^{p_1}, \\
A_2 &= \int \nu_2|u_2|^{p_1}, \\
D_1^{\pm} &= \int \mu_1|u_1^{\pm}|^{2*}, \\
D_2 &= \int \mu_2|u_2|^{2*}, \\
D_1^{\pm} &= \int \beta|u_1^{\pm}|^{2^*/2}|u_2|^{2^*/2}, \\
D &= \int \beta|u_1|^{2^*/2}|u_2|^{2^*/2}.
\end{aligned} \tag{69}$$

Then,

$$\begin{aligned}
a_{11} &:= F_{t_1t_1}\big|_{(1,1,1)} = -(p_1-2)A_1^+ - (2^*-2)D_1^+ - \left(2-\frac{2^*}{2}\right)D^+ < 0, \\
a_{22} &:= F_{t_2t_2}\big|_{(1,1,1)} = -(p_1-2)A_1^- - (2^*-2)D_1^- - \left(2-\frac{2^*}{2}\right)D^- < 0, \\
a_{33} &:= F_{ss}\big|_{(1,1,1)} = -(p_2-2)A_2 - (2^*-2)D_2 - \left(2-\frac{2^*}{2}\right)D < 0.
\end{aligned} \tag{70}$$

Note that $D = D^+ + D^-$, $D^2 \le (D_1^+ + D_1^-)\,D_2$, and $2\,D \le D_1^+ + D_1^- + D_2$. We can show that H_F is negative definite. Thus, $(1,1,1)$ is the unique maximum point of $F(t_1,t_2,s)$. □

Lemma 7. *Assume that $-\sqrt{\mu_1\mu_2} < \beta < 0$; then, for any $\mathbf{u} \in \mathbf{H}$ with $u_1, u_2 \equiv 0$, we have that there exist unique $t_u, s_u > 0$ such that $(t_u u_1, s_u u_2) \in \mathcal{M}$. Moreover, t_u and s_u are continuous with respect to $\mathbf{u}$.*

Proof. Proof. For any $\mathbf{u} \in \mathbf{H}$ with $u_1, u_2 \equiv 0$, we denote

$$\begin{aligned}
E_1 &= \|u_1\|_1^2, \\
E_2 &= \|u_2\|_2^2,
\end{aligned} \tag{71}$$

$$\begin{aligned}
A_1 &= \int \nu_1|u_1|^{p_1}, \\
A_2 &= \int \nu_2|u_2|^{p_2},
\end{aligned} \tag{72}$$

$$\begin{aligned}
D_1 &= \int \mu_1|u_1|^2, \\
D_2 &= \int \mu_2|u_2|^{2^*},
\end{aligned} \tag{73}$$

$$D = \int \beta|u_1|^{2^*/2}|u_2|^{2^*/2}. \tag{74}$$

Then, $D < 0$ and $D_1D_2 > D^2$. $(tu_1, su_2) \in \mathcal{M}$ for some $t, s > 0$ is equivalent to

$$t^2E_1 = t^{p_1}A_1 + t^{2^*}D_1 + t^{2^*/2}s^{2^*/2}D, \tag{75}$$

$$s^2E_2 = s^{p_2}A_2 + s^{2^*}D_2 + t^{2^*/2}s^{2^*/2}D, \quad t, s > 0. \tag{76}$$

By (75), we have

$$s^{2^*/2} = \frac{t^{2-(2^*/2)}E_1 - t^{p_1}A_1 - t^{2^*}D_1}{D} := \frac{G(t)}{D} := \frac{t^{2-(2^*/2)}\widetilde{G}(t)}{D}, \tag{77}$$

where $s > 0$ implies that $\widetilde{G}(t) < 0$. Since $\widetilde{G}'(t) < 0$ and for $t > 0$ small enough, $\widetilde{G}(t) > 0$, there exists a unique $t_0 > 0$ such that $\widetilde{G}(t_0) = 0$; thus, $t > t_0$. Combining (76) and (77), it is left to show that

$$\begin{aligned}
F(t) &:= \left(\frac{G(t)}{D}\right)^{(4/2^*)-1} t^{-(2^*/2)}E_2 - \left(\frac{G(t)}{D}\right)^{(2p_2/2^*)-1} t^{-(2^*/2)}A_2 \\
&\quad - \frac{G(t)}{D}t^{-(2^*/2)}D_2 - D,
\end{aligned} \tag{78}$$

has a solution $t > t_0$. Note that $2 - (2^*/2) < p_1 - (2^*/2) < (2^*/2)$ and $4/2^* - 1 < (2p_2/2^*) - 1 < 1$; by direct calculation, we can check that

$$\lim_{t\longrightarrow+\infty} F(t) = \frac{D_1D_2 - D^2}{D} < 0,$$
$$\lim_{t\longrightarrow t_0} F(t_0) = -D > 0. \tag{79}$$

This implies that (75) and (76) have a positive solution (t_u, s_u). The uniqueness of (t_u, s_u) follows from Lemma 6.

For the continuity of (t_u, s_u) with respect to $\mathbf{u}$, we take a sequence $\{\mathbf{u}_n\}$ with $\mathbf{u}_n \longrightarrow \mathbf{u}$ strongly in $\mathbf{H}$. We may assume that $(t_u, s_u) = (1,1)$ by replacing $\mathbf{u}_n$ by $(t_{u_n}\mathbf{u}_{1n}, s_{u_n}\mathbf{u}_{2n})$ if necessary. That is, $\mathbf{u} \in \mathscr{M}$. Using similar denotations to (74) and rewriting $(t_{u_n}, s_{u_n}) = (t_n, s_n)$ for convenience, we have

$$t_n^2 E_{1n} = t_n^{p_1} A_{1n} + t_n^{2^*} D_{1n} + t_n^{2^*/2} s_n^{2^*/2} D_n, \tag{80}$$

$$s_n^2 E_2 = s_n^{p_2} A_{2n} + s_n^{2^*} D_{2n} + t_n^{2^*/2} s_n^{2^*/2} D_n, \tag{81}$$

$$E_1 = A_1 + D_1 + D, E_2 = A_2 + D_2 + D. \tag{82}$$

If $t_n \longrightarrow \infty$, then by (81), we have $s_n \longrightarrow \infty$. Therefore,

$$\begin{aligned} D^2 &= \lim_{n\longrightarrow+\infty} D_n^2 \\ &= \lim_{n\longrightarrow+\infty} \frac{t_n^{2-(2^*/2)}E_{1n} - t_n^{p_1-(2^*/2)}A_{1n} - t_n^{2^*/2}D_{1n}}{s_n^{2^*/2}} \cdot \frac{s_n^{2-(2^*/2)}E_{2n} - s_n^{p_2-(2^*/2)}A_{2n} - s_n^{2^*/2}D_{2n}}{t_n^{2^*/2}} \\ &= \lim_{n\longrightarrow+\infty} \left(t_n^{2-2^*}E_{1n} - t_n^{p_1-2^*}A_{1n} - D_{1n}\right)\left(s_n^{2-2^*}E_{2n} - s_n^{p_2-2^*}A_{2n} - D_{2n}\right) \\ &= \lim_{n\longrightarrow+\infty} D_{1n}D_{2n} = D_1D_2 > D^2, \end{aligned} \tag{83}$$

is a contradiction. Thus, t_n and s_n are uniformly bounded and we may assume that $t_n \longrightarrow \hat{t}$, $s_n \longrightarrow \hat{s}$ as $n \longrightarrow +\infty$. By (80) and (81), we have $(\hat{t}u_1, \hat{s}u_2) \in \mathscr{M}$. Then, $\hat{t} = \hat{s} = 1$, that is, $(t_{u_n}, s_{u_n}) \longrightarrow (t_u, s_u)$.

Now, we show the following: □

Lemma 8. *Assume that* $-\sqrt{\mu_1\mu_2} < \beta < 0$, *then*

$$\mathscr{B} < \min\left\{B + \frac{1}{N}\mu_1^{-(N-2/2)}S^{N/2}, \mathscr{B}_1 + \frac{1}{N}\mu_2^{-((N-2)/2)}S^{N/2}\right\}. \tag{84}$$

Proof.

(i) We first show that $\mathscr{B} < B + (1/N)\mu_1^{-((N-2)/2)}S^{N/2}$. In [27], the authors show that (11) has a positive least energy solution; we denote it as (U_1, U_2) with the energy

$$\left\{I(U_1,U_2) = B < \min\left\{\frac{1}{N}\mu_1^{-(N-2/2)}S^{N/2} + B_2, B_1 + \frac{1}{N}\mu_2^{-(N-2/2)}S^{N/2}\right\}\right\}, \tag{85}$$

and $U_1, U_2 \in C^2(\Omega) \cap C(\bar{\Omega})$. For $r > 0$ small, take $x_r \in \Omega$ such that $dist(x_r, \partial\Omega) = 4r$ and

$$|U_1(x)| + |U_2(x)| \le Cr, \quad \forall x \in B(x_r, 3r). \tag{86}$$

Let $\phi \in C_0^\infty(B(0,2))$ with $0 \le \phi \le 1$ and $\phi \equiv 1$ on $B(0,1)$. Set $\phi_r(x) := 1 - \phi((x - x_r)/r)$. Then, there exists a positive constant such that $|\nabla\phi_r(x)| \le (c/r)$. Define $(U_{1r}, U_{2r}) := (\phi_r U_1, \phi_r U_2)$. Then, for $j = 1, 2$, we have

$$\begin{aligned} \int |\nabla U_{jr}|^2 &\le \int |\nabla U_j|^2 + Cr^N, \\ \int U_{jr}^q &\ge \int U_j^q - Cr^{N+q}, \quad q = 2, p_j, 2^*. \end{aligned} \tag{87}$$

By Lemma 7, there exist $t_r, s_r > 0$ such that $(t_r U_{1r}, s_r U_{2r}) \in \mathscr{M}$. That is,

$$\begin{aligned} t_r^2\|U_{1r}\|_1^2 &= t_r^p \nu_1 \int U_{1r}^{p_1} + t_r^{2^*}\mu_1 \int U_{1r}^{2^*} \\ &\quad + \beta t_r^{2^*/2} s_r^{2^*/2} \int U_{1r}^{2^*/2} U_{2r}^{2^*/2}, \end{aligned} \tag{88}$$

$$\begin{aligned} s_r^2\|U_{2r}\|_1^2 &= s_r^p \nu_2 \int U_{2r}^{p_2} + s_r^{2^*}\mu_1 \int U_{2r}^{2^*} \\ &\quad + \beta t_r^{2^*/2} s_r^{2^*/2} \int U_{1r}^{2^*/2} U_{2r}^{2^*/2}. \end{aligned} \tag{89}$$

Then similar to the proof of (83), it can be shown that t_r and s_r are uniformly bounded. Thus, up to a subsequence, there exist $t_0, s_0 > 0$ such that $t_r \longrightarrow t_0$ and $s_r \longrightarrow s_0$. Let $r \longrightarrow 0$ in (88) and (89); it implies that $(t_0U_1, s_0U_2) \in \mathscr{M}$. Then, $t_0 = s_0 = 1$, and we may assume that $(1/2) \le t_r$ and $s_r \le 2$ for $r > 0$ small. By (87), we have

$$\begin{aligned} I(t_rU_{1r}, s_rU_{2r}) &\le I(t_rU_1, s_rU_2) + Cr^N \le I(U_1, U_2) + Cr^N \\ &= B + Cr^N. \end{aligned} \tag{90}$$

Let V_{jr} $(j = 1, 2)$ be the solution of problem (8) when $\Omega = B(0, r)$; then, $(t_rU_{1r} - V_{1r}(\cdot - x_r), s_rU_{2r}) \in \mathscr{S}$ and from [40], it holds that

$$I_j\left(V_{jr}\right) \le \frac{1}{N}\mu_j^{-((N-2)/2)}S^{N/2} - C_1 r^{(2N-4)/(N-4)}, \quad j = 1, 2. \tag{91}$$

Thus,

$$\begin{aligned}\mathscr{B} &\le I\left(t_r U_{1r} - V_{1r}(\cdot - x_r), s_r U_{2r}\right)\\ &= I\left(t_r U_{1r}, s_r U_{2r}\right) + I_1\left(V_{1r}(\cdot - x_r)\right)\\ &\le B + Cr^N + \frac{1}{N}\mu_1^{-((N-2)/2)}S^{N/2} - C_1 r^{(2N-4)/(N-4)}\\ &< B + \frac{1}{N}\mu_1^{-((N-2)/2)}S^{N/2},\end{aligned} \tag{92}$$

for $r > 0$ small.

(ii) Next, we show that $\mathscr{B} < \mathscr{B}_1 + (1/N)\mu_2^{-(N-2)/2}S^{N/2}$. Let w_j be the sign changing solution of (8) with $I_j(w_j) = \mathscr{B}_j$ for $j = 1, 2$. Then, $w_j \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\left\{w_j^+ > 0\right\} := \left\{x \in \Omega\colon w_j(x) > 0\right\} \ne \varnothing$. For any $R > 0$, we may take $y_R \in \left\{w_j^+ > 0\right\}$ with $\operatorname{dist}(y_R, \partial(\left\{w_j^+ > 0\right\})) = 4R$. Then, $B(y_r, 3r) \subset \left\{w_j^+ > 0\right\}$ and $|w_j^+| \le CR, \forall x \in B(y_R, 3R)$. Recall ϕ and ϕ_R in the above step (i); we define $U_{1R} := \phi_R w_1^+$. Since $I_1'(w_1)w_1^+ = 0$, we have

$$\left\|w_1^+\right\|_1^2 = \nu_1 \int \left|w_1^+\right|^{p_1} + \mu_1 \int \left|w_1^+\right|^{2^*}. \tag{93}$$

Then, there exists $t_R > 0$ such that

$$\left\|t_R U_{1R}^+\right\|_1^2 = \nu_1 \int \left|t_R U_{1R}^+\right|^{p_1} + \mu_1 \int \left|t_R U_{1R}^+\right|^{2^*}, \tag{94}$$

and $I(t_R U_{1R}, 0) = \max I(tU_{1R}, 0)$. It is easy to see that $(t_R U_{1R} - w_1^-, V_{2R}(\cdot \overset{t>0}{-} y_R)) \in \mathscr{S}$, where V_{2R} is the positive solution of problem (8) in the ball $B(0, R)$ for $j = 2$. Then by (91), we have

$$\begin{aligned}\mathscr{B} &\le I\left(t_R U_{1R} - w_1^-, V_{2R}(\cdot - y_R)\right)\\ &= I\left(t_R U_{1R}, 0\right) + I_1\left(w_1^-\right) + I_2\left(V_{2R}(\cdot - y_R)\right)\\ &\le I_1\left(w_1^+\right) + CR^N + I_1\left(w_1^-\right) + \frac{1}{N}\mu_2^{-(N-2/2)}S^{N/2}\\ &\quad - C_1 R^{(2N-4)/N-4}\\ &< \mathscr{B}_1 + \frac{1}{N}\mu_2^{-(N-2/2)}S^{N/2},\end{aligned} \tag{95}$$

for $R > 0$ small.

For $(u_1, u_2) \in \mathscr{S}$, we define $u_{1\rho} := (1-\rho)u_1^+ - \rho u_1^-$, $\rho \in [0, 1]$. Let $\mathbf{u}_\rho := (u_{1\rho}, u_2)$ and recall that (t_{u_ρ}, s_{u_ρ}) is defined in Lemma 7; then, we can define a continuous map $\eta_u\colon [0, 1] \longrightarrow \mathscr{M}$ by

$$\eta_u(\rho) := \left(t_{u_\rho} u_{1\rho}, s_{u_\rho} u_2\right). \tag{96}$$

Then, we have the following conclusion and recall that ε_0 is defined in (60). □

Lemma 9. *For any $\mathbf{u} \in \mathscr{S}$ with $I(\mathbf{u}) < \mathscr{B} + \varepsilon_0$, there exists a small $\rho_o \in (0, (1/4))$ such that*

$$I'(\eta_u(\rho_0))\left(t_{u_{\rho_0}} u_{1\rho_0}^+, 0\right) < 0, \tag{97}$$

$$I'(\eta_u(1-\rho_0))\left(t_{u_{1-\rho_0}} u_{1(1-\rho_0)}^+, 0\right) > 0, \tag{98}$$

$$\begin{aligned} &I(\eta_u(\rho_0)),\\ &I(\eta_u(1-\rho_0)) < \mathscr{B} - \varepsilon_0.\end{aligned} \tag{99}$$

Proof. It is easy to see that there exist constants C_1 and $C_2 > 0$ such that

$$\begin{aligned} &C_1 \le t_{u_\rho},\\ &s_{u_\rho} \le C_2,\\ &\forall \rho \in [0, 1].\end{aligned} \tag{100}$$

Since $I'(\eta_u(\rho))(t_{u_\rho} u_{1\rho}, 0) = 0$ and $u_{1\rho}^- = \rho u_1^-$, we have

$$\begin{aligned} &I'(\eta_u(\rho))\left(t_{u_\rho} u_{1\rho}^+, 0\right)\\ &\quad = -I'(\eta_u(\rho))\left(t_{u_\rho}\rho u_1^-, 0\right)\\ &\quad \le -t_{u_\rho}^2\rho^2\left\|u_1^-\right\|_1^2 + t_{u_\rho}^{p_1}\rho^{p_1}\nu_1\int\left|u_1^-\right|^{p_1} + t_{u_\rho}^{2^*}\rho^{2^*}\mu_1\int\left|u_1^-\right|^{2^*}\\ &\quad \le C_1^2\rho^2 u_1^{-2} + C_2^{p_1}\rho^{p_1}\nu_1\int\left|u_1^-\right|^{p_1} + C_2^{2^*}\rho^{2^*}\mu\int\left|u_1^-\right|^{2^*}\\ &\quad < 0, \text{for } \rho > 0 \text{ small.}\end{aligned} \tag{101}$$

Similarly, note that $u_{1(1-\rho)}^+ = \rho u_1^+$; we have

$$\begin{aligned} &I'(\eta_u(1-\rho))\left(t_{u_{1-\rho}} u_{1(1-\rho)}^+, 0\right)\\ &\quad \ge t_{u_{1-\rho}}^2\rho^2\left\|u_1^+\right\|_1^2 - t_{u_{1-\rho}}^{p_1}\rho^{p_1}\nu_1\int\left|u_1^+\right|^{p_1} + t_{u_{1-\rho}}^{2^*}\rho^{2^*}\mu_1\int\left|u_1^+\right|^{2^*}\\ &\quad \ge C_1^2\rho^2\left\|u_1^-\right\|_1^2 - C_2^{p_1}\rho^{p_1}\nu_1\int\left|u_1^-\right|^{p_1} - C_2^{2^*}\rho^{2^*}\mu_1\int\left|u_1^-\right|^{2^*}\\ &\quad > 0, \text{for } \rho > 0 \text{ small.}\end{aligned} \tag{102}$$

Since $\mathbf{u} \in \mathscr{S}$, we have $(u_1^+, u_2), (-u_1^-, u_2) \in \mathscr{M}$. Then,

$$\begin{aligned}
I(\eta_u(0)) &= I(u_1^+, u_2) \\
&= I(u_1, u_2) - \left[\frac{1}{2}\|u_1^-\|_1^2 - \frac{\nu_1}{p_1}\int |u_1^-|^{p_1} - \frac{\mu_1}{2^*}\int |u_1^-|^{2^*}\right. \\
&\quad \left. + \frac{2|\beta|}{2^*}\int |u_1^-|^{2^*/2} u_2^{2^*/2}\right] \\
&= I(u_1, u_2) - \left(\frac{1}{2} - \frac{1}{p_1}\right)\|u_1^-\|_1^2 - \left[\left(\frac{1}{p_1} - \frac{1}{2^*}\right)\int \mu_1 |u_1^-|^{2^*}\right. \\
&\quad \left. + \left(\frac{2|\beta|}{2^*} - \frac{1}{p_1}\right)\int |u_1^-|^{2^*/2} u_2^{2^*/2}\right] \\
&\le I(u_1, u_2) - \left(\frac{1}{2} - \frac{1}{p_1}\right)\|u_1^-\|_1^2 \\
&< \mathscr{B} - 2\varepsilon_0.
\end{aligned} \tag{103}$$

Similarly, we have

$$\begin{aligned}
I(\eta_u(1)) &= I(-u_1^-, u_2) \\
&= I(u_1, u_2) - \left[\frac{1}{2}u_1^{+2} - \frac{\nu_1}{p_1}\int |u_1^+|^{p_1} - \frac{\mu_1}{2^*}\int |u_1^+|^{2^*}\right. \\
&\quad \left. + \frac{2|\beta|}{2^*}\int |u_1^+|^{2^*/2} u_2^{2^*/2}\right] \\
&\le I(u_1, u_2) - \left(\frac{1}{2} - \frac{1}{p_1}\right)u_1^{+2} \\
&< \mathscr{B} - 2\varepsilon_0.
\end{aligned} \tag{104}$$

Since I' and I are continuous with respect to ρ, (101), (102), (103), and (104) imply (97), (98), and (99), respectively. □

Now, we show the existence of a $(\mathrm{PS})_{\mathscr{B}}$ sequence of I. Precisely, we have the following important lemma.

Lemma 10. *There exist a sequence $\{\mathbf{u}_n\} \subset \mathscr{M}$ and a constant $C > 0$ such that*

$$\begin{gathered}
I(\mathbf{u}_n) \longrightarrow \mathscr{B}, \\
I'(\mathbf{u}_n) \longrightarrow 0, \\
\text{as } n \longrightarrow +\infty; \\
\|u_{1n}^{\pm}\|_1, \|u_{2n}\|_2 \ge C, \\
\forall n \ge 1.
\end{gathered} \tag{105}$$

Proof. By Lemma 5 and the Ekeland's variational principle, there exists a minimizing sequence $\{\tilde{u}_n\} = \{(\tilde{u}_{1n}, \tilde{u}_{2n})\} \subset \mathscr{S}$ such that

$$\begin{gathered}
I(\tilde{u}_n) < \mathscr{B} + \varepsilon_0, \\
I'(\tilde{u}_n) \longrightarrow 0.
\end{gathered} \tag{106}$$

Let $\eta_n(\rho) = (\eta_n^1(\rho), \eta_n^2(\rho)) := \eta_{\tilde{u}_n}(\rho)$; by Lemma 9, there exists $\rho_n \in (0, (1/4))$ such that

$$I'(\eta_n(\rho_n))(\eta_n^1(\rho_n)^+, 0) < 0, \tag{107}$$

$$I'(\eta_n(1-\rho_n))(\eta_n^1(1-\rho_n)^+, 0) > 0, \tag{108}$$

$$\begin{gathered}
I(\eta_n(\rho_n)), \\
I(\eta_n(1-\rho_n)) < \mathscr{B} - \varepsilon_0.
\end{gathered} \tag{109}$$

We claim that there exists a sequence $\{\mathbf{u}_n\} \subset \mathscr{M}$ such that

$$\operatorname{dist}(\mathbf{u}_n, \eta_n([\rho_n, 1-\rho_n])) \longrightarrow 0, \tag{110}$$

$$\begin{gathered}
I'|_{\mathscr{M}}(u_n) \longrightarrow 0, \\
I(u_n) \longrightarrow \mathscr{B}.
\end{gathered} \tag{111}$$

If not, then we assume that there exists small $\varepsilon_1 > 0$ such that

$$\forall \mathbf{u} \in I|_{\mathscr{M}}^{-1}([\mathscr{B} - \varepsilon_1, \mathscr{B} + \varepsilon_1]) \cap S_{\varepsilon_1}: \|I|_{\mathscr{M}}'(\mathbf{u})\| \ge \varepsilon_1, \tag{112}$$

where

$$\begin{gathered}
I|_{\mathscr{M}}^{-1}([\mathscr{B} - \varepsilon_1, \mathscr{B} + \varepsilon_1]) := \{\mathbf{u} \in \mathscr{M}: I(\mathbf{u}) \in [\mathscr{B} - \varepsilon_1, \mathscr{B} + \varepsilon_1]\}, \\
S := \eta_n([\rho_n, 1-\rho_n]),
\end{gathered} \tag{113}$$

and S_{ε_1} is the ε_1 neighborhood of S. Then, by the well-known deformation lemma, there exist a continuous map ξ: $[0,1] \times \mathscr{M} \longrightarrow \mathscr{M}$ and an $\varepsilon_2 \in (0, (\min\{\varepsilon_0, \varepsilon_1\})/4)$ such that

(1) $\xi(0, \mathbf{u}) = \mathbf{u}$, $\forall \mathbf{u} \in \mathscr{M}$

(2) $\xi(t, \mathbf{u}) = \mathbf{u}$, $\forall \mathbf{u} \notin I^{-1}([\mathscr{B} - 2\varepsilon_2, \mathscr{B} + 2\varepsilon_2])$, $t \in [0,1]$

(3) $\xi(1, I|_{\mathscr{M}}^{\mathscr{B}+\varepsilon_2} \cap S) \subset I|_{\mathscr{M}}^{\mathscr{B}-\varepsilon_2}$

By Lemma 6, for n large enough, we have that

$$\sup_{\rho \in [\rho_n, 1-\rho_n]} I(\eta_n(\rho)) \le I(\tilde{u}_n) < \mathscr{B} + \varepsilon_2. \tag{114}$$

Then, by the properties (2) and (3) of ξ and (109), we have

$$\sup_{\rho \in [\rho_n, 1-\rho_n]} I(\xi(1, \eta_n(\rho))) \le \mathscr{B} - \varepsilon_2, \tag{115}$$

$$\begin{gathered}
\xi(1, \eta_n(\rho_n)) = \eta_n(\rho_n), \\
\xi(1, \eta_n(1-\rho_n)) = \eta_n(1-\rho_n).
\end{gathered} \tag{116}$$

Similarly as in (20), we define

$$E(u_1,u_2) := \begin{cases} 0, & u_1 = 0, \\ \dfrac{\nu_1 \int |u_1|^{p_1} + \mu_1 \int |u_1|^{2^*} + \beta \int |u_1|^{2^*/2} |u_2|^{2^*/2}}{\|u_1\|_1^2}, & u_1 \neq 0. \end{cases} \tag{117}$$

We rewrite $\eta(\rho) := (\eta^1(\rho), \eta^2(\rho)) := \xi(1, \eta_n(\rho))$ for convenience. $\eta(\rho) \in \mathscr{M}$ implies that $E((\eta^1(\rho)), \eta^2(\rho)) = 1$. By (107) and (108), we have that

$$\begin{aligned} E\left(\left(\eta^1(\rho_n)\right)^+, \eta^2(\rho_n)\right) &> 1, \\ E\left(\left(\eta^1(1-\rho_n)\right)^+, \eta^2(1-\rho_n)\right) &< 1. \end{aligned} \tag{118}$$

Then, by the continuity of E, we can define

$$\rho_0 := \max\left\{\rho \in [\rho_n, 1-\rho_n] : E\left(\eta^1(\rho)^+, \eta^2(\rho)\right) = 1\right\}. \tag{119}$$

By definition of ρ_0, there exists $\delta_0 \in (0, 1-\rho_n-\rho_0)$ such that $E(\eta^1(\rho)^+, \eta^2(\rho)) < 1$ for any $\rho \in (\rho_0, \rho_0+\delta_0)$. Then, $E(\eta^1(\rho)^-, \eta^2(\rho)) > 1$ for any $\rho \in (\rho_0, \rho_0+\delta_0)$. Thus, $E(\eta^1(\rho_0)^-, \eta^2(\rho_0)) \geq 1$. Hence, $E(\eta^1(\rho_0)^\pm, \eta^2(\rho_0)) = 1$. That is, $\eta(\rho_0) \in \mathscr{S}$. But by (115), we have $I(\eta(\rho_0)) < \mathscr{B}$, a contradiction with the definition of $\mathscr{B}$. Therefore, the claim (111) becomes true.

Now, we can choose $\hat{\rho}_n \in [\rho_n, 1-\rho_n]$ such that $\|\mathbf{u}_n - \eta_n(\hat{\rho}_n)\| \longrightarrow 0, n \longrightarrow \infty$. Since $I(\mathbf{u}_n) \longrightarrow \mathscr{B}$, we have $I(\eta_n(\hat{\rho}_n)) \longrightarrow \mathscr{B}$. Recall that $\tilde{u}_n \in \mathscr{S}$ and $I(\tilde{u}_n) \longrightarrow \mathscr{B}$; then by Lemma 6, we have $\|\eta_n(\hat{\rho}_n) - \tilde{u}_n\| \longrightarrow 0$. By (59), we have that there exists a constant $C > 0$ independent of n such that $\|u_{1n}^\pm\|_1, \|u_{2n}\|_2 \geq C > 0$.

Set

$$\begin{aligned} J_1(u_1,u_2) &:= I'(u_1,u_2)(u_1,0), \\ J_2(u_1,u_2) &:= I'(u_1,u_2)(0,u_2). \end{aligned} \tag{120}$$

By (111), there exist two Lagrange multipliers $L_{1n}, L_{2n} \in \mathbb{R}$ such that

$$\begin{aligned} J(u_{1n},u_{2n}) := I'(u_{1n},u_{2n}) &+ L_{1n} J_1'(u_{1n},u_{2n}) \\ &+ L_{2n} J_2'(u_{1n},u_{2n}) \longrightarrow 0. \end{aligned} \tag{121}$$

Then, $(J(u_{1n},u_{2n}), (u_{1n},0)) = o(1)$ and $(J(u_{1n},u_{2n}), (0,u_{2n})) = o(1)$; i.e.,

$$\begin{aligned} &L_{1n}\left[(p_1-2)\int \nu_1 |u_{1n}|^{p_1} + (2^*-2)\int \mu_1 |u_{1n}|^{2^*} + \left(2-\frac{2^*}{2}\right)\int |\beta| |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2}\right] \\ &\quad = L_{2n}\frac{2^*}{2}\int |\beta| |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2} + o(1), \end{aligned} \tag{122}$$

$$\begin{aligned} &L_{2n}\left[(p_2-2)\int \nu_2 |u_{2n}|^{p_2} + (2^*-2)\int \mu_2 |u_{2n}|^{2^*} + \left(2-\frac{2^*}{2}\right)\int |\beta| |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2}\right] \\ &\quad = L_{1n}\frac{2^*}{2}\int |\beta| |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2} + o(1). \end{aligned} \tag{123}$$

Similarly as in (74), we denote

$$\begin{aligned} A_{1n} &= \int \nu_1 |u_{1n}|^{p_1}, \\ A_{2n} &= \int \nu_2 |u_{2n}|^{p_2}, \\ D_{1n} &= \int \mu_1 |u_{1n}|^{2^*}, \\ D_{2n} &= \int \mu_2 |u_{2n}|^{2^*}, \\ D_n &= \int |\beta| |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2}. \end{aligned} \tag{124}$$

Note that $(u_{1n}, u_{2n}) \subset \mathscr{M}$, we have

$$\begin{aligned} \|u_{1n}\|_1^2 &= A_{1n} + D_{1n} - D_n, \\ \|u_{2n}\|_2^2 &= A_{2n} + D_{2n} - D. \end{aligned} \tag{125}$$

Consequently, by (59), we have

$$\begin{aligned} A_{1n} + D_{1n} &> D_n, \\ A_{2n} + D_{2n} &> D. \end{aligned} \tag{126}$$

Set

$$\begin{aligned} S_{1n} := {} & (p_1-2)(p_2-2)A_{1n}A_{2n} + (p_1-2)(2^*-2)A_{1n}D_{2n} \\ & + (p_1-2)\left(2-\frac{2^*}{2}\right)A_{1n}D_n + (p_2-2)(2^*-2)A_{2n}D_{1n} \\ & + (p_2-2)\left(2-\frac{2^*}{2}\right)A_{2n}D_n, \\ S_{2n} := {} & (2^*-2)^2 D_{1n}D_{2n} + (2^*-2)\left(2-\frac{2^*}{2}\right)D_{1n}D_n \\ & + (2^*-2)\left(2-\frac{2^*}{2}\right)D_{2n}D_n - 2(2^*-2)D_n^2. \end{aligned} \tag{127}$$

Then, by (122) and (123), we have

$$o(1) = L_{1n}L_{2n}(S_{1n} + S_{2n}) := L_{1n}L_{2n}S_n. \tag{128}$$

Note that

$$D_n \le \frac{1}{2}(D_{1n} + D_{2n}),$$
$$D_n^2 \le D_{1n}D_{2n}. \tag{129}$$

We have

$$S_{2n} \ge (2^* - 2)^2 \left(D_{1n}D_{2n} - D_n^2\right) \ge 0. \tag{130}$$

Thus, $S_n > C > 0$ for some constant $C > 0$. By (128), we have $L_{1n}L_{2n} = o(1)$. Without loss of generality, we may assume that $L_{1n} = o(1)$; then by (123) and (59), we have $L_{2n} = o(1)$. Then, (121) implies that $I'(u_{1n}, u_{2n}) \longrightarrow 0$. Thus, we complete the proof.

We also need an important lemma which is proved in [25]. □

Lemma 11 (see [25]). *Assume that $(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2)$ weakly in $\mathbf{H}$ as $n \longrightarrow \infty$; then, passing to a subsequence, it holds that*

$$\lim_{n\longrightarrow\infty} \int \left(|u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2} - |u_1|^{2^*/2} |u_2|^{2^*/2} - |u_{1n} - u_1|^{2^*/2} |u_{2n} - u_2|^{2^*/2} \right) = 0. \tag{131}$$

Proof of Theorem 3. Proof of Theorem 3. Let $\{\mathbf{u}_n\} \subset \mathscr{M}$ be the sequence obtained in Lemma 10. Then, up to a subsequence, there exists $\mathbf{u} \in \mathbf{H}$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathbf{H}$. Then, $I'(\mathbf{u}) = 0$ and

$$\|u_{1n}\|_1^2 = \int \left(\nu_1 |u_{1n}|^{p_1} + \mu_1 |u_{1n}|^{2^*} + \beta |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2} \right), \tag{132}$$

$$\|u_{2n}\|_2^2 = \int \left(\nu_2 |u_{2n}|^{p_2} + \mu_2 |u_{1n}|^{2^*} + \beta |u_{1n}|^{2^*/2} |u_{2n}|^{2^*/2} \right), \tag{133}$$

$$\|u_{1n}^{\pm}\|_1^2 = \int \left(\nu_1 |u_{1n}^{\pm}|^{p_1} + \mu_1 |u_{1n}^{\pm}|^{2^*} + \beta |u_{1n}^{\pm}|^{2^*/2} |u_{2n}|^{2^*/2} \right) + o(1). \tag{134}$$

Set $\mathbf{w}_n = (w_{1n}, w_{2n}) := \mathbf{u}_n - \mathbf{u}$. By Lemma 11, (133), and (134), we have

$$\int |\nabla (w_{1n}^{\pm})|^2 = \int \left(\mu_1 |w_{1n}^{\pm}|^{2^*} + \beta |w_{1n}^{\pm}|^{2^*/2} |w_{2n}|^{2^*/2} \right) + o(1), \tag{135}$$

$$\int |\nabla w_{2n}|^2 = \int \left(\mu_2 |w_{2n}|^{2^*} + \beta |w_{1n}|^{2^*/2} |w_{2n}|^{2^*/2} \right) + o(1). \tag{136}$$

Then,

$$\begin{aligned} \mathscr{B} &= \lim_{n\longrightarrow\infty} I(\mathbf{u}_n) = \lim_{n\longrightarrow\infty} \left[\frac{1}{N} \|\mathbf{u}_n\|^2 + \int \sum_j \left(\frac{1}{2^*} - \frac{1}{p_j} \right) \nu_j |u_{jn}|^{p_j} \right] \\ &= \left[\frac{1}{N} \|\mathbf{u}\|^2 + \int \sum_j \left(\frac{1}{2^*} - \frac{1}{p_j} \right) \nu_j |u_j|^{p_j} \right] + \lim_{n\longrightarrow\infty} \frac{1}{N} \int \left(|\nabla (w_{1n}^+)|^2 + |\nabla (w_{1n}^-)|^2 + |\nabla w_{2n}|^2 \right). \end{aligned} \tag{137}$$

□

Case 1. 1. Assume that $u_1, u_2 \equiv 0$. Then, $u_{jn} \longrightarrow 0$ in $L^2(\Omega)$ and $L^{p_j}(\Omega)$ for $j = 1, 2$. From (135) and (136), we have

$$\int |\nabla (w_{1n}^{\pm})|^2 \le \mu_1 S^{-(2^*/2)} \left(\int |\nabla (w_{1n}^{\pm})|^2 \right)^{2^*/2},$$
$$\int |\nabla w_{2n}|^2 \le \mu_2 S^{-(2^*/2)} \left(\int |\nabla w_{2n}|^2 \right)^{2^*/2}. \tag{138}$$

Then,

$$\int |\nabla (w_{1n}^{\pm})|^2 \ge \mu_1^{-((N-2)/2)} S^{N/2}, \tag{139}$$

$$\int |\nabla w_{2n}|^2 \ge \mu_2^{-((N-2)/2)} S^{N/2}. \tag{140}$$

Thus, by (137), we have

$$\mathscr{B} \ge \frac{2}{N} \mu_1^{-((N-2)/2)} S^{N/2} + \frac{1}{N} \mu_2^{-(N-2/2)} S^{N/2}, \tag{141}$$

a contradiction with Lemma 8, (85), and (9). Therefore, Case 1 is impossible.

Case 2. Assume that $u_1 \equiv 0$ and $u_2 \equiv 0$. Then, (139) still holds and $I'(0, u_2) = 0$ which implies that u_2 is a nontrivial solution of (10) with $j = 2$. Thus, $I(0, u_2) \ge B_2$. By (137), we have

$$\mathscr{B} \ge B_2 + \frac{2}{N} \mu_1^{-(N-2/2)} S^{N/2}, \tag{142}$$

a contradiction with Lemma 8 and (85). Therefore, Case 2 is impossible.

Case 3. Assume that $u_1 \equiv 0$ and $u_2 \equiv 0$. Then, (140) still holds and $I'(u_1, 0) = 0$ which implies that u_1 is a nontrivial solution of (10) with $j = 1$. Thus, $I(u_1, 0) \ge B_1$.

(i) If $u_{1n} \longrightarrow u_1$ strongly in $H_0^1(\Omega)$, then u_1 is a sign changing solution of (10) with $j = 1$. So $I(u_1, 0) \ge \mathscr{B}_1$. By (137), we have

$$\mathscr{B} \ge \mathscr{B}_1 + \frac{1}{N} \mu_2^{-((N-2)/2)} S^{N/2}, \tag{143}$$

a contradiction with Lemma 8.

(ii) If $u_{1n} \nrightarrow u_1$ strongly in $H_0^1(\Omega)$, then by (132) and Brezis–Lieb lemma, we can also show that

$$\int \left|\nabla w_{1n}\right|^2 \geq \mu_1^{-((N-2)/2)} S^{N/2}. \tag{144}$$

By (137), we have

$$\mathscr{B} \geq B_1 + \frac{1}{N}\mu_1^{-((N-2)/2)} S^{N/2} + \frac{1}{N}\mu_2^{-((N-2)/2)} S^{N/2}, \tag{145}$$

a contradiction with Lemma 8 and (85).

Therefore, Case 3 is impossible.

Now, we have shown that $u_1, u_2 \equiv 0$ and then (u_1, u_2) is a nontrivial solution of (11). If $u_{1n} \nrightarrow u_1$ strongly in $H_0^1(\Omega)$, then (144) holds and

$$\mathscr{B} \geq B + \frac{1}{N}\mu_1^{-((N-2)/2)} S^{N/2}, \tag{146}$$

a contradiction with Lemma 8. Thus, $u_{1n} \longrightarrow u_1$ strongly in $H_0^1(\Omega)$ and then u_1 is sign changing, $I(\mathbf{u}) \geq \mathscr{B}$. On the other hand, by (137), we have

$$I(\mathbf{u}) \leq \lim_{n\longrightarrow\infty} I(\mathbf{u}_n) = \mathscr{B}. \tag{147}$$

Then, we have $I(\mathbf{u}) = \mathscr{B}$ and $\mathbf{u}_n \longrightarrow \mathbf{u}$ strongly in $\mathbf{H}$. Then $(u_1, |u_2|) \in \mathscr{S}$ and $I(u_1, |u_2|) = \mathscr{B}$. Similarly to the proof in Lemma 10, we have $I'(u_1, |u_2|) = 0$. By the maximum principle, we have $|u_2| > 0$ in Ω. Consequently, $(u_1, |u_2|)$ is a least energy sign changing solution of system (11) with u_1 sign changing and u_2 positive. Thus, we complete the proof.

5. Conclusion

In this paper, we study a coupled nonlinear Schrödinger system with critical exponents which arise in many physical problems. By the modified Nehari manifold method and some mathematical skills for estimations of the energy, we show the existence of the least energy sign changing solution for the general system. Besides, we construct multiple solutions of the system for the symmetrical case.

Data Availability

No data were used to support this study and all materials are available in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors studied this problem and read and approved the final manuscript.

Acknowledgments

This work was supported by the NSFC (11601109), Hainan Association of Science and Technology plans for Youth Innovation (201503), and Natural Science Fund of Hainan Province (20161001).

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