

## Research Article

# Mean Square Integral Inequalities for Generalized Convex Stochastic Processes via Beta Function

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The integral inequalities have become a very popular area of research in recent years. The present paper deals with some important generalizations of convex stochastic processes. Several mean square integral inequalities are derived for this generalization. The involvement of the beta function in the results makes the inequalities more convenient for applied sciences.

## 1. Introduction

Just as the probability theory is regarded as the study of mathematical models of random phenomena, the theory of stochastic processes plays an important role in the investigation of random phenomena depending on time. A random phenomenon that arises through a process which is developing in time and controlled by some probability law is called a stochastic process. Thus, stochastic processes can be referred to as the dynamic part of the probability theory. We will now give a formal definition of a stochastic process.

Various collections of random variables  $X(l, \cdot)$ ,  $l \in J$ , have the property in some sense that  $X(l)$  is stochastically convex (or  $-X(l, \cdot)$  is stochastically concave). The stochastic process with convexity properties has a large number of applications. In [1], the authors demonstrated the use of a stochastically convex function in different areas of probability and statistics.

In queueing theory, the convexity of steady-state waiting time is used in [2]. More in [1], the authors used the convexity of payoff in the success rate to obtain an imperfect repair.

In 1980, Nikodem introduced the study of quadratic and convex stochastic processes (see [3, 4]). In [5, 6], Skowronski explained the properties of the Wright-convex and Jensen-convex stochastic process. Also, Kotrys described results on convex and strongly convex stochastic processes, together

with a Hermite-Hadamard-type inequality for convex stochastic processes (see [7–9]).

The Hermite-Hadamard inequality for the convex stochastic process is defined as follows:

Let  $X : J \times \Omega \rightarrow \mathbb{R}$  be a convex and mean square continuous in the interval  $T \times \Omega$ ; then, the inequality holds almost everywhere:

$$X\left(\frac{r+s}{2}, \cdot\right) \leq \frac{1}{r-s} \int_r^s X(l, \cdot) dl \leq \frac{X(r, \cdot) + X(s, \cdot)}{2}, \quad (1)$$

for any  $r, s \in J$ . For more details on Hermite-Hadamard-type inequalities for the stochastic process, we may refer the reader to [10–12].

**Definition 1** (see [13]). A stochastic process is a collection of random variables  $X(l)$  parameterized by  $l \in J$ , where  $J \subset \mathbb{R}$ . When  $J = \{1, 2, \dots\}$ , then  $X(l)$  is said to be a stochastic process in discrete time (i.e., a sequence of random variables). When  $J$  is an interval in  $\mathbb{R}$  ( $J = [0, \infty)$ ), then we say that  $X(l)$  is a stochastic process in continuous time.

For every  $\omega \in \Omega$ , the function

$$J \ni l \mapsto X(l, \omega) \quad (2)$$

is said to be a path or sample path of  $X(l)$ .

**Definition 2** (see [13]). A family  $F_l$  of  $\alpha$ -fields on  $\Omega$  parametrized by  $l \in J$ , where  $J \subset \mathbb{R}$ , is said to be a filtration if

$$F_s \subset F_l \subset F, \quad (3)$$

for any  $s, l \in J$  such that  $s \leq l$ .

**Definition 3** (see [13]). A stochastic process  $X(l)$  parametrized by  $l \in T$  is said to be a martingale (supermartingale, submartingale) with respect to a filtration  $F_l$  if

- (1)  $X(l)$  is integrable for each  $l \in J$
- (2)  $X(l)$  is  $F_l$ -measurable for each  $l \in J$
- (3)  $X(s) = E(X(l) | F_s)$  (respectively,  $\leq$  or  $\geq$ ) for every  $s, l \in J$  such that  $s \leq l$ .

**Definition 4** (see [7]). Let  $(\Omega, A, P)$  be an arbitrary probability space and  $J \subset \mathbb{R}$  be an interval. A stochastic process  $X : \Omega \rightarrow \mathbb{R}$  is called as follows:

- (1) Stochastically continuous in interval  $J$ , if  $\forall l_0 \in J$

$$P - \lim_{l \rightarrow l_0} X(l, \cdot) = X(l_0, \cdot), \quad (4)$$

where  $P - \lim$  denotes the limit in probability.

- (2) Mean square continuous in  $J$ , if  $\forall l_0 \in J$

$$P - \lim_{l \rightarrow l_0} E(X(l, \cdot) - X(l_0, \cdot))^2 = 0, \quad (5)$$

where  $E(X(l, \cdot))$  denotes the expectation value of the random variable  $X(l, \cdot)$ .

- (3) Increasing (decreasing) if  $\forall \mu, \nu \in J$  such that

$$X(\mu, \cdot) \leq X(\nu, \cdot), X(\mu, \cdot) \geq X(\nu, \cdot). \quad (6)$$

- (4) Monotonic if it is increasing or decreasing

- (5) If there exists a random variable  $X'(l, \cdot) : J \times \Omega \rightarrow \mathbb{R}$ , then we say that it is differentiable at a point  $l \in J$ , such that

$$X'(l, \cdot) = P - \lim_{l \rightarrow l_0} \frac{X(l, \cdot) - X(l_0, \cdot)}{l - l_0}. \quad (7)$$

A stochastic process  $X : J \times \Omega \rightarrow \mathbb{R}$  is continuous (differentiable) if it is continuous (differentiable) at every point of interval  $J$ .

**Definition 5** (see [7, 14]). Suppose that  $(\Omega, A, P)$  be a probability space and  $J \subset \mathbb{R}$  be an interval with  $E(X(\vartheta)^2) < \infty \forall \vartheta \in J$ . If  $[r, s] \subset J$ ,  $r = \vartheta_0 < \vartheta_1 < \vartheta_2 < \dots < \vartheta_n = s$  is a partition of  $[r, s]$  and  $\Theta \in [\vartheta_{\kappa-1}, \vartheta_\kappa]$  for  $\kappa = 1, 2, \dots, n$ . A random variable

$Z : \Omega \rightarrow \mathbb{R}$  is known as mean square integral of the process  $X(\vartheta, \cdot)$  on  $[r, s]$  if

$$\lim_{n \rightarrow \infty} E \left[ \sum_{\kappa=1}^n X(\Theta_\kappa, \cdot) (\vartheta_\kappa - \vartheta_{\kappa-1}) - Z(\cdot) \right]^2 = 0, \quad (8)$$

then, we have

$$\int_r^s X(\vartheta, \cdot) d\vartheta = Z(\cdot) (a.e.). \quad (9)$$

Also, the mean square integral operator is increasing; thus,

$$\int_r^s X(\vartheta, \cdot) d\vartheta \leq \int_r^s Y(\vartheta, \cdot) d\vartheta (a.e.), \quad (10)$$

where  $X(\vartheta, \cdot) \leq Y(\vartheta, \cdot)$  in  $[r, s]$ .

For more details on stochastic processes, we may refer the reader to [15, 16].

Next, we write some basic definitions which will be used in this work:

**Definition 6** (see [4]). Let  $(\Omega, A, P)$  be a probability space and  $J \subseteq \mathbb{R}$  be an interval. A stochastic process  $X : J \times \Omega \rightarrow \mathbb{R}$  is called a convex stochastic process; then, the inequality holds almost everywhere:

$$X(\vartheta r + (1 - \vartheta)s, \cdot) \leq \vartheta X(r, \cdot) + (1 - \vartheta)X(s, \cdot), \quad (11)$$

$$\forall r, s \in J \text{ and } \vartheta \in [0, 1].$$

**Definition 7** (see [17]). A process  $X : J \times \Omega \rightarrow \mathbb{R}$  is said to be a  $p$ -convex stochastic process, if the following inequality holds:

$$X \left( [\vartheta r^p + (1 - \vartheta)s^p]^{1/p}, \cdot \right) \leq \vartheta X(r, \cdot) + (1 - \vartheta)X(s, \cdot) (a.e.), \quad (12)$$

for all  $r, s \in J$  and  $\vartheta \in [0, 1]$ .

In [18], Barr  ez et al. defined the definition of the  $h$ -convex stochastic process as follows:

**Definition 8** (see [18]). Let  $h : (0, 1) \rightarrow \mathbb{R}$  be a nonnegative function,  $h \neq 0$ . A stochastic process  $X : J \times \Omega \rightarrow \mathbb{R}$  is a  $h$ -convex stochastic process, if the inequality holds:

$$X(\vartheta r + (1 - \vartheta)s, \cdot) \leq h(\vartheta)X(r, \cdot) + h(1 - \vartheta)X(s, \cdot) (a.e.), \quad (13)$$

for every  $r, s \in J$  and  $\vartheta \in [0, 1]$ .

Obviously, by taking  $h(\vartheta) = \vartheta$  in (13), then the definition of the  $h$ -convex stochastic process reduces to the definition of the convex stochastic process [4].

**Definition 9** (see [9]). Let  $c : \Omega \longrightarrow \mathbb{R}$  be a positive random variable. A stochastic process  $X : J \times \Omega \longrightarrow \mathbb{R}$  is known as strongly convex with modulus  $c(\cdot) > 0$ , if the following inequality holds:

$$X(\vartheta r + (1 - \vartheta)s, \cdot) \leq \vartheta X(r, \cdot) + (1 - \vartheta)X(s, \cdot) - c(\cdot)\vartheta(1 - \vartheta)(r - s)^2(a.e.), \quad (14)$$

for all  $r, s \in J$  and  $\vartheta \in [0, 1]$ .

For more details on the strongly convex stochastic process, we refer to [9], and for some interesting properties of some special function, see [19, 20]. Obviously, if we omit the term  $c(\cdot)\vartheta(1 - \vartheta)(r - s)^2$  in (14), then we get the definition of a convex stochastic process (see [4]). On the other hand, if we set  $c = 0$ , then we get it from (14) in limit case. Also, we use the beta function in this present work which is expressed as

$$\beta(r, s) = \int_0^1 \vartheta^{r-1} (1 - \vartheta)^{s-1} d\vartheta, \operatorname{Re}(r) > 0, \operatorname{Re}(s) > 0. \quad (15)$$

## 2. Main Results

**Lemma 10** (see [21]). Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. Then, the following equality holds almost everywhere:

$$\int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega = (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu X(\vartheta r + (1 - \vartheta)s, \cdot) d\vartheta, \quad (16)$$

for some fixed  $\mu, \nu > 0$ .

**Lemma 11.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. Then, the following equality holds almost everywhere:

$$\begin{aligned} & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\ &= \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu X\left([\vartheta r^p + (1 - \vartheta)s^p]^{1/p}, \cdot\right) d\vartheta, \end{aligned} \quad (17)$$

for some fixed  $\mu, \nu > 0$ .

*Proof.* Let  $\omega = [\vartheta r^p + (1 - \vartheta)s^p]^{1/p}$ . Then,  $\vartheta = (s^p - \omega^p)/(s^p - r^p)$ ,  $1 - \vartheta = (\omega^p - r^p)/(s^p - r^p)$ , and  $d\vartheta = -p/((s^p - r^p)\omega^{1-p})d\omega$ , so

$$\begin{aligned} & \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu X\left([\vartheta r^p + (1 - \vartheta)s^p]^{1/p}, \cdot\right) d\vartheta \\ &= \frac{p}{(s^p - r^p)^{\mu+\nu+1}} \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega(a.e.), \end{aligned} \quad (18)$$

which completes the proof.  $\square$

**Remark 12.** If we take  $p = 1$  in Lemma 11, then we obtain Lemma 3.1 of [21].

The following results are derived for  $p$ -convex stochastic processes.

**Theorem 13.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|$  is  $p$ -convex on  $[r, s]$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned} & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\ & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\mu + 1, \nu + 2)|X(r, \cdot)| + \beta(\mu + 2, \nu + 1)|X(s, \cdot)|). \end{aligned} \quad (19)$$

*Proof.* By using Lemma 11, the definition of the  $p$ -convexity of  $|X|$  and the beta function yield that

$$\begin{aligned} & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\ & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu \left| X\left([\vartheta r^p + (1 - \vartheta)s^p]^{1/p}, \cdot\right) \right| d\vartheta \\ & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu (\vartheta |X(r, \cdot)| + (1 - \vartheta) |X(s, \cdot)|) d\vartheta \\ & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\mu + 1, \nu + 2)|X(r, \cdot)| + \beta(\mu + 2, \nu + 1)|X(s, \cdot)|)(a.e.), \end{aligned} \quad (20)$$

which completes the proof.  $\square$

**Remark 14.** If we take  $p = 1$  in Theorem 13, then we obtain Theorem 3.1 of [21].

**Theorem 15.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is  $p$ -convex on  $[r, s]$  for  $q > 1$  with  $1/\kappa + 1/q = 1$ , where  $r, s \in J$ ,  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned} & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\ & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1/\kappa} \left( \frac{|X(r, \cdot)|^q + |X(s, \cdot)|^q}{2} \right)^{1/q}. \end{aligned} \quad (21)$$

*Proof.* Employing Lemma 11 and Hölder's integral inequality, we have (a.e.)

$$\begin{aligned}
 & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu \left| X\left([\vartheta r^p + (1-\vartheta)s^p]^{1/p}, \cdot\right) \right| d\vartheta \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \left( \int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1/\kappa} \\
 & \quad \times \left( \int_0^1 \left| X\left([\vartheta r^p + (1-\vartheta)s^p]^{1/p}, \cdot\right) \right|^q d\vartheta \right)^{1/q}. \tag{22}
 \end{aligned}$$

Since  $|X|^q$  is a  $p$ -convex stochastic process, one can yield that

$$\begin{aligned}
 & \int_0^1 \left| X\left([\vartheta r^p + (1-\vartheta)s^p]^{1/p}, \cdot\right) \right|^q d\vartheta \\
 & \leq \int_0^1 (\vartheta |X(r, \cdot)|^q + (1-\vartheta) |X(s, \cdot)|^q) d\vartheta \tag{23} \\
 & = \frac{|X(r, \cdot)|^q + |X(s, \cdot)|^q}{2} \text{ (a.e.)},
 \end{aligned}$$

and by the definition of the beta function, we can write

$$\int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta = \beta(\kappa\mu + 1, \kappa\nu + 1). \tag{24}$$

Inserting (23) and (24) in (22) yields the required inequality (21).  $\square$

*Remark 16.* If we take  $p = 1$  in Theorem 15, then we get Theorem 3.2 of [21].

**Theorem 17.** Let  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is  $p$ -convex on  $[r, s]$  for  $q > 1$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned}
 & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1-(1/q)} \\
 & \quad \times (\beta(\mu + 1, \nu + 2) |X(r, \cdot)|^q + \beta(\mu + 2, \nu + 1) |X(s, \cdot)|^q)^{1/q}. \tag{25}
 \end{aligned}$$

*Proof.* Making use of Lemma 11 and the power-mean integral inequality for  $\kappa \geq 1$  yields that

$$\begin{aligned}
 & \int_{r^p}^{s^p} (\omega^p - r^p)^\mu (s^p - \omega^p)^\nu \frac{X(\omega, \cdot)}{\omega^{1-p}} d\omega \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \int_0^1 (1-\vartheta)^\mu \vartheta^\nu \left| X\left([\vartheta r^p + (1-\vartheta)s^p]^{1/p}, \cdot\right) \right| d\vartheta \text{ (a.e.)} \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \left( \int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-(1/q)} \\
 & \quad \times \left( \int_0^1 (1-\vartheta)^\mu \vartheta^\nu \left| X\left([\vartheta r^p + (1-\vartheta)s^p]^{1/p}, \cdot\right) \right|^q d\vartheta \right)^{1/q} \text{ (a.e.)}. \tag{26}
 \end{aligned}$$

By using the  $p$ -convexity of the stochastic process  $|X|^q$  and by the definition of the beta function, we have (a.e.)

$$\begin{aligned}
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} \left( \int_0^1 (1-\vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-(1/q)} \\
 & \quad \times \left( \int_0^1 (1-\vartheta)^\mu \vartheta^\nu (\vartheta |X(r, \cdot)|^q + (1-\vartheta) |X(s, \cdot)|^q) d\vartheta \right)^{1/q} \\
 & \leq \frac{(s^p - r^p)^{\mu+\nu+1}}{p} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1-(1/q)} \\
 & \quad \times (\beta(\mu + 1, \nu + 2) |X(r, \cdot)|^q + \beta(\mu + 2, \nu + 1) |X(s, \cdot)|^q)^{1/q}. \tag{27}
 \end{aligned}$$

which completes the proof.  $\square$

*Remark 18.* If we take  $p = 1$  in Theorem 17, then we obtain Theorem 3.3 of [21].

The following results are derived for  $h$ -convex stochastic processes.

**Theorem 19.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|$  is  $h$ -convex on  $[r, s]$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned}
 & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
 & \leq (s - r)^{\mu+\nu+1} (|X(r, \cdot)| \beta_h(\vartheta) + |X(s, \cdot)| \beta_h(1 - \vartheta)), \tag{28}
 \end{aligned}$$

where

$$\beta_h(\vartheta)^{\mu\nu} = \int_0^1 (1-\vartheta)^\mu \vartheta^\nu h(\vartheta) d\vartheta, \tag{29}$$

$$\beta_h(1 - \vartheta)^{\mu\nu} = \int_0^1 (1-\vartheta)^\mu \vartheta^\nu h(1 - \vartheta) d\vartheta. \tag{30}$$

*Proof.* By Lemma 10, the definition of the  $h$ -convexity of  $|X|$  and the beta function yield that

$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta \\
& \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu (h(\vartheta) |X(r, \cdot)| + h(1 - \vartheta) |X(s, \cdot)|) d\vartheta \\
& \leq (s - r)^{\mu+\nu+1} \left( |X(r, \cdot)| \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu h(\vartheta) d\vartheta + |X(s, \cdot)| \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu h(1 - \vartheta) d\vartheta \right) \\
& = (s - r)^{\mu+\nu+1} (|X(r, \cdot)| \beta_h(\vartheta)^{\mu, \nu} + |X(s, \cdot)| \beta_h(1 - \vartheta)^{\mu, \nu}) (a.e.),
\end{aligned} \tag{31}$$

which completes the proof.  $\square$

**Remark 20.** If we take  $h(\vartheta) = \vartheta$  in Theorem 19, then we obtain Theorem 3.1 of [21].

**Theorem 21.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is  $h$ -convex on  $[r, s]$  for  $q > 1$  with  $1/\kappa + 1/q = 1$ , where  $r, s \in J$ ,  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1/\kappa} (|X(r, \cdot)|^q \beta_h(\vartheta) + |X(s, \cdot)|^q \beta_h(1 - \vartheta))^{1/q},
\end{aligned} \tag{32}$$

where  $\beta_h(\vartheta) = \int_0^1 h(\vartheta) d\vartheta$  and  $\beta_h(1 - \vartheta) = \int_0^1 h(1 - \vartheta) d\vartheta$ .

*Proof.* Employing Lemma 10 and Hölder's integral inequality, we have (a.e.)

$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta \\
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1/\kappa} \left( \int_0^1 |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \right)^{1/q}.
\end{aligned} \tag{33}$$

Since  $|X|^q$  is an  $h$ -convex stochastic process, one can yield that

$$\begin{aligned}
& \int_0^1 |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \\
& \leq \int_0^1 (h(\vartheta) |X(r, \cdot)|^q + h(1 - \vartheta) |X(s, \cdot)|^q) d\vartheta \\
& \leq |X(r, \cdot)|^q \int_0^1 h(\vartheta) d\vartheta + |X(s, \cdot)|^q \int_0^1 h(1 - \vartheta) d\vartheta \\
& \leq |X(r, \cdot)|^q \beta_h(\vartheta) + |X(s, \cdot)|^q \beta_h(1 - \vartheta) (a.e.),
\end{aligned} \tag{34}$$

and by the definition of the beta function, we can write

$$\int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta = \beta(\kappa\mu + 1, \kappa\nu + 1). \tag{35}$$

Inserting (34) and (35) in (33) yields the desired inequality (32).  $\square$

**Remark 22.** If we take  $h(\vartheta) = \vartheta$  in Theorem 21, then we obtain Theorem 3.2 of [21].

**Theorem 23.** Let  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is  $h$ -convex on  $[r, s]$  for  $q > 1$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1-(1/q)} \\
& \quad \cdot (\beta_h(\vartheta)^{\mu, \nu} |X(r, \cdot)|^q + \beta_h(1 - \vartheta)^{\mu, \nu} |X(s, \cdot)|^q)^{1/q},
\end{aligned} \tag{36}$$

where

$$\beta_h(\vartheta)^{\mu, \nu} = \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu h(\vartheta) d\vartheta, \tag{37}$$

$$\beta_h(1 - \vartheta)^{\mu, \nu} = \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu h(1 - \vartheta) d\vartheta. \tag{38}$$

*Proof.* By Lemma 10 and the power-mean integral inequality for  $\kappa \geq 1$ , one can yield that

$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta (a.e.) \\
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-(1/q)} \\
& \quad \times \left( \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \right)^{1/q} (a.e.).
\end{aligned} \tag{39}$$

By using the  $h$ -convexity of the stochastic process  $|X|^q$  and by the definition of the beta function, we have (a.e.)

$$\begin{aligned}
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-1/q} \\
& \quad \times \left( \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu (h(\vartheta) |X(r, \cdot)|^q + h(1 - \vartheta) |X(s, \cdot)|^q) d\vartheta \right)^{1/q} \\
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-1/q} \\
& \quad \times \left( |X(r, \cdot)|^q \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu h(\vartheta) d\vartheta + |X(s, \cdot)|^q \int_0^1 h(1 - \vartheta) d\vartheta \right)^{1/q} \\
& \leq (s - r)^{\mu+\nu+1} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1-(1/q)} (\beta_h(\vartheta)^{\mu, \nu} |X(r, \cdot)|^q \\
& \quad + \beta_h(1 - \vartheta)^{\mu, \nu} |X(s, \cdot)|^q)^{1/q},
\end{aligned} \tag{40}$$

which completes the proof.  $\square$

**Remark 24.** If we take  $h(\vartheta) = \vartheta$  in Theorem 23, then we obtain Theorem 3.3 of [21].

The following results are derived for strongly convex stochastic processes.

**Theorem 25.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|$  is strongly convex on  $[r, s]$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned} & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\ & \leq (s - r)^{\mu+\nu+1} (\beta(\mu + 1, \nu + 2) |X(r, \cdot)| + \beta(\mu + 2, \nu + 1) \\ & \quad \cdot |X(s, \cdot)| - c(\cdot)(r - s)^2 \beta(\mu + 2, \nu + 2)). \end{aligned} \quad (41)$$

*Proof.* From Lemma 10, the definition of the strong convexity of  $|X|$  and the beta function yield that

$$\begin{aligned} & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\ & \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta \\ & \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu (\vartheta |X(r, \cdot)| \\ & \quad + 1 - \vartheta |X(s, \cdot)| - c(\cdot)\vartheta(1 - \vartheta)(r - s)^2) d\vartheta \\ & \leq (s - r)^{\mu+\nu+1} (\beta(\mu + 1, \nu + 2) |X(r, \cdot)| \\ & \quad + \beta(\mu + 2, \nu + 1) |X(s, \cdot)| - c(\cdot)(r - s)^2 \\ & \quad \cdot \beta(\mu + 2, \nu + 2))(a.e.), \end{aligned} \quad (42)$$

which completes the proof.  $\square$

**Remark 26.** If we take  $c = 0$  in Theorem 25, then we obtain Theorem 3.1 of [21].

**Theorem 27.** Suppose that  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is strongly convex on  $[r, s]$  for  $q > 1$  with  $1/\kappa + 1/q = 1$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned} & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\ & \leq (s - r)^{\mu+\nu+1} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1/\kappa} \\ & \quad \times \left( \frac{1}{2} (|X(r, \cdot)|^q + |X(s, \cdot)|^q) - \frac{1}{6} c(\cdot)(r - s)^2 \right)^{1/q}. \end{aligned} \quad (43)$$

*Proof.* By Lemma 10 and Hölder's integral inequality, we have (a.e.)

$$\begin{aligned} & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\ & \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta \\ & \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1/\kappa} \\ & \quad \cdot \left( \int_0^1 |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \right)^{1/q}. \end{aligned} \quad (44)$$

Since  $|X|^q$  is a strongly convex stochastic process, one can yield that

$$\begin{aligned} & \int_0^1 |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \\ & \leq \int_0^1 (\vartheta |X(r, \cdot)|^q + 1 - \vartheta |X(s, \cdot)|^q - c(\cdot)\vartheta(1 - \vartheta)(r - s)^2) d\vartheta \\ & \leq \frac{1}{2} (|X(r, \cdot)|^q + |X(s, \cdot)|^q) - \frac{1}{6} (c(\cdot)(r - s)^2) (a.e.), \end{aligned} \quad (45)$$

and taking the definition of the beta function, we can write

$$\int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta = \beta(\kappa\mu + 1, \kappa\nu + 1). \quad (46)$$

Replacing (45) and (46) in (44) yields the desired inequality (43).  $\square$

**Remark 28.** If we take  $c = 0$  in Theorem 27, then we get Theorem 3.2 of [21].

**Theorem 29.** Let  $X : J \times \Omega \longrightarrow \mathbb{R}$  be a mean square continuous and mean square integrable stochastic process. If  $|X|^q$  is strongly convex on  $[r, s]$  for  $q > 1$ , where  $r, s \in J$  with  $r < s$ , and  $\mu, \nu > 0$  is taken, then the inequality holds almost everywhere:

$$\begin{aligned} & \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\ & \leq (s - r)^{\mu+\nu+1} \beta(\kappa\mu + 1, \kappa\nu + 1)^{1-(1/q)} \\ & \quad \times (\beta(\mu + 1, \nu + 2) |X(r, \cdot)|^q + \beta(\mu + 2, \nu + 1) \\ & \quad \cdot |X(s, \cdot)|^q - c(\cdot)(r - s)^2 \beta(\mu + 2, \nu + 2))^{1/q}. \end{aligned} \quad (47)$$

*Proof.* By making use of Lemma 11 and the power-mean integral inequality for  $\kappa \geq 1$ , one can yield that



$$\begin{aligned}
& \int_r^s (\omega - r)^\mu (s - \omega)^\nu X(\omega, \cdot) d\omega \\
& \leq (s - r)^{\mu+\nu+1} \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)| d\vartheta (a.e.) \\
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-(1/q)} \\
& \quad \times \left( \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu |X(\vartheta r + (1 - \vartheta)s, \cdot)|^q d\vartheta \right)^{1/q} (a.e.).
\end{aligned} \tag{48}$$

By using the strong convexity of the stochastic process  $|X|^q$  and taking the definition of the beta function, we have (a.e.)

$$\begin{aligned}
& \leq (s - r)^{\mu+\nu+1} \left( \int_0^1 (1 - \vartheta)^{\kappa\mu} \vartheta^{\kappa\nu} d\vartheta \right)^{1-(1/q)} \\
& \quad \times \left( \int_0^1 (1 - \vartheta)^\mu \vartheta^\nu (\vartheta |X(r, \cdot)|^q + (1 - \vartheta) \right. \\
& \quad \cdot |X(s, \cdot)|^q - c(\cdot) \vartheta(1 - \vartheta)(r - s)^2) d\vartheta \Big)^{1/q} \\
& \leq (s - r)^{\mu+\nu+1} (\beta(\kappa\mu + 1, \kappa\nu + 1))^{1-(1/q)} \\
& \quad \times (\beta(\mu + 1, \nu + 2) |X(r, \cdot)|^q + \beta(\mu + 2, \nu + 1) \\
& \quad \cdot |X(s, \cdot)|^q - c(\cdot)(r - s)^2 \beta(\mu + 2, \nu + 2))^{1/q},
\end{aligned} \tag{49}$$

which completes the proof.  $\square$

**Remark 30.** If we take  $c = 0$  in Theorem 29, then we obtain Theorem 3.1 of [21].

### 3. Conclusions

Stochastic processes have applications in many disciplines such as biology, chemistry, ecology, neuroscience, physics, image processing, signal processing, control theory, information theory, computer science, cryptography, and telecommunications. In this paper, we studied the generalized convex stochastic processes via a special function “beta function.” We established mean square integral inequalities for these generalized convex stochastic processes.

### 4. Future Directions

It will be interesting for researchers to work on the generalized convex stochastic processes via different fractional integral operators.

### Data Availability

All data required for this research are included within this paper.

### Conflicts of Interest

The authors declare that they do not have any competing interests.

### Authors' Contributions

All authors contributed equally in this paper.

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