Sign Changing Solutions for Coupled Critical Elliptic Equations

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In this paper, we consider the coupled elliptic system with a Sobolev critical exponent. We show the existence of a sign changing solution for problem (\(\mathcal{P}\)) for the coupling parameter \(-\sqrt{\mu_1\mu_2} < \beta < 0\). We also construct multiple sign changing solutions for the symmetric case.

1. Introduction

In this paper, we consider the following coupled elliptic system with a Sobolev critical exponent:

\[
\begin{aligned}
-\Delta u_1 + \lambda_1 u_1 &= v_1 |u_1|^{p_1-2} u_1 + \mu_1 |u_1|^{2^*-2} u_1 \\
+ \beta_1 |u_1|^{(2^*/2)-2} u_1 |u_2|^{2^*/2}, & \quad \text{in } \Omega, \\
-\Delta u_2 + \lambda_2 u_2 &= v_2 |u_2|^{p_2-2} u_2 + \mu_2 |u_2|^{2^*-2} u_2 \\
+ \beta_2 |u_1|^{(2^*/2)-2} u_1 |u_2|^{2^*/2}, & \quad \text{in } \Omega, \\
u_1 = u_2 &= 0 \text{ on } \partial \Omega, \\
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(N \geq 6\), \(2 < p_j < 2^*\), \(\lambda_j \in (-\lambda_1(\Omega), 0), v_j, \mu_j > 0\) for \(j = 1, 2\), and \(\lambda_1(\Omega)\) is the first eigenvalue of \(-\Delta\) with the Dirichlet boundary condition.

In recent years, the following coupled elliptic system has attracted much interest:

\[
\begin{aligned}
-\Delta u_1 + \lambda_1 u_1 &= \mu_1 |u_1|^3 + \beta u_2^2 u_1, & \quad x \in \Omega, \\
-\Delta u_2 + \lambda_2 u_2 &= \mu_2 |u_2|^3 + \beta u_1^2 u_2, & \quad x \in \Omega, \\
u_1 = u_2 &= 0, & \quad x \in \partial \Omega. \\
\end{aligned}
\]

The system above has applications in many physical problems (see [1, 2]). It also arises in the Hartree–Fock theory for a double condensate, which is a binary mixture of Bose–Einstein condensates in two different hyperfine states [3] and the references therein. Considering the solitary wave solutions of system (3), we set \(\Phi_j(x, t) = e^{itj}u_j(x)\) for \(j = 1, 2\). Then, it is reduced to system (2). For the subcritical case, i.e., \(N = 3\), the existence of least energy and other finite energy solutions as well as the existence and multiplicity of positive and sign-changing solutions are studied in [2, 4–23] and the references therein. For the critical case, i.e., \(N = 4(2^* := ((2N)/(N-2)) = 4)\), the existence of a positive least energy solution is proved when \(\beta\) is negative, positive small, and positive large in [24]. For the higher dimension \(N \geq 5\), the authors in [25, 26] also consider the following critical case:
\[\begin{align*}
-\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^{2^* - 1} + \beta u_1^{(2^*/2) - 1} u_2^{2^*/2}, \quad \text{in } \Omega, \\
-\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^{2^* - 1} + \beta u_2^{(2^*/2) - 1} u_1^{2^*/2}, \quad \text{in } \Omega,
\end{align*}\]
\[u_1, u_2 \geq 0 \text{ in } \Omega, \quad u_1 = u_2 = 0 \text{ on } \partial \Omega.\]  
(4)

Note that when \(N = 4\), system (4) is the same as system (2). But interestingly, the authors in [25] find different results for the higher dimension case \(N \geq 5\) from that of \(N = 4\). The authors in [27] also consider the even case:

\[\begin{align*}
-\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^{2^* - 1} + \beta u_1^{(2^*/2) - 1} u_1^{2^*/2}, \quad \text{in } \Omega, \\
-\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^{2^* - 1} + \beta u_2^{(2^*/2) - 1} u_2^{2^*/2}, \quad \text{in } \Omega,
\end{align*}\]
\[u_1 = u_2 = 0 \text{ on } \partial \Omega.\]  
(5)

They show that when \(N \geq 6\), system (5) has a sign changing solution for any \(\beta < 0\).

Recall that when \(\beta = 0\), system (4) becomes the famous Brezis–Nirenberg problem:

\[\begin{align*}
-\Delta u_j + \lambda_j u_j &= \mu_j |u_j|^{2^* - 2} u_j, \quad \text{in } \Omega, \\
u_j \geq 0 \text{ in } \Omega, \quad u_j = 0 \text{ on } \partial \Omega, \quad j = 1, 2,
\end{align*}\]  
(6)

which are two different independent scalar equations. In [28], the authors show that when \(N \geq 4, -\lambda_j (\Omega) < \lambda_j < 0\), and \(\mu_j = 1\), equation (6) has a ground state solution. For the even case,

\[\begin{align*}
-\Delta u + \lambda_j u &= \mu_j |u|^{2^* - 2} u, \quad \text{in } \Omega, \\
u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad j = 1, 2.
\end{align*}\]  
(7)

The authors in [29] show that when \(N \geq 6, -\lambda_1 (\Omega) < \lambda_j < 0, \text{ and } \mu_j > 0\), equation (7) has sign changing solutions for \(j = 1, 2\). In [30, 31], when \(N \geq 4, \lambda_j < 0\), and \(\mu_j > 0\), the authors obtained nontrivial solutions of (7) for \(j = 1, 2\).

In fact, in the pioneering paper [28] of Brezis and Nirenberg, they also study a more general equation including the following classical case:

\[\begin{align*}
-\Delta u + \lambda_j u &= v_j u^{1 - p_j} + \mu_j u^{2^* - 1}, \quad \text{in } \Omega, \\
u_j \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad j = 1, 2,
\end{align*}\]  
(8)

where \(2 < p_j < 2^*\) and \(v_j > 0\). From [28], we know that when \(N \geq 4, v_j, \mu_j > 0\), and \(\lambda_j \in (\lambda_1 (\Omega), 0)\), then problem (8) has positive ground state solutions \(v_1, v_2 \in C^2(\Omega) \cap C(\Omega)\) with the positive energy:

\[0 < B_j := \frac{1}{2} \int_\Omega \left( |\nabla v_j|^2 + \lambda_j v_j^2 \right) - \int_\Omega \frac{v_j^{1 + p_j}}{p_j} - \frac{1}{2^*} \int_\Omega \mu_j v_j^{2^*} < \frac{1}{N^{2p_j}} \left( \frac{N - 2}{2} \right)^{1/2} S^{N/2}, \quad j = 1, 2,\]  
(9)

where \(S\) is the sharp imbedding constant from \(D^{1,2} (\mathbb{R}^N)\) into \(L^{2^*} (\mathbb{R}^N)\).

Based on these, firstly, we try to show similar results in [29–31] for the even case of the general equation (8), i.e.,

\[\begin{align*}
-\Delta u + \lambda_j u &= v_j |u|^{1 - p_j} + u_j |u|^{2^* - 2} u, \quad \text{in } \Omega, \\
u = 0 \text{ on } \partial \Omega, \quad j = 1, 2,
\end{align*}\]  
(10)

Then, naturally and interestingly, we guess that the more general system corresponding to system (5) also has similar results of the existence of sign changing solutions. That is, we consider the more general critical elliptic system as follows:

\[\begin{align*}
-\Delta u_1 + \lambda_1 u_1 &= v_1 |u_1|^{1 - p_1} - u_1 + u_1 |u_1|^{2^* - 2} u_1 + \beta |u_1|^{(2^*/2) - 2} u_1^2, \quad \text{in } \Omega, \\
-\Delta u_2 + \lambda_2 u_2 &= v_2 |u_2|^{1 - p_2} - u_2 + u_2 |u_2|^{2^* - 2} u_2 + \beta |u_2|^{(2^*/2) - 2} u_2^2, \quad \text{in } \Omega,
\end{align*}\]  
(11)

\[u_1 = u_2 = 0 \text{ on } \partial \Omega.\]

Precisely, we get the following results.

**Theorem 1.** Assume that \(N \geq 6, 2 < p_j < 2^*, \lambda_j \in (-\lambda_1 (\Omega), 0)\), and \(v_j, \mu_j > 0\), for \(j = 1, 2\); then, (10) has sign changing solutions for \(j = 1, 2\) and the energy is the least one among all sign changing solutions.

Then, we can consider the case \(\lambda_1 = \lambda_2 = \lambda \text{ and } p_1 = p_2 = p\). The following equation is one case of (10):

\[\begin{align*}
-\Delta u + \lambda u &= |u|^{p - 2} u + |u|^{2^* - 2} u, \quad \text{in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{align*}\]  
(12)

where \(p > 0\). Let \(v\) and \(w\) be the sign changing solutions of (12) when \(N \geq 6, \lambda \in (-\lambda_1 (\Omega), 0)\), and \(N \geq 4, \lambda \leq - \lambda_1 (\Omega)\), respectively (the latter case really exists and we shall show it later).

By Lemma 1 in [25], the system below has a positive solution:

\[\begin{align*}
\mu_j a_j^{2^* - 2} + \beta a_j^{(2^*/2) - 2} a_j^{2^*/2} = 1, \\
\mu_j a_j^{2^* - 2} + \beta a_j^{(2^*/2) - 2} a_j^{2^*/2} = 1.
\end{align*}\]  
(13)

Then, we can construct solutions of (11) by using \(v\) and \(w\) as in [2, 24, 25].

**Theorem 2.** Let \((a_1, a_2)\) be a solution of (13). Assume that \(v_1 = a_1 a_j^{p - 2}\) and \(v_2 = a_2 a_j^{p - 2}\), \(\mu_j, \mu_k, \beta > 0\) and one of the following happens:

1. \(N \geq 6, -\lambda_1 (\Omega) < \lambda_2 < 0\),
2. \(N \geq 4, \lambda_1 = \lambda_2 \leq - \lambda_1 (\Omega)\),

Then \((a_1 v, a_2 v)\) and \((a_1 w, a_2 w)\) are sign changing solutions of system (11).

We also get a least energy semi-sign changing solution for system (11).

**Theorem 3.** Assume that \(N \geq 6, 2 < p_j < 2^*, -\lambda_1 (\Omega) < \lambda_j < 0, v_j, \mu_j > 0, \text{ and } - \sqrt{\mu_j} < \beta < 0\). Then, system (11) has a semi-sign changing solution with one component changing sign and the other one positive, and the energy is the least one among all these kinds of solutions.
Next, we give the proof of Theorem 1 in Section 2. And we shall show Theorems 2 and 3 in Sections 3 and 4, respectively.

2. Proof of Theorem 1

In this section, we consider scalar case (10). Fixing \( j = 1 \) or \( 2 \), we show the existence result. The proof is similar to that in [29]. Considering we need this result in the following two Theorems 2 and 3, we verify it also for the completeness of the current paper. The working space and some notations shall be given firstly. We assume that all the integrations below are taken over \( \Omega \) if without special specification. Since \( \lambda_j - \lambda_1(\Omega) \), we can define the equivalent inner product in \( H_0^1(\Omega) \) by

\[
(u, v) \rangle_f = \langle \nabla u, \nabla v \rangle + \lambda_j u v,
\]

which gives rise to a norm denoted by \( \| \cdot \|_{j} \). We also use \( |u|_q = \left( \int |u|^q dx \right)^{1/q} (1 \leq q < \infty) \) and \( |u|_0^2 = \int \nabla u^2 dx \) for convenience. Then, the energy functional of equation (10) is

\[
I_j(u) = \frac{1}{2} \|u\|_{j}^2 - \frac{1}{p_j} \int v_j |u|^p_j + \frac{1}{2} \int \mu_j |u|^2_j.
\]

Recall \( v_j \) is a ground state solution of (8), that is,

\[
B_j := I_j(v_j) = \inf \{ I_j(u) : u \in H_0^1(\Omega) \setminus \{0\}, I_j(u)u = 0 \}.
\]

For the sign changing case, we define the manifold as follows:

\[
\delta_j := \{ u \in H_0^1(\Omega) : u^+ \equiv 0, I_j(u^+)u^+ = 0 \}.
\]

where \( u^\pm := \pm \max\{u, 0\} \) and the condition of the definition of (17) is that

\[
\|u^\pm\|_{j}^2 = v_j \int |u^\pm|^{p_j} + \mu_j \int |u^\pm|^2.
\]

It is easy to check that \( \delta_j \neq \emptyset \). Then, we define

\[
\mathcal{B}_j := \inf_{u \in \delta_j} \mathcal{I}_j(u).
\]

We need a conclusion in [32].

Lemma 1 (see [32]). Consider a rectangle \( R = [a_j, b_j] \subset \mathbb{R}^s \) and a continuous function \( \Phi = (\Phi_1, \ldots, \Phi_s) : \mathbb{R} \rightarrow \mathbb{R}^s \). If \( \Phi |_{x_{a_j}} > 0 > \Phi |_{x_{b_j}} \) hold for all \( i = 1, \ldots, s \), then \( \Phi \) has a zero inside \( R \).

Similarly as in [29], we set

\[
E_j(u) = \begin{cases} 
0, & u = 0, \\
\frac{1}{2} \|u\|_{j}^2 - \frac{1}{p_j} \int v_j |u|^p_j + \mu_j \int |u|^2, & u \neq 0.
\end{cases}
\]

Then, (18) is equivalent to \( E_j(u^+) = E_j(u^-) = 1 \). Define

\[
\mathcal{H}_j := \{ h \in C([0, 1] \times [0, 1], H^1_0(\Omega)) : h(t, 0) = 0, \\
h(0, s) \equiv 0, h(1, s) \equiv 0, I_j(h(t, 1)) \leq 0, \\
E_j(h(s, 1)) \geq 2, \quad \forall t, s \in [0, 1] \}.
\]

Since \( \mu_j > 0 \), it is easy to see that \( \mathcal{H}_j \neq \emptyset \). Then, we have the following lemma.

Lemma 2. \( \mathcal{B}_j = \inf_{h \in \mathcal{H}_j} \sup_{t \in [0, 1]} I_j(h(t)). \)

Proof. For any \( u \in \delta_j \), \( t, s \geq 0 \), we have

\[
I_j(tu^+ - su^-) = \left[ \frac{1}{2} \|u^+\|_{j}^2 - \frac{1}{p_j} \int v_j |u|^p_j - \frac{1}{2} \int |u|^2 \right]
\]

\[
+ \left[ \frac{1}{2} \|u^-\|_{j}^2 - \frac{1}{p_j} \int v_j |u|^p_j - \frac{1}{2} \int |u|^2 \right]
\]

\[
:= f_+(t) + f_-(s).
\]

Then,

\[
\sup_{t \in [0, 1]} I_j(tu^+ - su^-) = f_+(1) + f_-(1) = I_j(u).
\]

Denote the map \( h \in \mathcal{H}_j \) satisfying

\[
h([0, 1] \times [0, 1]) \subset \{ tu^+ - su^- : t, s \geq 0 \}.
\]

By (23), we have

\[\mathcal{B}_j \leq \inf_{h \in \mathcal{H}_j} \sup_{t \in [0, 1]} I_j(h(t)).\]

On the other hand, for any \( h \in \mathcal{H}_j \), we have

\[
E_j(h(t^+)) + E_j(h(t^-)) - 2 \geq 0, \quad t \in [0, 1],
\]

\[
\leq 0, \quad t \in [1, s] : s \in [0, 1],
\]

\[
E_j(h(t^+)) + E_j(h(t^-)) - 2 \geq 0, \quad t \in [0, 1],
\]

\[
\geq 0, \quad t \in [1, 1] : t \in [0, 1].
\]

Then, by Lemma 1, we have that there exists \( t_0 \in [0, 1] \times [0, 1] \) such that \( E_j(h(t_0^+)) = E_j(h(t_0^-)) = 1 \), i.e., \( u_{t_0} := h(t_0) \in \delta_j \). Consequently,

\[\mathcal{B}_j \leq \inf_{h \in \mathcal{H}_j} \sup_{t \in [0, 1]} I_j(h(t)).\]

This completes the proof.

Now, we have an upper estimate for \( \mathcal{B}_j \).

Lemma 3. \( \mathcal{B}_j < c_j := B_j + (1/N) \mu_j^{-(N-2)/2} S_{N/2}. \)

Proof. Since \( v_j \in C(\bar{\Omega}) \) and \( v_j = 0 \) on \( \partial \Omega \), there exists \( B(x_0, 2r_0) := \{ x : |x - x_0| \leq 2r \} \subset \Omega \) for some \( x_0 \) near \( \partial \Omega \) and \( r > 0 \) small to be decided later. Take a cutoff function \( \phi \in C_0^\infty(B(x_0, 2r)) \) with \( 0 \leq \phi \leq 1 \) and \( \phi(x) \equiv 1 \) for \( |x - x_0| \leq r \). Let
Then (see [33, 34]), $U_{ε, λ}$ is the solution of the following equation:

$$-Δu = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N).$$  

Set $u_* := φU_{ε, λ}$; then (see [35]),

$$\int |Δu_*|^2 = S^{N/2} + O(ε^{-2}),$$

$$\int |u_*|^{2^*} = S^{N/2} + O(ε),$$

$$\int |u_*|^2 ≥ Cε^2 + O(ε^{-2}).$$

Conversely

$$\int |u_*|^{2^*-1} ≥ C(N-2)/2.$$  

Now, we show that

$$\sup_{t, τ \in \mathbb{R}} I_j(tv_j + su_*) \leq c_j^*.$$  

In fact, by (30) and $((N - 2)(3 - p_j) + 4)/(2) \geq (N - 2)/2$ since $p_j < 2^*$, we have

$$I_j(tv_j + su_*) ≤ \left[ \frac{1}{2} \left[ \int |v_j|^2 \right]^{p_j/p_j} \int |v_j|^{p_j} \right] + \frac{1}{2} \left( \int |v_j|^{2^*} \right)^{p_j/p_j} \int |v_j|^{2^*}

+ C_1 \int |\Delta v_j|_{L^∞(ς)} \left| |u_*| \right|_{L^∞(ς)}

+ C_2 \left| |v_j|_{L^∞(ς)} \right| \left( \left| \left| s_0 \right| \right|_{L^∞(ς)} \right)^{p_j/p_j} \int |s_0|^{2^*-1}

≤ B_j + \frac{1}{Nμj} ε^{j(N-2)/2} S^{N/2}

+ C_3 \left| |v_j|_{L^∞(ς)} \right| ε^{j(N-2)/2} - C_4 ε^2

< B_j + \frac{1}{Nμj} ε^{j(N-2)/2} S^{N/2},$$

for $ε > 0$ small enough.  

Then by Lemma 2, the conclusion follows.

Set

$$\mathcal{E} := \left\{ u \in H^1_0(Ω) : |E_j(u^δ) - 1| < \frac{1}{2} \right\}.  

Then, we show that $I_j$ satisfies a local (PS) condition in the following sense.
Then, by Lemma 2, we have
\[ \lim_{n \to \infty} \sup_{u \in h_n([0,1] \times [0,1])} I_j(u) = \lim_{n \to \infty} I_j(u_n) = \mathcal{B}_j. \]
(40)

By the well-known deformation lemma, we have that there exists a sequence \( \{u_n\} \subset H^1_0(\Omega) \) such that
\[ \text{dist}(u_n, h_n([0,1] \times [0,1])) \to 0, \]
(41)
\[ I_j(u_n) \to \mathcal{B}_j, I'_j(u_n) \to 0, \text{ as } n \to \infty. \]
(42)

Obviously, \( \{u_n\} \) is bounded and there exists \( u \in H^1_0(\Omega) \) such that \( u_n \to u \) weakly in \( H^1_0(\Omega) \); then, \( I'_j(u) = 0 \) by the definition of weak convergence.

Now, we claim that \( \{u_n\} \subset \mathcal{B} \). In fact, by (41), there exists a sequence \( \{\tilde{u}_n\} \) with
\[ \tilde{u}_n = t_n u_n^+ - s_n u_n^- \in h_n([0,1] \times [0,1]), \]
\[ \text{dist}(\tilde{u}_n, u_n) \to 0. \]
(43)

Then, for \( n \) large enough depending on \( \xi > 0 \),
\[ I_j(\tilde{u}_n^+) = I_j(t_n u_n^+) > I_j(u_n^+) - \xi \geq B_j - \xi, \]
(44)
\[ I_j(\tilde{u}_n^-) = I_j(s_n u_n^-) > I_j(u_n^-) - \xi \geq B_j - \xi. \]
(45)

(44), (45), and (43) implies that \( \tilde{u}_n \neq 0 \). Since \( I'_j(\tilde{u}_n)\tilde{u}_n^+ = o(1) \), the claim comes true.

Therefore, by Lemmas 3 and 4, we have that \( u_n \to u \) strongly in \( H^1_0(\Omega) \) with \( u^\# \neq 0 \), i.e., \( u \) is a sign changing solution of equation (10) and the energy \( I_j(u) \) is the least one among all sign changing solutions of (10).

\[ \square \]

3. Proof of Theorem 2

Firstly, inspired by [30, 31], we show the following theorem about the scalar equation by the minimax method (see [39]), and we would like to just give the sketch for the completeness of the current paper. We also use the same notations as in Section 2 whenever no confusion arises.

Denote \( \varphi_1, \varphi_2, \varphi_3, \ldots \) as the \( L^2 \) normalized eigenfunctions of \( -\Delta \), corresponding to positive eigenvalues \( 0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots \uparrow +\infty \), counted with their multiplicity. We set \( H_k := \langle \varphi_1, \varphi_2, \ldots, \varphi_k \rangle \) as the \( k \) dimensional space in \( H^1_0(\Omega) \). For the fixed \( \lambda_j < 0 \), define
\[ \lambda_j^* := \min \{ \lambda_k(\Omega) : -\lambda_j < \lambda_k(\Omega) \}. \]
(46)

We may suppose that \( \lambda_j^* = \lambda_{k_0}(\Omega) \) for some \( k_0 \in \mathbb{N} \). Especially, we set \( \lambda_{0}(\Omega) \equiv 0 \).

**Theorem 4.** Assume that \( N \geq 4, 2 < p < 2^*, \lambda_j < 0, \) and \( \nu_j, \mu_j > 0 \), for \( j = 1, 2 \), then equation (10) has a nontrivial solution. Moreover, when \( \lambda_j \leq -\lambda_1(\Omega) \), the solution changes sign.

\[ h_n([0,1] \times [0,1]) \subset \{ tu_n^+ - su_n^- : t, s \geq 0 \}. \]
(39)

**Proof.** It is easy to check that \( I_j \) satisfies the geometry.

(a) For \( u \in (H^{k_0-1}_0)^\varepsilon \), there exist \( \alpha_0, \beta_0 > 0 \) such that
\[ I_j(u) > \alpha_0 \quad \text{with} \quad \| u \|_0 = r_0 \quad \text{and} \quad I_j(u) > 0 \quad \text{for} \quad 0 < \| u \|_0 < r_0. \]

(b) For any finite dimensional subspace \( \tilde{H} \subset H^1_0(\Omega) \), it holds that \( \tilde{I} \cap \tilde{H} \) is bounded in \( H^1_0(\Omega) \), where
\[ \tilde{I}_j := \{ u \in H^1_0(\Omega) : I_j(u) \geq 0 \}. \]

As in [36], we define the minimax value:
\[ \hat{c}_j := \sup_{u \in \Lambda} \inf_{u \in B_{r_0}} I_j(h(u)), \]
(47)
where
\[ \Lambda := \{ h \in C(H^1_0(\Omega)) : h(IntB_1) \subset \tilde{I}_j \cup B_{r_0} \}, \]
\[ h \text{ is an odd homeomorphism of } H^1_0(\Omega) \text{ onto } H^1_0(\Omega). \]
(48)

Obviously, \( h = r_j \text{id} \in \Lambda \). We also set another minimax value:
\[ \hat{b}_j := \inf_{K \in \Lambda} \sup_{u \in K} I_j(u), \]
(49)
where
\[ \Lambda := \{ K \subset H^1_0(\Omega) : K \text{ is compact, symmetric, and} \}
\[ \gamma(K \cap h(\partial B_1)) \geq k_0, \text{ for all } h \in \Lambda \}, \]
\[ \text{and } \gamma \text{ is the Krasnosel'skii genus. It can be verified that } \Lambda \neq \emptyset. \]

By properties (a) and (b), we can show that
\[ 0 < \alpha_0 \leq c_j \leq \hat{b}_j \leq \hat{c}_j : = \frac{1}{N^j} \mu_j^{(N-2)/2} S^{N/2}, \]
(51)
and \( I_j \) satisfies the (PS)\(_j\) condition. Also by a deformation lemma, there exists a (PS)\(_j\) sequence \( \{u_n\} \) of \( I_j \) and then \( u_n \to u \) in \( H^1_0(\Omega) \). That is, \( u \) is a nontrivial solution of equation (10).

Multiply the equations in (10) with the first eigenfunction \( \varphi_1 \) and integrate over \( \Omega \); we have
\[ \left( \lambda_1(\Omega) + \lambda_j \right) \int_{\Omega} \varphi_1 \varphi = \int_{\Omega} \left( \nu_j |u|^{p-2} \varphi_1 + \mu_j |u|^{2^* - 2} \varphi \right). \]
(52)

Then, it is easy to see that when \( \lambda_j \leq -\lambda_1(\Omega), \nu_j, \mu_j > 0 \) any solution of (10) must change sign.

\[ \square \]

**Proof of Theorem 2.** The proof is by direct computation. Recall that \( \nu \) and \( u \) are sign changing solutions of (12) being well defined. Since \( \langle a_1, a_2 \rangle \) satisfy (13), for any \( \varphi \in C_0^\infty (\Omega) \), we have
\[ a_1 \int_{\Omega} |\varphi|^{p-2} a_1 \varphi \varphi_1 + \mu_1 \int_{\Omega} |\varphi|^{2^* - 2} a_1 \varphi \varphi_1 \]
\[ + \beta \int_{\Omega} |\varphi|^{(2^*/2)-2} a_1 \varphi a_2 \varphi_1^{2^*/2} \varphi \]
\[ = a_1 \left( \nu \int_{\Omega} |\varphi|^{p-2} \varphi \varphi_1 + \int_{\Omega} |\varphi|^{2^* - 2} \varphi \right) = (a_1 \nu, \varphi_1), \]
(53)
That is, \((a_1, v, a_2, v)\) satisfy the first equation in system (11). Similarly, we can verify that Theorem 2 holds.

4. Proof of Theorem 3

Inspired by [27, 29], we now consider the general case of system (11), i.e., we do not assume that \(\lambda_1 = \lambda_2\). We always assume that \(\beta < 0\) and \(\gamma_j, \mu_j > 0\) for \(j = 1, 2\). Firstly, we introduce the product space \(H := H_0^2(\Omega) \times H_0^2(\Omega)\) as the working space. Define the inner product as

\[
\langle u, v \rangle = \sum_{j=1}^{2} \langle u_j, v_j \rangle, \quad \text{for} \ u = (u_1, u_2), \ v = (v_1, v_2) \in H,
\]

which gives rise to a norm on \(H\) denoted by \(\| \cdot \|\). Recall that solutions of (11) are critical points of the following energy functional:

\[
I(u) = \frac{1}{2} \| u \|^2 - \int \sum_{j=1}^{2} \frac{\gamma_j}{p_j} |u_j|^{p_j} - \frac{1}{2} \sum_{j=1}^{2} \left( \mu_j |u_j|^2 + 2\beta |u_j|^{2/2} |u_{2j}|^{2/2} \right).
\]

(54)

For \(a, b \in \mathbb{R}\) with \(a \leq b\), we denote

\[
I^b := \{ u \in H : I(u) \leq b \},
\]

\[
I^a := \{ u \in H : I(u) \geq a \},
\]

(56)

\[
I^{-1}((a, b)) = I^b \cap I^a.
\]

Similar to \(\delta_j\) related to the scalar equation, we define

\[
\delta := \{ (u_1, u_2) \in H : u_1^0, u_2^0 \equiv 0, I(u_1^0, u_2^0)(u_1, 0) = I(u_1, u_2)(0, u_2) = 0 \}.
\]

(57)

That is,

\[
\| u_1 \|_{2}^2 = \int \left( \gamma_1 |u_1|^p_1 + \mu_1 |u_1|^{2/2} + \beta |u_1|^{2/2} |u_2|^{2/2} \right),
\]

\[
\| u_2 \|_{2}^2 = \int \left( \gamma_2 |u_2|^p_2 + \mu_2 |u_2|^{2/2} + \beta |u_2|^{2/2} |u_1|^{2/2} \right).
\]

(58)

Let \(U_1, U_2 \subset C_0^\infty(\Omega)\) with \(U_1^+, U_2^+ \equiv 0\) and \(\text{supp} U_1 \cap \text{supp} U_2 = \emptyset\), it is easy to see that there exist \(t_1, t_2, s > 0\) such that \((t_1U_1^+ - t_2U_2^+, SU_2) \in \delta\). Thus, \(\delta \neq \emptyset\). By Sobolev inequality, it is easy to see that for any \(u \in \delta\), there exists a constant \(C_0 > 0\) such that

\[
\| u_1 \|_{1}, \| u_2 \|_{2}, \| u_1^0 \|_{2}, \| u_2^0 \|_{2} \geq C_0 > 0.
\]

(59)

Set

\[
\epsilon_0 := \min \left( \frac{1}{2} - \frac{1}{P_1}, \frac{1}{2} - \frac{1}{P_2} \right) C_0^2 \left( \frac{1}{2} - \frac{1}{P_1} \right) C_0^2
\]

(60)

Then, we define

\[
\mathcal{B} := \inf_{u \in \delta} I(u).
\]

(61)

We firstly have a lower bound for \(\mathcal{B}\).

Lemma 5. Assume that \(-\sqrt{\mu_1\mu_2} \leq \beta < 0\); then \(I\) is coercive on \(\delta\) and \(\mathcal{B} > 0\).

Proof. We may assume that \(p_1 \geq p_2\). For \(u \in \delta\), note that for \(\beta \geq -\sqrt{\mu_1\mu_2}\), it holds that

\[
\int \sum_{j} \mu_j |u_j|^{2} + 2\beta |u_1|^{2/2} |u_2|^{2/2} \geq 0.
\]

(62)

Then, we have

\[
I(u) = I(u) - \frac{1}{p_2} \int \gamma_j |u_j|^{p_j}
\]

\[
+ \left( \frac{1}{2} - \frac{1}{P_2} \right) \int \left( \sum_{j} \mu_j |u_j|^{2} + 2\beta |u_1|^{2/2} |u_2|^{2/2} \right)
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{P_2} \right) \| u \|^2.
\]

(63)

(63) implies that \(I\) is coercive on \(\delta\). Then, by (59), we have \(\mathcal{B} > 0\).

We shall give an upper bound for \(\mathcal{B}\) later by defining another manifold and the infimum on it. The idea is similar to that in [27], but since the corresponding equation (10) is different and more complex as well as system (11), some new tricks should be used in the current paper, so we shall give the details of the proofs. Let

\[
\mathcal{M} := \{ (u_1, u_2) \in H : u_1, u_2 \equiv 0, I(u_1, u_2)(0, u_2) = 0 \}.
\]

(64)

Obviously, \(\delta \subset \mathcal{M}\). Then, we define

\[
B := \inf_{u \in \mathcal{M}} I(u).
\]

(65)

Before the estimate for the bound of \(\mathcal{B}\), we need some preliminaries.

Lemma 6. Assume that \(-\sqrt{\mu_1\mu_2} \leq \beta < 0\); then, for any \(u \in \delta\), \(\sup_{s \geq t_1, t_2} I(t_1u_1^+ - t_2u_2^+, u_2)\) is attained only by \(t_1 = t_2, s = 1\). Similarly, for any \(u \in \mathcal{M}\), \(\sup_{s \geq t_1, t_2} I(tu_1, su_2)\) is attained only by \(t = s = 1\).

Proof. For \(u \in \delta, t_1, t_2, s \geq 0\); set \(F(t_1, t_2, s) := I(t_1u_1^+ - t_2u_2^+, u_2)\)

\[
= \frac{1}{2} \left( t_1 \| u_1 \|^2 + t_2^2 \| u_1 \|^2 + s^2 \| u_2 \|^2 
\]

\[
- \int \left( \frac{\gamma_1}{P_1} |u_1|^p_1 + \frac{\gamma_1}{P_1} |u_1|^p_1 + \frac{\gamma_2}{P_2} |u_2|^p_2 \right)
\]

\[
- \frac{1}{2} \sum \left( \mu_j |u_j|^{2} + \beta |u_1|^{2/2} |u_2|^{2/2} \right)
\]

\[
- \frac{2\beta}{s^2} \sum |u_2|^{2/2} \left( \frac{t_1^{2/2} |u_1|^{2/2} + t_2^{2/2} |u_2|^{2/2}}{s^{2/2}} \right) |u_2|^{2/2}.
\]

(66)
Then, $F_{t_1|_{\{1,1,1\}}} = F_{t_2|_{\{1,1,1\}}} = F_{t_3|_{\{1,1,1\}}} = 0$ and

$$F_{t_1|_{\{1,1,1\}}} = \left\| u_1 \right\|^2 - (p_1 - 1) \int_\mathbb{R} v_1 |u_1|^{p_1} - (2^* - 1) \int_\mathbb{R} \mu_1 |u_1|^{2^*}$$

$$- \frac{2^*}{2} \int_\mathbb{R} |u_1|^{2^*/2} |\mu_1|^{2^*/2} ,$$

$$F_{t_2|_{\{1,1,1\}}} = \left\| u_2 \right\|^2 - (p_2 - 1) \int_\mathbb{R} v_1 |u_2|^{p_2} - (2^* - 1) \int_\mathbb{R} \mu_2 |u_2|^{2^*}$$

$$- \frac{2^*}{2} \int_\mathbb{R} |u_2|^{2^*/2} |\mu_2|^{2^*/2} ,$$

$$F_{t_3|_{\{1,1,1\}}} = \left\| u_3 \right\|^2 - (p_3 - 1) \int_\mathbb{R} v_2 |u_3|^{p_3} - (2^* - 1) \int_\mathbb{R} \mu_3 |u_3|^{2^*}$$

$$- \frac{2^*}{2} \int_\mathbb{R} |u_3|^{2^*/2} |\mu_3|^{2^*/2} .$$

Note that $D = D + D^2 \leq (D^1 + D^2) D_2$, and $2 D \leq D^1 + D^2$. We can show that $H_F$ is negative definite. Thus, $(1,1,1)$ is the unique maximum point of $F(t_1,t_2,s)$. \qed

**Lemma 7.** Assume that $-\sqrt{\mu_2} < \beta < 0$; then, for any $u \in H$ with $u_1, u_2 \equiv 0$, we have that there exist unique $t_u, s_u > 0$ such that $(t_u u_1, s_u u_2) \in \mathscr{M}$. Moreover, $t_u$ and $s_u$ are continuous with respect to $u$.

**Proof.** For any $u \in H$ with $u_1, u_2 \equiv 0$, we denote

$$E_1 = \left\| u_1 \right\|^2 ,$$

$$E_2 = \left\| u_2 \right\|^2 ,$$

$$A_1 = \int_\mathbb{R} v_1 |u_1|^{p_1} ,$$

$$A_2 = \int_\mathbb{R} v_2 |u_2|^{p_2} ,$$

$$D_1 = \int_\mathbb{R} \mu_1 |u_1|^{2^*} ,$$

$$D_2 = \int_\mathbb{R} \mu_2 |u_2|^{2^*} .$$

Set

$$H_F := \begin{bmatrix} F_{t_1|_{\{1,1,1\}}} & F_{t_2|_{\{1,1,1\}}} & F_{t_3|_{\{1,1,1\}}} \\ F_{t_1|_{\{1,1,1\}}} & F_{t_2|_{\{1,1,1\}}} & F_{t_3|_{\{1,1,1\}}} \\ F_{s_1|_{\{1,1,1\}}} & F_{s_2|_{\{1,1,1\}}} & F_{s_3|_{\{1,1,1\}}} \end{bmatrix} .$$

Then, $D < 0$ and $D_1 D_2 > D^2$. $(t_1 u_1, s_1 u_2) \in \mathscr{M}$ for some $t, s > 0$ is equivalent to

$$t^2 E_1 = t^{p_1} A_1 + t^{2^*} D_1 + t^{2^*/2} s^{2^*/2} D, \quad (75)$$

$$s^2 E_2 = s^{p_2} A_2 + s^{2^*} D_2 + t^{2^*/2} s^{2^*/2} D, \quad (76)$$

By (75), we have

$$s^{2^*/2} = \frac{t^{2-(2^*/2)} E_1 - t^{p_1} A_1 - t^{2^*} D_1}{D} := \frac{G(t)}{D} := \frac{t^{2-(2^*/2)} \bar{G}(t)}{D}, \quad (77)$$

where $s > 0$ implies that $\bar{G}(t) < 0$. Since $\bar{G}(t) < 0$ and for $t > 0$ small enough, $\bar{G}(t) > 0$, there exists a unique $t_0 > 0$ such that $\bar{G}(t_0) = 0$; thus, $t > t_0$. Combining (76) and (77), it is left to show that

$$F(t) := \left( \frac{G(t)}{D} \right)^{(4/2^*)-1} t^{-(2^*/2)} E_2 - \left( \frac{G(t)}{D} \right)^{(2p_2/2^*)-1} t^{-(2^*/2)} A_2 - \frac{G(t)}{D} t^{-(2^*/2)} D_2 - D, \quad (78)$$

has a solution $t > t_0$. Note that $2 - (2^*/2) < p_1 - (2^*/2) < (2^*/2)$ and $4/2^* - 1 < (2p_2/2^*) - 1 < 1$; by direct calculation, we can check that
\[
\lim_{t \to +\infty} F(t) = \frac{D_1D_2 - D^2}{D} < 0, \quad \text{(79)}
\]
\[
\lim_{t \to t_0} F(t) = -D > 0.
\]

This implies that (75) and (76) have a positive solution \((t_+, s_+).\) The uniqueness of \((t_+, s_+).\) follows from Lemma 6.

For the continuity of \((t_+, s_+),\) with respect to \(u,\) we take a sequence \([u_n]_{n=1}^{\infty}\) with \(u_n \to u\) strongly in \(H.\) We may assume that \((t_n, s_n) = (1, 1)\) by replacing \(u_n\) by \((t_n u_n, s_n u_n, u_n)\) if necessary. That is, \(u \in M.\) Using similar denotations to (74) and rewriting \((t_n, s_n) = (t_n, s_n)\) for convenience, we have

\[
t^2_2 E_{1n} = t^{\alpha}_1 A_{1n} + t^2\|D_{1n}\|_{L^2} + t^{2/2}_n s^2/n D_{2n}, \quad \text{(80)}
\]
\[
\dot{s}_n^2 E_2 = s^{\alpha}_2 A_{2n} + s^2 D_{2n} + t^{2/2}_n s^2/n D_{2n}, \quad \text{(81)}
\]
\[
E_1 = A_1 + D_1 + D_2 E_2 = A_2 + D_2 + D. \quad \text{(82)}
\]

If \(t_n \to \infty\) then by (81), we have \(s_n \to \infty.\) Therefore,

\[
\int \|\nabla U_j\|^2 \leq \int \|\nabla U\|^2 + C^N, \quad \text{(87)}
\]
\[
\int U^2_j \geq \int U^2_1 - C^{N-\omega}, \quad q = 2, p, \nu^2. \quad \text{(88)}
\]

By Lemma 7, there exist \(t_r, s_r > 0\) such that \((t_r U_1, s_r U_2)\in M.\) That is,

\[
t^2_0 \|U_{1r}\|_2 = t^{\alpha}_1 \|U_{1r}\|_2 + t^{2/2}_r \|\nabla U_{1r}\|_2, \quad \text{(89)}
\]
\[
\dot{s}_2 \|U_{2r}\|_2 = s^{\alpha}_2 \|U_{2r}\|_2 + \dot{s}_2 \|\nabla U_{2r}\|_2. \quad \text{(89)}
\]

Then similar to the proof of (83), it can be shown that \(t_r\) and \(s_r\) are uniformly bounded. Thus, up to a subsequence, there exist \(t_0, s_0 > 0\) such that \(t_r \to t_0\) and \(s_r \to s_0.\) Let \(r \to 0\) in (88) and (89); it implies that \((t_0 U_{1r}, s_0 U_{2r})\in M.\) Then, \(t_0 = s_0 = 1,\) and we may assume that \((1/2) \leq t,\) and \(s \leq 2\) for \(r > 0\) small. By (87), we have

\[
I(t, U_1, U_2) \leq I(t_0 U_{1r}, s_0 U_{2r}) + C^N \leq I(U_1, U_2) + C^N = B + C^N. \quad \text{(90)}
\]

Let \(V_j (j = 1, 2)\) be the solution of problem (8) when \(\Omega = B(0, r);\) then, \((t U_{1r} - V_1, (s - x_r), s U_{2r}) \in \delta^\prime\) and from [40], it holds that
\[ I_j(V_j) \leq \frac{1}{N^{\eta_j}}u_j^{-(N-2)/2}S^{N/2} - C_1 r^{(2N-4)/(N-4)}, \quad j = 1, 2. \] (91)

Thus,
\[ \mathcal{B} \leq I(t, U_1, -V_1, \cdot - x_i, s, U_2) \]
\[ = I(t, U_1, s, U_2) + I_1(V_1, \cdot - x_i) \]
\[ \leq B + Cr^N + \frac{1}{N^{\eta_1}}u_1^{-(N-2)/2}S^{N/2} - C_1 r^{(2N-4)/(N-4)} \] (92)
\[ < B + \frac{1}{N^{\eta_2}}u_2^{-(N-2)/2}S^{N/2}, \]
for \( r > 0 \) small.

(ii) Next, we show that \( \mathcal{B} < \mathcal{B}_1 + (1/N)^{\eta_2}(N-2)/2S^{N/2} \).
Let \( w_j \) be the sign changing solution of (8) with \( I_j(w_j) = \mathcal{B} \) for \( j = 1, 2 \). Then, \( w_j \in C^2(\Omega) \cap C(\Omega) \) and \( \{ w_j > 0 \} = \{ x \in \Omega : w_j(x) > 0 \} \neq \emptyset \). For any \( R > 0 \), we may take \( y_R \in \{ w_j > 0 \} \) with \( \partial(\{ w_j > 0 \}) = 4R \). Then, \( B(y_R, 3R) \) and \( |w_j| \leq CR, \forall x \in B(y_R, 3R) \). Recall \( \phi \) and \( \phi_R \) in the above steps (i); we define \( U_{1R} = \phi_R w_{1R} \). Since \( I_1'(w_1)w_1 = 0 \), we have
\[ \|w_{1R}\| = \gamma_1 \int |w_{1R}|^2 + \mu_1 \int |w_{1R}|^2. \] (93)

Then, there exists \( t_R > 0 \) such that
\[ \|t_R U_{1R}\| = \gamma_1 \int |t_R U_{1R}|^2 + \mu_1 \int |t_R U_{1R}|^2, \] (94)
and \( I(t_R U_{1R}, 0) = \max_I(t_R U_{1R}, 0) \). It is easy to see that \( (t_R U_{1R} - w_{1R}, V_{2R}) \in \mathcal{B}_2 \), where \( V_{2R} \) is the positive solution of problem (8) in the ball \( B(0, R) \) for \( j = 2 \). Then by (91), we have
\[ \mathcal{B} \leq I(t_R U_{1R}, 0) + I_1(|w_{1R}|) \]
\[ = I(t_R U_{1R}, 0) + I_1(|w_{1R}|) \]
\[ \leq I_1(|w_{1R}|) + C R^N + \frac{1}{N^{\eta_1}}u_1^{-(N-2)/2}S^{N/2} - C_1 R^{(2N-4)/(N-4)} \]
\[ < \mathcal{B}_1 + \frac{1}{N^{\eta_2}}u_2^{-(N-2)/2}S^{N/2}, \]
for \( R > 0 \) small.

For \( \mathcal{B}_1 \), we define \( u_{1R} = (1 - \rho)u_{1R} - \rho u_{1R} \), \( \rho \in [0, 1] \). Let \( u_{1R} := (u_{1R}, u_{2R}) \) and recall that \( (t_{u_{1}, s_{u_{1}}}) \) is defined in Lemma 7; then, we can define a continuous map \( \eta_{u_{1}} : [0, 1] \rightarrow \mathcal{M} \) by
\[ \eta_{u_{1}}(\rho) := \left( t_{u_{1}}, u_{1R} \right). \] (96)

Then, we have the following conclusion and recall that \( \epsilon_0 \) is defined in (60).
\[ \Box \]

**Lemma 9.** For any \( u \in \mathcal{B} \) with \( I(u) < \mathcal{B} + \epsilon_0 \), there exists a small \( \rho_0 \in (0, (1/4)) \) such that
\[ I' \left( \eta_{u_{1}}(\rho_0) \right) \left( t_{u_{1}}, u_{1R} \right) < 0, \]
\[ I' \left( \eta_{u_{1}}(1 - \rho_0) \right) \left( t_{u_{1}}, u_{1R} \right) > 0, \]
\[ I(\eta_{u_{1}}(\rho_0)), \]
\[ I(\eta_{u_{1}}(1 - \rho_0)) < \mathcal{B} - \epsilon_0. \] (99)

**Proof:** It is easy to see that there exist constants \( C_1 \) and \( C_2 > 0 \) such that
\[ C_1 \leq t_{u_{1}}', \]
\[ s_{u_{1}} \leq C_2, \]
\[ \forall \rho \in [0, 1]. \] (100)

Since
\[ I' \left( \eta_{u_{1}}(\rho) \right) \left( t_{u_{1}}, u_{1R} \right) = 0 \quad \text{and} \quad u_{1R} = \rho u_{1R}, \]
we have
\[ I' \left( \eta_{u_{1}}(\rho) \right) \left( t_{u_{1}}, u_{1R} \right) = -I' \left( \eta_{u_{1}}(\rho) \right) \left( t_{u_{1}}, \rho u_{1R} \right) \]
\[ \leq -t^2 \rho^2 \int |u_{1R}|^2 - t^2 \rho^2 \int |u_{1R}|^2 - t^2 \rho^2 \int |u_{1R}|^2 \]
\[ \leq C_1 \rho^2 u_{1R}^2 + C_2 \rho^2 \int |u_{1R}|^2 + C_1 \rho^2 \mu_1 \int |u_{1R}|^2 \]
\[ < 0, \] for \( \rho > 0 \) small. \hspace{1cm} (101)

Similarly, note that \( u_{1R}(1 - \rho) = \rho u_{1R} \); we have
\[ I' \left( \eta_{u_{1}}(1 - \rho) \right) \left( t_{u_{1}}, u_{1R}(1 - \rho) \right) \]
\[ \geq t^2 \rho^2 \int |u_{1R}|^2 - t^2 \rho^2 \int |u_{1R}|^2 - t^2 \rho^2 \int |u_{1R}|^2 \]
\[ \geq C_1 \rho^2 \int |u_{1R}|^2 + C_2 \rho^2 \int |u_{1R}|^2 \]
\[ > 0, \] for \( \rho > 0 \) small. \hspace{1cm} (102)

Since \( u \in \mathcal{B} \), we have \( (u_{1R}, u_{2R}), (-u_{1R}, u_{2R}) \in \mathcal{M} \). Then,
Now, we show the existence of a sequence of \( \eta_n \) such that
\begin{align}
I(\eta_n(0)) &= I(u_1, u_2) \\
&= I(u_1, u_2) - \left[ \frac{1}{2} |u_1|^2 - \frac{\eta_1}{p_1} \right] |u_1|^{p_1} - \frac{1}{2^*} \int |u_1|^2^* \\
&\quad + \left( \frac{2|\beta|}{2^*} - \frac{1}{p_1} \right) \int |u_1|^{2^*/2} u_2^{2^*/2} \\
&= I(u_1, u_2) - \left[ \frac{1}{2} |u_1|^2 - \frac{\eta_1}{p_1} \right] |u_1|^{p_1} - \frac{1}{2^*} \int |u_1|^2^* \\
&\quad + \left( \frac{2|\beta|}{2^*} - \frac{1}{p_1} \right) \int |u_1|^{2^*/2} u_2^{2^*/2} \\
&\leq I(u_1, u_2) - \left( \frac{1}{2} - \frac{1}{p_1} \right) |u_1|^{p_1} + 2^{*} \\
&< B - 2 \varepsilon_0.
\end{align}

Similarly, we have
\begin{align}
I(\eta_n(1)) &= I(-u_1, u_2) \\
&= I(-u_1, u_2) - \left[ \frac{1}{2} |u_1|^2 - \frac{\eta_1}{p_1} \right] |u_1|^{p_1} - \frac{1}{2^*} \int |u_1|^2^* \\
&\quad + \left( \frac{2|\beta|}{2^*} - \frac{1}{p_1} \right) \int |u_1|^{2^*/2} u_2^{2^*/2} \\
&= I(-u_1, u_2) - \left[ \frac{1}{2} |u_1|^2 - \frac{\eta_1}{p_1} \right] |u_1|^{p_1} - \frac{1}{2^*} \int |u_1|^2^* \\
&\quad + \left( \frac{2|\beta|}{2^*} - \frac{1}{p_1} \right) \int |u_1|^{2^*/2} u_2^{2^*/2} \\
&\leq I(u_1, u_2) - \left( \frac{1}{2} - \frac{1}{p_1} \right) |u_1|^{p_1} + 2^{*} \\
&< B - 2 \varepsilon_0.
\end{align}

Since \( I' \) and \( I \) are continuous with respect to \( \rho \), (101), (102), (103), and (104) imply (97), (98), and (99), respectively.

Now, we show the existence of a (PS)\( _{B} \) sequence of \( \eta_n \). Precisely, we have the following important lemma.

**Lemma 10.** There exist a sequence \( \{u_n\} \subset M \) and a constant \( C > 0 \) such that
\begin{align}
I(u_n) &\rightarrow B, \\
I'(u_n) &\rightarrow 0, \\
as \rho \rightarrow +\infty; &\quad \|u_n\|_{1}, \|u_n\|_{2} \geq C, \\
\forall n \geq 1.
\end{align}

**Proof.** By Lemma 5 and the Ekeland’s variational principle, there exists a minimizing sequence \( \{u_n\} \subset M \) such that
\begin{align}
I(u_n) < B + \varepsilon_0, \\
I'(u_n) \rightarrow 0.
\end{align}

Let \( \eta_n(\rho) = (\eta_{n1}(\rho), \eta_{n2}(\rho)) := \eta_n(\rho) \); by Lemma 9, there exists \( \rho_n \in (0, (1/4)) \) such that
\begin{align}
I'(\eta_n(\rho_n))(\eta_{n1}(\rho_n)^{+}, 0) < 0, \\
I'(\eta_n(1 - \rho_n))(\eta_{n1}(1 - \rho_n)^{+}, 0) > 0,
\end{align}
\begin{align}
I(\eta_n(\rho_n)), \\
I(\eta_n(1 - \rho_n)) < B - \varepsilon_0.
\end{align}

We claim that there exists a sequence \( \{u_n\} \subset M \) such that
\begin{align}
dist (u_n, \eta_n([\rho_n, 1 - \rho_n])) \rightarrow 0, \\
I'(u_n) \rightarrow 0, \\
I(u_n) \rightarrow B.
\end{align}

If not, then we assume that there exists small \( \varepsilon_1 > 0 \) such that
\begin{align}
\forall u \in I^{-1}_{\delta}([B - \varepsilon_1, B + \varepsilon_1]) \cap S_\delta; \quad \|I(u)\| \geq \varepsilon_1,
\end{align}
where
\begin{align}
\delta \delta (B - \varepsilon_1, B + \varepsilon_1) := \{u \in M; \quad \|u\| \in [B - \varepsilon_1, B + \varepsilon_1]\},
\end{align}
\begin{align}
S := \eta_n([\rho_n, 1 - \rho_n]),
\end{align}
and \( S_\delta \) is the \( \varepsilon_1 \) neighborhood of \( S \). Then, by the well-known deformation lemma, there exist a continuous map \( \xi : [0, 1] \times M \rightarrow M \) and an \( \delta_1 \in (0, (\min[\varepsilon_0, \varepsilon_1])/4) \) such that
\begin{enumerate}
\item \( \xi(0, u) = u, \forall u \in M \),
\item \( \xi(t, u) \neq I^{-1}([B - 2\varepsilon_2, B + 2\varepsilon_2]), t \in [0, 1] \),
\item \( \xi(1, I_{\delta}^{-1}(S_\delta)) \subset I_{\delta}^{-1} \).
\end{enumerate}

By Lemma 6, for \( n \) large enough, we have that
\begin{align}
\sup_{\rho \in [\rho_n, 1 - \rho_n]} I(\eta_n(\rho)) \leq I(\eta_n(\rho_n)) < B + \varepsilon_2.
\end{align}

Then, by the properties (2) and (3) of \( \xi \) and (109), we have
\begin{align}
\sup_{\rho \in [\rho_n, 1 - \rho_n]} I(\xi(1, \eta_n(\rho_n))) \leq B - \varepsilon_2,
\end{align}
\begin{align}
\xi(1, \eta_n(\rho_n)) = \eta_n(\rho_n), \\
\xi(1, \eta_n(1 - \rho_n)) = \eta_n(1 - \rho_n).
\end{align}

Similarly as in (20), we define
We rewrite $\eta(\rho) := (\eta^1(\rho), \eta^2(\rho)) := \xi(1, \eta_n(\rho))$ for convenience. $\eta(\rho) \in \mathcal{M}$ implies that $E((\eta^1(\rho)), (\eta^2(\rho))) = 1$. By (107) and (108), we have that

$$E\left((\eta^1(\rho_0))^+, \eta^2(\rho_0)\right) > 1,$$

$$E\left((\eta^1(1 - \rho_0))^+, \eta^2(1 - \rho_0)\right) < 1.$$  

Then, by the continuity of $E$, we can define $\rho_0 := \max\{\rho \in [\rho_{\text{min}}, 1 - \rho_{\text{min}}]: E((\eta^1(\rho))^+, \eta^2(\rho)) = 1\}$.

By definition of $\rho_0$, there exists $\delta_0 \in (0, 1 - \rho_{\text{min}})$ such that $E((\eta^1(\rho))^+, \eta^2(\rho)) < 1$ for any $\rho \in (\rho_{\text{min}}, \rho_0 + \delta_0)$. Then, $E((\eta^1(\rho))^+, \eta^2(\rho)) > 1$ for any $\rho \in (\rho_{\text{min}}, \rho_0 - \delta_0)$. Thus, $E((\eta^1(\rho_0))^+, \eta^2(\rho_0)) \geq 1$. Hence, $E((\eta^1(\rho_0))^+, \eta^2(\rho_0)) = 1$. That is, $\eta(\rho_0) \in \mathcal{B}$. But by (115), we have $I(\eta(\rho_0)) < \mathcal{B}$, a contradiction with the definition of $\mathcal{B}$. Therefore, the claim (111) becomes true.

Now, we can choose $\rho_{\text{opt}} \in [\rho_{\text{min}}, 1 - \rho_{\text{min}}]$ such that $\|u_n - \eta_n(\rho_{\text{opt}})\| \rightarrow 0, n \rightarrow \infty$. Since $I(u_n) \rightarrow \mathcal{B}$, we have $I(\eta_n(\rho_{\text{opt}})) \rightarrow \mathcal{B}$. Therefore, we can choose two Lagrange multipliers $L_{1n}, L_{2n} \in \mathbb{R}$ such that

$$J_1(u_1, u_2) := J(u_1, u_2)(u_1, 0),$$

$$J_2(u_1, u_2) := J(u_1, u_2)(0, u_2).$$

By (111), there exist two Lagrange multipliers $L_{1n}, L_{2n} \in \mathbb{R}$ such that

$$J(u_1, u_2) := J(u_1, u_2) + L_{1n}J_1(u_1, u_2) + L_{2n}J_2(u_1, u_2) \rightarrow 0.$$  

Then, $J(u_1, u_2) = o(1)$ and $J(u_1, u_2) = o(1)$; i.e.,

$$L_{1n}^\infty \left[(p_1 - 1) \int \nabla u_1 \nabla p_1 + (2^* - 2) \int \nabla u_1 \nabla u_2 \right]$$

$$= L_{2n}^\infty \left[2^* \int \nabla u_1 \nabla u_2 \right] + o(1),$$

$$L_{2n}^\infty \left[(p_2 - 2) \int \nabla u_2 \nabla p_2 + (2^* - 2) \int \nabla u_2 \nabla u_2 \right]$$

$$= L_{2n}^\infty \left[2^* \int \nabla u_2 \nabla u_2 \right] + o(1).$$  

Similarly as in (74), we denote

$$A_{1n} = \int \nabla u_1 \nabla p_1,$$

$$A_{2n} = \int \nabla u_2 \nabla p_2,$$

$$D_{1n} = \int \nabla u_1 \nabla u_1,$$

$$D_{2n} = \int \nabla u_2 \nabla u_2,$$

$$D_n = \int \nabla u_1 \nabla u_2 \nabla u_2.$$

Note that $(u_{1n}, u_{2n}) \in \mathcal{M}$, we have

$$\|u_{1n}\|_2^2 = A_{1n} + D_{1n} - D_n,$$

$$\|u_{2n}\|_2^2 = A_{2n} + D_{2n} - D.$$  

Consequently, by (59), we have

$$A_{1n} + D_{1n} > D_n,$$

$$A_{2n} + D_{2n} > D.$$  

Set

$$S_{1n} := (p_1 - 2)A_{1n}A_{2n} + (p_1 - 2)(2^* - 2)A_{1n}D_{2n} + (p_2 - 2)(2^* - 2)A_{2n}D_{1n} + (p_2 - 2)(2^* - 2)A_{2n}D_{2n}$$

$$+ (2^* - 2)D_{1n}D_{2n} + (2^* - 2)(2^* - 2)D_{1n}D_{2n} + (2^* - 2)(2^* - 2)D_{1n}D_{2n}.$$  

Then, by (122) and (123), we have

$$\sigma(1) = L_{1n}L_{2n}(S_{1n} + S_{2n}) := L_{1n}L_{2n}S_n.$$  

Note that
\[ D_n \leq \frac{1}{2} (D_{1n} + D_{2n}), \quad (129) \]

\[ D_n^2 \leq D_{1n} D_{2n}. \]

We have
\[ S_{2n} \leq (2^* - 2)^2 (D_{1n} D_{2n} - D_n^2) \geq 0. \quad (130) \]

Thus, \( S_n \geq C > 0 \) for some constant \( C > 0 \). By (128), we have \( L_{1n} L_{2n} = o(1) \). Without loss of generality, we may assume that \( L_{1n} = o(1) \); then by (123) and (59), we have \( L_{2n} = o(1) \). Then, (121) implies that \( I' (u_{1n}, u_{2n}) \to 0 \). Thus, we complete the proof.

We also need an important lemma which is proved in [25].

\[ \text{Lemma 11 (see [25]). Assume that } (u_{1n}, u_{2n}) \to (u_1, u_2) \text{ weakly in } H \text{ as } n \to \infty; \text{ then, passing to a subsequence, it holds that} \]
\[
\lim_{n \to \infty} \left( \left| \left| u_{1n} \right|^{2^*} \right| u_{2n} \right|^{2^*} - \left| u \right|^{2^*} |u_2|^{2^*} - \left| u_{1n} - u \right|^{2^*} |u_{2n} - u|^{2^*} \right) = 0. \quad (131) \]

\[ \text{Proof of Theorem 3. Proof of Theorem 3. Let } \{u_n\} \subset \mathcal{M} \text{ be the sequence obtained in Lemma 10. Then, up to a} \]

\[ B = \lim_{n \to \infty} I (u_n) = \lim_{n \to \infty} \left[ \frac{1}{N} \sum_{j} \left( \frac{1}{2^*} - \frac{1}{p_j} \right) \gamma_j |u_j|^p \right] \]
\[ = \left[ \frac{1}{N} \|u\|^2 + \sum_{j} \left( \frac{1}{2^*} - \frac{1}{p_j} \right) \gamma_j |u_j|^p \right] + \lim_{n \to \infty} \frac{1}{N} \left( \|\nabla (w_{1n})\|^2 + \|\nabla (w_{2n})\|^2 + \|\nabla w_{2n}\|^2 \right). \quad (137) \]

\[ \text{Case 1. Assume that } u_1, u_2 \equiv 0. \text{ Then, } u_n \to 0 \text{ in } L^2 (\Omega) \text{ and } L^p (\Omega) \text{ for } j = 1, 2. \text{ From (135) and (136), we have} \]
\[ \int \|\nabla (w_{1n})\|^2 \leq \mu_1 S^{(2^*/2)} \left( \int \|\nabla (w_{1n})\|^2 \right)^{2^*/2}, \quad (138) \]
\[ \int \|\nabla w_{2n}\|^2 \leq \mu_2 S^{(2^*/2)} \left( \int \|\nabla w_{2n}\|^2 \right)^{2^*/2}. \]

Then,
\[ \int \|\nabla (w_{1n})\|^2 \geq \mu_1^{s(2^*/2)} S^{N/2}, \quad (139) \]
\[ \int \|\nabla w_{2n}\|^2 \geq \mu_2^{s(2^*/2)} S^{N/2}. \quad (140) \]

Thus, by (137), we have
\[ B \geq \frac{2}{N} \mu_1^{s(2^*/2)} S^{N/2} + \frac{1}{N} \mu_2^{s(2^*/2)} S^{N/2}, \quad (141) \]
a contradiction with Lemma 8, (85), and (9). Therefore, Case 1 is impossible.

\[ \text{Case 2. Assume that } u_1 \equiv 0 \text{ and } u_2 \equiv 0. \text{ Then, (139) still holds and } I' (0, u_2) = 0 \text{ which implies that } u_2 \text{ is a nontrivial solution of (10) with } j = 2. \text{ Thus, } I (0, u_2) \geq B_2. \text{ By (137), we have} \]
\[ B \geq B_2 + \frac{2}{N} \mu_1^{s(2^*/2)} S^{N/2}, \quad (142) \]
a contradiction with Lemma 8 and (85). Therefore, Case 2 is impossible.

\[ \text{Case 3. Assume that } u_1 \equiv 0 \text{ and } u_2 \equiv 0. \text{ Then, (140) still holds and } I' (u_1, 0) = 0 \text{ which implies that } u_1 \text{ is a nontrivial solution of (10) with } j = 1. \text{ Thus, } I (u_1, 0) \geq B_1. \]

(i) If \( u_{1n} \to u_1 \) strongly in \( H_0^1 (\Omega) \), then \( u_1 \) is a sign changing solution of (10) with \( j = 1 \). So \( I (u_1, 0) \geq B_1 \).

By (137), we have
\[ B \geq B_1 + \frac{1}{N} \mu_2^{s(2^*/2)} S^{N/2}, \quad (143) \]
a contradiction with Lemma 8.
(ii) If $u_{1n} \rightharpoonup u_1$ strongly in $H_0^1(\Omega)$, then by (132) and Brezis–Lieb lemma, we can also show that
\[
\int |\nabla w_{1n}|^2 \geq \mu_1^{-(N-2)/2} S^{N/2}.
\] (144)

By (137), we have
\[
\mathcal{B} \geq \mathcal{B}_1 + \frac{1}{N} \mu_1^{-(N-2)/2} S^{N/2},
\] (145)
a contradiction with Lemma 8 and (85).

Therefore, Case 3 is impossible.

Now, we have shown that $u_1, u_2 \equiv 0$ and then $(u_1, u_2)$ is a nontrivial solution of (11). If $u_{1n} \rightharpoonup u_1$ strongly in $H_0^1(\Omega)$, then (144) holds and
\[
\mathcal{B} \geq \mathcal{B}_1 + \frac{1}{N} \mu_1^{-(N-2)/2} S^{N/2},
\] (146)
a contradiction with Lemma 8. Thus, $u_{1n} \rightharpoonup u_1$ strongly in $H_0^1(\Omega)$ and then $u_1$ is sign changing, $I(u) \geq \mathcal{B}$. On the other hand, by (137), we have
\[
I(u) \leq \lim_{n \to \infty} I(u_n) = \mathcal{B}.
\] (147)

Then, we have $I(u) = \mathcal{B}$ and $u_{1n} \rightharpoonup u$ strongly in $H$. Then $(u_{1n}, |u_{1n}|) \in \mathcal{B}$ and $I(u_{1n}, |u_{1n}|) = \mathcal{B}$. Similarly to the proof in Lemma 10, we have $I(u_{1n}, |u_{1n}|) = 0$. By the maximum principle, we have $|u_{1n}| > 0$ in $\Omega$. Consequently, $(u_1, |u_1|)$ is a least energy sign changing solution of system (11) with $u_1$ sign changing and $u_2$ positive. Thus, we complete the proof.

5. Conclusion

In this paper, we study a coupled nonlinear Schrödinger system with critical exponents which arise in many physical problems. By the modified Nehari manifold method and some mathematical skills for estimations of the energy, we show the existence of the least energy sign changing solution for the general system. Besides, we construct multiple solutions of the system for the symmetrical case.

Data Availability

No data were used to support this study and all materials are available in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors’ Contributions

All authors studied this problem and read and approved the final manuscript.

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