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Research Article

A New Type of Sturm-Liouville Equation in the Non-Newtonian Calculus

Sertac Goktas

Department of Mathematics, Faculty of Science and Letters, Mersin University, Mersin, Turkey

Correspondence should be addressed to Sertac Goktas; srtcgoktas@gmail.com

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In mathematical physics (such as the one-dimensional time-independent Schrödinger equation), Sturm-Liouville problems occur very frequently. We construct, with a different perspective, a Sturm-Liouville problem in multiplicative calculus by some algebraic structures. Then, some asymptotic estimates for eigenfunctions of the multiplicative Sturm-Liouville problem are obtained by some techniques. Finally, some basic spectral properties of this multiplicative problem are examined in detail.

1. Introduction

In the 1960's, Grossman and Katz [1, 2] constructed a comprehensive family of calculus that includes classical calculus as well an infinite subbranches of non-Newtonian calculus. Arithmetics, a complete ordered field on $A \subset \mathbb{R}$, are of great importance in the construction of non-Newtonian calculus. The real number system is a classical arithmetic. Every arithmetic produces one generator, which is one to one on the domain and range of $A \subset \mathbb{R}$. Conversely, every generator produces one arithmetic. For instance, I, exp and $\sigma(x) = (e^x - 1)/(e^x + 1)$ are generators. So, I generates usual arithmetic, exp produces geometric arithmetic, and the function $\sigma(x)$ generates sigmoidal arithmetic mathematically describing the sigmoidal curves that occur in the study of population and biological growth.

Non-Newtonian calculus is divided into many subbranches as geometric, anageometric, biogeometric, quadratic, and harmonic calculus. Geometric calculus, which is one of these, is also defined as multiplicative calculus. Changes of arguments and values of a function are measured by differences and ratios in multiplicative calculus, respectively, while they are measured by differences in the classical case. Multiplicative calculus is especially useful in situations where products and ratios provide the natural methods of combining and comparing magnitudes. There are actually many reasons to study multiplicative calculus. It improves the work of additive calculations indirectly. Problems that are difficult to solve in the usual case can be solved with incredible ease in here.

Many events such as the levels of sound signals, the acidities of chemicals, and the magnitudes of earthquakes change exponentially. For this reason, examining these problems in nature using multiplicative calculus offers great convenience and benefits. It allows the physical properties of the events dealt with physically to be examined from different angles. The problems encountered in the study of these physical properties can be expressed with multiplicative differential equations [3–5]. It has applications in many areas required by mathematical modeling, especially in applied mathematics [6–11], engineering [3], economics [12, 13], business [14], and medicine [15] (see also [16–22]). Different alternative analyses have been developed to solve the problems that arise while working with these problems and to achieve better results in solving the problems. For example, the analytical solution of a differential equation that is very difficult in classical calculus can be obtained more easily in multiplicative calculus. Moreover, one of the importance of this theory is to find positive solutions of nonlinear differential equations. The investigation of various properties of positive solutions has an important place in the spectral theory of differential operators (see [23, 24]). This theory has few applications to spectral analysis. For this reason, we think that the results we will obtain will have very significant reflections in spectral theory and will open new fields.

The concepts and methods developed during the study of the Sturm-Liouville (SL) equation led to the development of many important directions of mathematics and physics. In kindred areas of analysis and the SL theory that studies some properties such as asymptotic behavior of eigenvalues and eigenfunctions, these are a source of new problems and ideas [25]. It is a very important equation used to explain many phenomena in nature. The one-dimensional time-independent Schrödinger equation in quantum mechanics can be given as an example of SL equation. Significant results have been obtained by many mathematicians over the years regarding the SL equation (see [25–36]). This equation has not yet been addressed in multiplicative calculus. The results we will obtain will make important contributions to mathematical physics. Therefore, we examine the multiplicative SL problem other than the Newtonian calculus. Multiplicative analysis techniques can also be applied to different operators that have a significant impact on spectral theory.

2. Preliminaries

In this section, we will express the notions and theorems in multiplicative analysis, which are extremely important in solving the problem and examining its properties. There are many other features of this new theory that are available, other than the ones below. However, expressing the properties of the multiplicative derivative and integral is especially important for the rest of our study. This derivative and integral are structurally quite different from classical derivative and integral. In fact, it makes a great difference in logic. These concepts will make a great difference in physics, biology, spectral theory, and economics.

Definition 1 (see [37]). Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}^+$. *Derivative of f is expressed by

$$f^*(x) = \lim_{h \longrightarrow 0} \left[\frac{f(x+h)}{f(x)} \right]^{1/h}, \tag{1}$$

if the above limit exists and is positive. Indeed, *derivative is also called as the multiplicative (or geometric) derivative. Moreover, f is usual differentiable at x, and then,

$$f^*(x) = e^{(\ln \circ f)'(x)}. \tag{2}$$

Theorem 2 (see [37]). *Let f, g be *differentiable and h be classical differentiable at x.*

The following equalities hold for *derivative.

$$\begin{aligned}
(cf)^*(x) &= f^*(x), \\
(fg)^*(x) &= f^*(x) g^*(x), \\
\left(\frac{f}{g}\right)^*(x) &= \frac{f^*(x)}{g^*(x)}, \\
\left(f^h\right)^*(x) &= f^*(x)^{h(x)} f(x)^{h'(x)}, \\
(f \circ h)^*(x) &= f^*(h(x))^{h'(x)}, \\
(f+g)^*(x) &= f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}},
\end{aligned} \tag{3}$$

where c is a positive constant.

Definition 3 (see [37]). Let $f \in \mathbb{R}^+$ be bounded on $[a, b]$. Consider the partition $\mathscr{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and the numbers $\xi_1, \xi_2, \cdots, \xi_n$ associated with the partition $\mathscr{P}$.f is said to be *integrable if there exists a number $\mathbf{P}$ having the following property: for every $\varepsilon > 0$, there exists a partition $\mathscr{P}_\varepsilon$ of $[a, b]$ such that $|\mathbf{P}(f, \mathscr{P}) - \mathbf{P}| < \varepsilon$ for every refinement $\mathscr{P}$ of $\mathscr{P}_\varepsilon$ independently on the selection of the numbers associated with the partition $\mathscr{P}$ where

$$\mathbf{P}(f, \mathscr{P}) = \prod_{i=1}^{n} f(\xi_i)^{(x_i - x_{i-1})}. \tag{4}$$

Then, symbol $\int_a^b f(x)^{dx}$ is called *integral of f on $[a, b]$.

Considering this definition, if $f \in \mathbb{R}^+$ is integrable on $[a, b]$, it is *integrable on $[a, b]$,

$$\int_a^b f(x)^{dx} = \exp\left\{ \int_a^b (\ln \circ f)(x) dx \right\}. \tag{5}$$

Conversely, *integrability of f on $[a, b]$ implies

$$\int_a^b f(x) dx = \ln \int_a^b \left(e^{f(x)} \right)^{dx}. \tag{6}$$

Indeed, *integral is also called as multiplicative integral.

Theorem 4 (see [37]). *Let $f, g \in \mathbb{R}^+$ be bounded and *integrable and $h \in \mathbb{R}^+$ be usual differentiable on $[a, b]$. Then, the following expression holds*

$$\begin{aligned}
&\int_a^b \left[f(x)^k\right]^{dx} = \left[\int_a^b f(x)^{dx}\right]^k,\\
&\int_a^b [f(x)g(x)]^{dx} = \int_a^b f(x)^{dx}.\int_a^b g(x)^{dx},\\
&\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}},\\
&\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx}.\int_c^b f(x)^{dx},\\
&\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}}\left\{\int_a^b \left[f(x)^{g'(x)}\right]^{dx}\right\}^{-1}.
\end{aligned} \tag{7}$$

where $k \in \mathbb{R}$ is a constant and $c \in [a, b]$. The expression v is known as **integration by parts formula.*

3. Multiplicative SL Equation

In this section, the multiplicative SL equation will be established by using some algebraic structures and the eigenfunctions of the constructed problem will be obtained.

Firstly, let us express some concepts that form the basis of the SL equation in the multiplicative case. nth-order *multiplicative linear differential expression* is in the form of

$$l(y) = \left[y^{*(n)}\right]^{s_n(x)}\left[y^{*(n-1)}\right]^{s_{n-1}(x)}\cdots y^{s_0(x)}, \tag{8}$$

where $s_n(x), s_{n-1}(x), \cdots, s_0(x)$ are the continuous exponents on $[a, b]$. Let

$$u(y) = \left[y_a^{*(n-1)}\right]^{\alpha_{n-1}}\cdots [y_a^*]^{\alpha_1} y_a^{\alpha_0}\cdot\left[y_b^{*(n-1)}\right]^{\beta_{n-1}}\cdots [y_b^*]^{\beta_1} y_b^{\beta_0} \tag{9}$$

be the linear form when y_a and y_b are the values of y at end points of $[a, b]$. If such forms $u_\nu(y)$ have been specified for $\nu = \overline{1, m}$ and the conditions $u_\nu(y) = 1$ are imposed into $y(x) \in C^{*(n)}$, it must satisfy these boundary conditions, where $C^{*(n)}$ shows the set of the functions which are nth-order multiplicative differentiable and continuous. Let us consider a certain multiplicative differential expression $l(y)$ with $u_\nu(y) = 1$ on $D \subset C^{*(n)}$. Assume that $u = l(y)$ is a function where $y(x) \in D$. This relation is denoted by L whose domain is D. The operator L is called multiplicative differential operator generated by $l(y) = \omega(x)$ and $u_\nu(y) = 1$. The problem of determination $y(x) \in C^{*(n)}$ which satisfies the conditions $l(y) = 1$ and $u_\nu(y) = 1$ is called the homogeneous multiplicative boundary value problem.

Definition 5. Let $Ly = y^\lambda$. $y \neq 1$ is called multiplicative eigenfunction (*eigenfunction) of the operator L. Here, λ is a multiplicative eigenvalue (*eigenvalue) of L. That is, the * eigenvalues of an operator L are the values of λ when the multiplicative boundary value problem

$$\begin{aligned}
l(y) &= y^\lambda,\\
u_\nu(y) = 1, &\quad \nu = \overline{1, n}
\end{aligned} \tag{10}$$

has nontrivial solutions.

We will soon construct the multiplicative SL problem. That way, let us express multiplicative algebraic structures that we will encounter while establishing and solving the multiplicative SL equation. Arithmetic operations created with exponential functions are called multiplicative algebraic operations. Let us show some properties of these operations with a multiplicative arithmetic table for $f, g \in \mathbb{R}^+$ [37].

$$\begin{aligned}
f \oplus g &= fg,\\
f \ominus g &= \frac{f}{g},\\
f \odot g &= f^{\ln g} = g^{\ln f}.
\end{aligned} \tag{11}$$

These operations create some algebraic structures. If $\oplus : A \times A \longrightarrow A$ is an operation where $A \neq \varnothing$ and $A \subset \mathbb{R}^+$, the algebraic structure $(A, \oplus)$ is called a multiplicative group. Similarly, $(A, \oplus, \odot)$ is a multiplicative ring. This situation gives us the opportunity to use these processes easily and define different structures.

Consider the following multiplicative SL equation for $x \in [a, b]$

$$L[y] = \left(e^{-1} \odot y^{**}(x)\right) \oplus \left(e^{q(x)} \odot y(x)\right) = e^\lambda \odot y(x), \tag{12}$$

with the conditions

$$\begin{aligned}
&\left(e^{\cos\alpha} \odot y(a)\right) \oplus \left(e^{\sin\alpha} \odot y^*(a)\right) = 1,\\
&\left(e^{\cos\beta} \odot y(b)\right) \oplus \left(e^{\sin\beta} \odot y^*(b)\right) = 1,
\end{aligned} \tag{13}$$

where q is real valued on $[a, b]$ and α, β arbitrary real numbers. If we expand and simplify this problem by using the properties of multiplicative calculus, the multiplicative SL problem

$$\begin{aligned}
&\left(y^{**}\right)^{-1} y^{q(x)} = y^\lambda,\\
&(y(a))^{\cos\alpha}(y^*(a))^{\sin\alpha} = 1,\\
&(y(b))^{\cos\beta}(y^*(b))^{\sin\beta} = 1
\end{aligned} \tag{14}$$

is obtained.

In usual case, (12) is equivalent to the following nonlinear equation

$$y''y - \left(y'\right)^2 + [(\lambda - q(x))\ln y]y^2 = 0. \tag{15}$$

The solutions of this nonlinear equation coincide with the solutions of multiplicative equation (12). This shows how important the multiplicative calculus is.

We assume that $a = 0, b = \pi$ throughout this study without the loss of generality. In fact, $[a, b]$ is mapped to $[0, \pi]$ by the substitution $t = (x - a)^{\log_{b-a}\pi}$.

By setting $\cot\alpha = -h$ and $\cot\beta = H$, the boundary conditions in (13) are converted to

$$\begin{aligned} &y^*(0)y^{-h}(0) = 1, \\ &y^*(\pi)y^{H}(\pi) = 1. \end{aligned} \tag{16}$$

Let us denote the solutions of (12) by $u(x,\lambda)$ and $v(x,\lambda)$ which satisfies

$$\begin{aligned} &u(0,\lambda) = e, \\ &u_x^*(0,\lambda) = e^h, \end{aligned} \tag{17}$$

$$\begin{aligned} &v(0,\lambda) = 1, \\ &v_x^*(0,\lambda) = e. \end{aligned} \tag{18}$$

In order to avoid any difficulties in expressing the main parts of the study, the multiplicative inner product will be defined and the spaces used throughout the study will be given in the multiplicative case.

Definition 6. Let $S \neq \varnothing$ and $<,>_* : S \times S \longrightarrow \mathbb{R}^+$ be a mapping such that the following axioms hold for each $x, y, z \in X$:

(i) $<f, f>_* \geq 1$

(ii) $<f, f>_* = 1$ if $f = 1$

(iii) $<f \oplus g, h>_* = <f, h>_* \oplus <g, h>_*$

(iv) $<e^\alpha \odot f, g>_* = e^\alpha \odot <f, g>_*, \alpha \in \mathbb{R}$

(v) $<f, g>_* = <g, f>_*$

Here, $(S, <,>_*)$ is called the *inner product space and $<,>_*$ is the *inner product on S.

Lemma 7. *The space $L_2^*[a, b] = \{f : \int_a^b [f(x) \odot f(x)]^{dx} < \infty\}$ is an *inner product space with*

$$<,>_* : L_2^*[a,b] \times L_2^*[a,b] \longrightarrow \mathbb{R}^+, <f, g>_* = \int_a^b [f(x) \odot g(x)]^{dx}, \tag{19}$$

where $f, g \in L_2^*[a, b]$ are positive functions.

Proof. Using the properties of the multiplicative inner product and the definition of the given space, it can be easily proved. □

Theorem 8. *Let $\lambda = \mu^2$. The asymptotic formulas of *eigenfunction of problems (12) and (13) are*

$$\begin{aligned} u(x,\lambda) &= e^{\cos\mu x + (h/\mu)\sin\mu x} \cdot \left[\int_0^x u(t,\lambda)^{q(t)\sin[\mu(x-t)]dt}\right]^{1/\mu}, \\ v(x,\lambda) &= e^{(1/\mu)\sin\mu x} \cdot \left[\int_0^x v(t,\lambda)^{q(t)\sin[\mu(x-t)]dt}\right]^{1/\mu}. \end{aligned} \tag{20}$$

Proof. The first estimate will only be proved because the other can be proved similarly. Since $u(x,\lambda)$ satisfies (12), we get

$$\int_0^x u(t,\lambda)^{q(t)\sin[\mu(x-t)]^{dt}} = \int_0^x u^{**}(t,\lambda)^{\sin[\mu(x-t)]^{dt}} \left[\int_0^x u(t,\lambda)^{\sin[\mu(x-t)]^{dt}}\right]^{\mu^2}. \tag{21}$$

If the *integration by parts method is applied to the first multiplicative integral on the right twice in a row, we get

$$\begin{aligned} \int_0^x u^{**}(t,\lambda)^{\sin[\mu(x-t)]^{dt}} &= \frac{u^\mu(x,\lambda)}{u^*(0,\lambda)^{\sin\mu x} u(0,\lambda)^{\mu\cos\mu x}} \\ &\quad \left[\int_0^x u(t,\lambda)^{\sin[\mu(x-t)]^{dt}}\right]^{-\mu^2}. \end{aligned} \tag{22}$$

Then, by considering the conditions in (17) in (21) with above relation,

$$\int_0^x u(t,\lambda)^{q(t)\sin[\mu(x-t)]^{dt}} = u^\mu(x,\lambda) e^{-(h\sin\mu x + \mu\cos\mu x)}. \tag{23}$$

It completes the proof.□ □

Remark 9. The more general 2nd-order multiplicative differential equation

$$y^{**}(y^*)^{p(x)} y^{r(x)+\lambda w(x)} = 1 \tag{24}$$

can be transformed into the following multiplicative SL equation

$$\left[(y^*)^{\mu(x)}\right]^* y^{r(x)\mu(x)+\lambda\mu(x)w(x)} = 1, \tag{25}$$

where $\mu(x) = \int (e^{p(x)})^{dx}$. If the multiplicative Liouville transform $u = y^{\int_a^x (p(t)/2)^{dt}}$ is used here, the multiplicative differential equation above turns to the following multiplicative SL equation:

$$u^{**} u^{q(x)+\lambda w(x)} = 1, \tag{26}$$

where $q(x) = r(x) - (1/4)(2p'(x) + p^2(x))$.

Example 10. Consider the nonlinear eigenvalue problem in a usual case:

$$y''y-\left(y'\right)^{2}+\lambda(\ln y)y^{2}=0,\quad 0<x<S,\\ y(0)=y(S)=1. \tag{27}$$

It is quite difficult to solve this problem in the usual case. For this reason, we will obtain the eigenvalues and eigenfunctions of the problem by using multiplicative calculus techniques. By the relation $y^{*(n)}=e^{(\ln y)^{(n)}}$, $n=1,2$, (27) turns into the multiplicative linear eigenvalue problem

$$y^{**}y^{\lambda}=1,\\ y(0)=y(S)=1. \tag{28}$$

If $\lambda\le 0$, the trivial eigenfunction $y(x,\lambda)=1$ is obtained. Suppose that $\lambda>0$. Then, the solution of (28) is

$$\lambda_n=\left(\frac{n\pi}{S}\right)^{2},\\ y_n(x)=e^{\alpha_n\sin\left(\frac{n\pi}{S}x\right)}, \tag{29}$$

where $n=1,2,3,\cdots$, and α_n are constants. This solution is also the solution to nonlinear eigenvalue problem (27). This situation shows how effective multiplicative calculus can be in mathematical physics.

Example 11. Now let us consider the periodic nonlinear eigenvalue problem in the usual case:

$$y''y-\left(y'\right)^{2}+\lambda(\ln y)y^{2}=0,\quad -S<x<S,\\ y(-S)=y(S),\\ y'(-S)=y'(S). \tag{30}$$

This problem is very important in mathematical physics and its solution is extremely difficult in the usual case. Now, let us turn this problem into an equation that is more solvable in multiplicative calculus by relation $y^{*(n)}=e^{(\ln y)^{(n)}}$, $n=1,2$:

$$y^{**}y^{\lambda}=1,\\ y(-S)=y(S),\\ y^{*}(-S)=y^{*}(S). \tag{31}$$

This problem is a multiplicative periodic linear eigenvalue problem. Here, we get $y(x,\lambda)=1$ and $y(x,\lambda)=e$ when $\lambda<0$ and $\lambda=0$, respectively. Assume that $\lambda>0$. If the similar operations to the above solution are performed and periodic conditions are taken into account, we get

$$\lambda_n=\left(\frac{n\pi}{S}\right)^{2},\\ y_n(x)=\{e\}\cup\left\{e^{\cos\left(\frac{n\pi}{S}x\right)}\right\}\cup\left\{e^{\sin\left(\frac{n\pi}{S}x\right)}\right\},\quad n=0,1,2,3,\cdots, \tag{32}$$

where a_n are constants.

Now, we will establish the above equation with new conditions and examine its solutions with another method. For this solution, the multiplicative Laplace transform will be expressed and all the necessary properties will be given. Then, the multiplicative SL problem will be solved using this multiplicative transformation. The flawless operation of this transformation in multiplicative analysis is important in terms of carrying many concepts and theorems present in the classical case to this field. We can guess from this situation that transformations used for different purposes in mathematical physics can also be carried. This is important in terms of considering many theories in mathematical physics from a different perspective and obtaining different results.

Definition 12 (see [10]). Let $f(t)\in\mathbb{R}^{+}$ on $[0,\infty)$. Multiplicative Laplace transform for f is expressed by

$$\mathscr{L}_m\{f(t)\}=F(s)=e^{\mathscr{L}\{f(t)\}}, \tag{33}$$

where $\mathscr{L}$ denotes usual Laplace transform.

Lemma 13 (see [10]). *The multiplicative Laplace transform is multiplicatively linear. Namely,*

$$\mathscr{L}_m\left\{f_1^{c_1}(t)f_2^{c_2}(t)\right\}=\mathscr{L}_m\{f_1(t)\}^{c_1}\mathscr{L}_m\{f_2(t)\}^{c_2}, \tag{34}$$

where c_1, c_2 are arbitrary exponents.

Definition 14 (see [10]). Let $f, f^{*}, f^{**},\cdots,f^{*(n-1)}$ be continuous and $f^{*(n)}$ be piece-wise continuous on $0\le t\le A$. Also, assume that there exists positive real numbers K,α, and t_0 such that

$$\left|f^{*(n-1)}(t)\right|\le Ke^{e^{\alpha t}},\quad t\ge t_0. \tag{35}$$

Furthermore, $\mathscr{L}_m\{f^{*(n)}(t)\}$ exists and can be calculated by

$$\mathscr{L}_m\left\{f^{*(n)}(t)\right\}=\frac{1}{f(0)^{s^{n-1}}f^{*}(0)^{s^{n-2}}f^{**}(0)^{s^{n-3}}\cdots f^{*(n-1)}(0)}F(s)^{s^{n}}. \tag{36}$$

Definition 15 (see [10]). Let $F(s)$ be a multiplicative Laplace transform of continuous function f, i.e., $\mathscr{L}_m\{f(t)\}=F(s)$. And $\mathscr{L}_m^{-1}\{F\}$ is called the inverse multiplicative Laplace transform. Here, we have the following relation

$$\mathscr{L}_m\left\{f(t)^{t^n}\right\}=\left(F^{*(n)}(s)\right)^{(-1)^n}. \tag{37}$$

Now, let us solve a nonlinear initial value problem in the usual case by the multiplicative Laplace transform.

Example 16. Consider the below nonlinear IVP.

$$\begin{gathered} y''y-\left(y'\right)^2+\{(\lambda-q(x))\ln y\}y^2=0,\\ y(0)=e^{\alpha},\\ y'(0)-\beta y(0)=0, \end{gathered} \tag{38}$$

where $q(x)=c$ and c is a constant. By substitution $y^{*(n)}(x)=e^{(\ln y)^{(n)}(x)}$, $n=1,2$, (38) turns into the following multiplicative IVP.

$$\begin{gathered} \left(y^{**}\right)^{-1}y^{q(x)}=y^{\lambda},\\ y(0)=e^{\alpha},\\ y^{*}(0)=e^{\beta}. \end{gathered} \tag{39}$$

If the multiplicative Laplace transform is applied to both sides of the obtained equation (8) and necessary adjustments are made, we get

$$Y(s)=e^{\frac{\alpha s+\beta}{s^2+\lambda-c}}. \tag{40}$$

Finally, using the multiplicative inverse Laplace transform, the solution of (39) is

$$y(x,\lambda)=\begin{cases} e^{\alpha\cos\left(\sqrt{\lambda-c}x\right)+\frac{\beta}{\sqrt{\lambda-c}}\sin\left(\sqrt{\lambda-c}x\right)}, & \lambda>c,\\ e^{\alpha+\beta x}, \quad \lambda=c,\\ e^{\alpha\cos h\left(\sqrt{c-\lambda}x\right)+\frac{\beta}{\sqrt{c-\lambda}}\sin h\left(\sqrt{c-\lambda}x\right)}, & \lambda=c. \end{cases} \tag{41}$$

4. Some Spectral Properties of the Multiplicative SL Problem

We examine some properties of the multiplicative SL operator as self-adjointness, orthogonality, reality, and simplicity in this section. Especially, the concepts of operator self-adjointness and simplicity of eigenvalues have a very important place in physics. The self-adjointness of an operator provides a great advantage in explaining the problem and event. In addition, the simplicity of the eigenvalues is useful in resolving complex physical structures. Orthogonality of eigenfunctions and realism of eigenvalues also have different and important meanings physically. For all these reasons, these features will be examined mathematically.

Lemma 17. *The multiplicative Sturm-Liouville operator L in (12) is formally self-adjoint on $L_2^*[0,\pi]$.*

Proof. Let us use the notion of the multiplicative inner product on $L_2^*[0,\pi]$. It gives

$$\begin{aligned}\langle Lu,v\rangle_* &=\int_0^{\pi}\left[(Lu)^{\ln v}\right]^{dx}=\int_0^{\pi}\left[\left(\left(u^{**}\right)^{-1}u^{q(x)}\right)^{\ln v}\right]^{dx}\\ &=\int_0^{\pi}\left[\left(u^{**}\right)^{-\ln v}\right]^{dx}\int_0^{\pi}\left[u^{q(x)\ln v}\right]^{dx}.\end{aligned}$$

Using *integration by the parts formula two times for the first factor of the right side,

$$\int_0^{\pi}\left[\left(u^{**}\right)^{-\ln v}\right]^{dx}=\left.\frac{u^{\ln v^*}}{\left(u^*\right)^{\ln v}}\right|_0^{\pi}\cdot\int_0^{\pi}\left[u^{-\ln v^{**}}\right]^{dx}, \tag{43}$$

where $F(x)|_\alpha^\beta=F(\beta)/F(\alpha)$. Setting this result in (42) implies that

$$\begin{aligned}\langle Lu,v\rangle_* &=W_m\{u,v\}(x)|_0^{\pi}\cdot\int_0^{\pi}\left[u^{\ln\left(\left(v^{**}\right)^{-1}v^{q(x)}\right)}\right]^{dx}\\ &=W_m\{u,v\}(x)|_0^{\pi}\cdot\int_0^{\pi}\left[u^{\ln(Lv)}\right]^{dx}\\ &=W_m\{u,v\}(x)|_0^{\pi}\cdot\langle Lu,v\rangle_*,\end{aligned} \tag{44}$$

where$W_m\{u,v\}(x)=(u\odot v^*)\ominus(v\odot u^*)$. It follows by the conditions in (13) that $W_m\{u,v\}(0)=W_m\{u,v\}(\pi)=1$. Therefore, we get

$$\langle Lu,v\rangle_*=\langle u,Lv\rangle_*. \tag{45}$$

It completes the proof of self-adjointness.□ □

Lemma 18. *The *eigenfunctions $\varphi(x,\lambda)$ and $\psi(x,\mu)$ related to distinct eigenvalues λ and μ are orthogonal, i.e.,*

$$\int_a^b\left[\varphi(x,\lambda)^{\ln\psi(x,\mu)}\right]^{dx}=1. \tag{46}$$

Proof. By the self-adjointness of the SL operator L, we get

$$1=\frac{\langle Lu,v\rangle_*}{\langle u,Lv\rangle_*}=\frac{\left\langle\varphi^{\lambda}(x,\lambda),\psi(x,\mu)\right\rangle_*}{\left\langle\varphi(x,\lambda),\psi^{\mu}(x,\mu)\right\rangle_*}=\left[\int_0^{\pi}\left[\varphi(x,\lambda)^{\ln\psi(x,\mu)}\right]^{dx}\right]^{\lambda-\mu}, \tag{47}$$

where $u(x)=\varphi(x,\lambda)$, $v(x)=\psi(x,\mu)$. Since $\lambda\neq\mu$ and the right-side multiplicative integral is positive, it gives orthogonality of the *eigenfunctions.□ □

Lemma 19. *All eigenvalues of multiplicative SL problems (12) and (13) are real.*

Proof. Let $\lambda=u+iv$ be a complex eigenvalue for the given problem. $\mu=\bar{\lambda}=u-iv$ is also an eigenvalue for (12) and (13) corresponding to $y(\bar{x},\lambda)$. By previous the lemma, we acquire

$$\int_a^b \left[y(x,\lambda)^{\ln \bar{y(x,\lambda)}} \right]^{dx} = 1. \quad (48)$$

By definition of the multiplicative integral,

$$\begin{aligned} &e^{\int_a^b \ln \bar{y(x,\lambda)} \ln y(x,\lambda) dx} = 1, \\ &\int_a^b |\ln y(x,\lambda)|^2 dx = 0 \Rightarrow y(x,\lambda) = 1. \end{aligned} \quad (49)$$

This is a contradiction. It is due to our assumption. So, the chosen eigenvalue is real. Since this eigenvalue is arbitrary, all eigenvalues of the problem are real.

Now, let us examine another important spectral property of this problem. The simplicity of eigenvalues is a very important feature in mathematical physics, and there are many proof techniques for simplicity. The algebraic multiplicity of an eigenvalue is the number of times it repeats as a root of the characteristic polynomial. If the algebraic multiplicity of an eigenvalue is 1, that eigenvalue is called a simple eigenvalue.□

□

Lemma 20. *All eigenvalues of multiplicative SL equations (12) and (13) are simple.*

Proof. Let $y_1(x,\lambda)$, $y_2(x,\lambda)$ be eigenfunctions of (12) and (13) corresponding to λ. Therefore, both of these eigenfunctions satisfy the given equation.

$$\begin{aligned} &(y_1^{**})^{-1} y_1^{q(x)} = y_1^{\lambda}, \\ &(y_2^{**})^{-1} y_2^{q(x)} = y_2^{\lambda}. \end{aligned} \quad (50)$$

After some straightforward operations,

$$(y_1^{**})^{\ln y_2} (y_2^{**})^{-\ln y_1} = 1. \quad (51)$$

By using multiplicative integral from a to x,

$$\frac{y_1(x)^{\ln y_2^*(x)}}{(y_1^*(x))^{\ln y_2(x)}} \frac{y_1^*(a)^{\ln y_2(a)}}{(y_1(a))^{\ln y_2^*(a)}} = 1. \quad (52)$$

Since $y_1(x,\lambda)$, $y_2(x,\lambda)$ satisfy the given conditions,

$$\begin{aligned} &y_1^*(a) = y_1(a)^{-\cot \alpha}, \\ &y_2^*(a) = y_2(a)^{-\cot \alpha}. \end{aligned} \quad (53)$$

If we use this result in (52), it yields $W_m\{y_1, y_2\}(a) = 1$. It gives that y_1 and y_2 are linearly dependent on $[a,b]$. It completes the proof.□ □

5. Conclusion

In this study, we have constructed the multiplicative SL problem and obtained *eigenfunctions of that problem by using some techniques. Later, this problem was investigated in terms of spectral theory in the multiplicative case. This study shows that multiplicative calculus methods can be applied to problems in spectral analysis and give solutions more effectively. This situation will make great contributions to the theory if many important theorems and problems in spectral theory are dealt with in multiplicative calculus. The foundations of multiplicative analysis in spectral theory established with this study can then be applied to different topics of mathematical physics. For example, inverse problems in spectral theory can be identified in this analysis and quality results can be obtained for applications of inverse problems in engineering and medicine. Problems that are difficult to solve in medicine and engineering and situations that cause time loss during application can be reduced by using multiplicative analysis. The current methods and techniques in medicine and engineering can be developed by multiplicative analysis. These developments can be made in many application areas other than reverse problems. Some numerical computation techniques used in spectral analysis can be reestablished in this new theory, and different evaluations can be made.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no competing interests.

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