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Research Article

On the Oscillation Criteria for Fourth-Order p-Laplacian Differential Equations with Middle Term

Dandan Yang and Chuanzhi Bai

Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

Correspondence should be addressed to Chuanzhi Bai; czbai@hytc.edu.cn

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In this paper, we study the oscillatory properties of the solutions of a class of fourth-order p-Laplacian differential equations with middle term. The new oscillation criteria obtained by using the theory of comparison with first- and second-order differential equations and a refinement of the Riccati transformations. The results in this paper improve and generalize the corresponding results in the literatures. Three examples are provided to illustrate our results.

1. Introduction

In this paper, we are concerned with the oscillation behavior of solutions of the fourth-order p-Laplacian differential equations with middle term

$$\begin{aligned}&\left(r(t)\left|z'''(t)\right|^{p_1-2}z'''(t)\right)'+p(t)\left|z'''(t)\right|^{p_1-2}z'''(t)\\&\quad+q(t)\left|x(\sigma(t))\right|^{p_2-2}x(\sigma(t))=0,\end{aligned}\tag{1}$$

where $t\geq t_0$, $p_i>1i=1,2$, are real numbers, $z(t):=x(t)+a(t)x(\tau(t))$, $r\in C^1([t_0,\infty))$, $p,q,a\in C([t_0,\infty))$, $r(t)>0$, $q(t)>0$, $r'(t)\geq 0$, $0\leq a(t)\leq a_0<\infty$, $\tau\in C^1[t_0,\infty)$, $\sigma\in C[t_0,\infty)$, $\tau'(t)>0$, $\tau(t)\leq t$, and $\lim_{t\longrightarrow\infty}\tau(t)=\lim_{t\longrightarrow\infty}\sigma(t)=\infty$. Furthermore, we investigate (1) under the condition.

$$(H_1)\lim_{t\longrightarrow\infty}\int_{t_0}^{t}\left[\frac{1}{r(s)}\exp\left(-\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\right]^{1/(p_1-1)}ds=\infty.\tag{2}$$

A function $z\in C'''[t_x,\infty)$, $t_x\geq t_0$, is called a solution of equation (1), if $r(t)|z'''(t)|^{p_1-2}z'''(t)\in C^1([t_x,\infty)$, and x satisfies (1) on $[t_x,\infty)$.

Since the fourth-order delay differential equations described many real-life applications, such as models related to physical, chemical, and biological phenomena, in the last decades, a lot of research has been done on the oscillatory behavior of fourth-order delay differential equations, see [1–14] and the references cited therein. On the other hand, the p-Laplacian differential equations there are some important applications in continuum mechanics and elasticity theory [15–23], the oscillatory behavior of the solutions of fourth-order differential equations with p-Laplacian like operator have been investigated in recent years by using different methods and various techniques, for example, [24–26].

In this paper, motivated by [26–28], we will give some new sufficient conditions for that oscillatory behavior of (1). In Section 2, we will provide some lemmas that will help us to prove our main results. In Section 3, based on the comparison with first- and second-order differential equations and a refinement of the Riccati transformations, we establish some new oscillation criteria of (1).

2. Preliminaries

First, we give the following lemmas that can discuss our main results.

Lemma 1 ([1], Lemma 2.21). *Let $x(t)$ be a positive and n-times differentiable function on an $[T,\infty)$ with its nth derivative $x^{(n)}(t)$ nonpositive on $[T,\infty)$ and not identically zero on*

any interval of the form $[T', \infty)$, $T' \geq T$. *Then, there exists an integer* l, $0 \leq l \leq n-1$, *with* $n+l$ *odd such that for some large* $T^* \geq T$,

$$\begin{aligned} &(-1)^{l+j}x^j > 0 \text{ on } [T^*,\infty)(j = l, l+1, \cdots, n-1), \\ &x^{(i)} > 0 \text{ on } [T^*,\infty)(i = 1, 2, \cdots, l-1) \text{ when } l > 1. \end{aligned} \tag{3}$$

Lemma 2 ([29]). *Let the function* x *satisfies* $x^{(i)}(t) > 0$, $i = 0, 1, \cdots, n$, *and* $x^{(n+1)}(t) < 0$ *eventually. Then,*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}. \tag{4}$$

Lemma 3 ([30], Lemma 2.2.3). *Let* $h \in C^n([T,\infty))$ *and* $h(t) > 0$. *Assume that* $h^{(n)}(t)$ *is of a fixed sign, on* $[t_0, \infty)$, $h^{(n)}(t)$ *not identically zero and that there exists a* $t_1 \geq t_0$ *such that, for all* $t \geq t_1$, $h^{(n-1)}(t)h^{(n)}(t) \leq 0$. *If we have* $\lim_{t\longrightarrow\infty} h(t) \neq 0$, *then there exists* $t_\lambda \geq t_0$ *such that*

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| h^{(n-1)}(t) \right|, \tag{5}$$

for every $\lambda \in (0, 1)$ *and* $t \geq t_\lambda$.

Lemma 4 ([31], Lemma 2.3). *Assume that* α *is a quotient of odd positive integers;* $V > 0$ *and* $U \in \mathbb{R}$ *are constants. Then*

$$Uy - Vy^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} U^{\alpha+1} V^{-\alpha}. \tag{6}$$

The following lemma will be used in the proof of our main results in the next section.

Lemma 5. *Let* $x(t)$ *be an eventually positive solution of (1); if* (H_1) *holds, then* $z' > 0$ *and* $z''' > 0$.

Proof. Since $x(t)$ is an eventually positive solution of (1), then $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Thus, $z(t) > 0$, $t \geq t_1$. From (1), we obtain

$$\begin{aligned} &\left(r(t)\left|z'''(t)\right|^{p_1-2}z'''(t)\right)' + p(t)\left|z'''(t)\right|^{p_1-2}z'''(t) \\ &\quad = -q(t)|z(\sigma(t))|^{p_2-1}z(\sigma(t)) < 0. \end{aligned} \tag{7}$$

Multiplying by $\exp\left(\int_{t_0}^t (p(s)/r(s))ds\right)$ on both sides of the above equation, we get

$$\begin{aligned} &\frac{d}{dt}\left[\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}ds\right) r(t)\left|z'''(t)\right|^{p_1-2}z'''(t)\right] \\ &\quad = -q(t)\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}ds\right)|z(\sigma(t))|^{p_2-1}z(\sigma(t)) < 0. \end{aligned} \tag{8}$$

Thus

$$\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}ds\right) r(t)\left|z'''(t)\right|^{p_1-2}z'''(t), \tag{9}$$

is decreasing, and hence, $z'''(t)$ is eventually of one sign. Hence, we assert that $z'''(t) > 0$ for any $t \geq t_1$. Otherwise, if $z'''(t) < 0$ for any $t \geq t_1$, we get by (8) that

$$\begin{aligned} &\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}ds\right) r(t)\left|z'''(t)\right|^{p_1-2}z'''(t) \\ &\quad \leq \exp\left(\int_{t_0}^{t_1} \frac{p(s)}{r(s)}ds\right) r(t_1)\left|z'''(t_1)\right|^{p_1-2}z'''(t_1) \\ &\quad = -M^{p_1-1}\exp\left(\int_{t_0}^{t_1} \frac{p(s)}{r(s)}ds\right), \end{aligned} \tag{10}$$

where $M = r(t_1)^{1/p_1-1}|z'''(t_1)| > 0$. From (10), one has

$$\left(-z'''(t)\right)^{p_1-1} \geq \frac{M^{p_1-1}}{r(t)}\exp\left(-\int_{t_1}^t \frac{p(s)}{r(s)}ds\right), \tag{11}$$

that is,

$$z'''(t) \leq -M\left[\frac{1}{r(t)}\exp\left(-\int_{t_1}^t \frac{p(s)}{r(s)}ds\right)\right]^{1/(p_1-1)}. \tag{12}$$

Consequently,

$$\begin{aligned} z''(t) \leq z''(t_1) - M\int_{t_1}^t &\left[\frac{1}{r(s)}\exp\right. \\ &\left.\cdot\left(-\int_{t_1}^s \frac{p(\xi)}{r(\xi)}d\xi\right)\right]^{1/(p_1-1)} ds \longrightarrow -\infty, \text{as } t \longrightarrow \infty. \end{aligned} \tag{13}$$

So $z(t)$ is an eventually negative function which contradicts $z(t) > 0$. Thus, we have $z'''(t) > 0$. From (1), we get

$$\left(r(t)z'''(t)^{p_1-1}\right)' = r'(t)z'''(t)^{p_1-1} + (p_1-1)r(t)z'''(t)^{p_1-2}z^{(4)}(t) \leq 0, \tag{14}$$

from which it follows that

$$z^{(4)}(t) \leq 0, t \geq t_1 \geq t_0. \tag{15}$$

By (15) and Lemma 1 (set $n = 4$ and $l = 3$), one has that $z'(t) > 0$, $t \geq t_1 \geq t_0$. The proof is completed.

Lemma 6 ([32]). *Let* $\alpha \in (0, 1]$ *be a quotient of two positive integers. Assume that* R *is a positive continuous function on* (t_0, ∞). *If*

$$\liminf_{t\longrightarrow\infty}\int_{\sigma(t)}^{t} R(s)ds > \frac{1}{e}, \tag{16}$$

then the first-order delay differential equation

$$y'(t) + R(t)y^{\alpha}(\sigma(t)) = 0, \tag{17}$$

is oscillatory.

3. Main Results

In the following theorem, we then by using a comparison strategy involving first-order differential equations to provide an oscillation criterion for equation (1).

For convenience, let

$$A(t) = \frac{1}{a(\tau^{-1}(t))}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{3}}{\left(\tau^{-1}(t)\right)^{3} a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right), t \geq t_0. \tag{18}$$

Theorem 7. *Assume that (H_1) and*

$$(H_2)\ \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{3}}{\left(\tau^{-1}(t)\right)^{3} a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)} < 1, t \geq t_0, \tag{19}$$

hold. If the differential equation

$$\omega'(t) + \left(\frac{\mu}{6}\right)^{p_2-1} \frac{q(t)t^{3(p_2-1)} \exp\left(\int_{t_0}^{t}(p(s)/r(s))ds\right) A^{p_2-1}(\sigma(t))}{r^{(p_2-1)/(p_1-1)}(\tau^{-1}(\sigma(t))) \exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\tau^{-1}(\sigma(t))}(p(s)/r(s))ds\right)} \omega^{(p_2-1)/(p_1-1)}\left(\tau^{-1}(\sigma(t))\right) = 0, \tag{20}$$

is oscillatory for some $\mu \in (0, 1)$, then the differential equation (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we may let x be an eventually positive solution of (1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Let

$$\omega(t) := r(t) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\right) z'''(t)^{p_1-1} > 0 \text{ [from Lemma 5]}, \tag{21}$$

which having in mind (1) gives

$$\omega'(t) + q(t) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\right) x(\sigma(t))^{p_2-1} = 0. \tag{22}$$

From the definition of $z(t)$, one has

$$x(t) = \frac{1}{a(\tau^{-1}(t))}\left(z\left(\tau^{-1}(t)\right) - x\left(\tau^{-1}(t)\right)\right). \tag{23}$$

By repeating the same process, we have

$$\begin{aligned} x(t) &= \frac{z\left(\tau^{-1}(t)\right)}{a(\tau^{-1}(t))} - \frac{1}{a(\tau^{-1}(t))}\left(\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a(\tau^{-1}(\tau^{-1}(t)))} - \frac{x\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\geq \frac{z\left(\tau^{-1}(t)\right)}{a(\tau^{-1}(t))} - \frac{1}{a(\tau^{-1}(t))}\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{24}$$

Set $n = 3$ in Lemma 2, we obtain $z(t) \geq (1/3)tz'(t)$, which implies that $z(t)/t^3$ is nonincreasing. Moreover, by the fact $\tau(t) \leq t$ that gives

$$\left(\tau^{-1}(t)\right)^{3} z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{3} z\left(\tau^{-1}(t)\right). \tag{25}$$

Combining (24) and (25), which yields

$$\begin{aligned} x(t) &\geq \frac{1}{a(\tau^{-1}(t))}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{3}}{\left(\tau^{-1}(t)\right)^{3} a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) z\left(\tau^{-1}(t)\right) \\ &= A(t) z\left(\tau^{-1}(t)\right). \end{aligned} \tag{26}$$

Between equations (1) and (26), we obtain

$$\omega'(t) + q(t) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\right) A^{p_2-1}(\sigma(t)) z^{p_2-1}\left(\tau^{-1}(\sigma(t))\right) \leq 0. \tag{27}$$

Since z is positive and increasing (by Lemma 5), we have $\lim_{t\longrightarrow\infty} z(t)=0$. So, from Lemma 3, one has

$$z(t) \geq \frac{\mu}{6} t^3 z'''(t), \tag{28}$$

for some $\mu \in (0, 1)$. It follows between (27) and (28) that, for all $\mu \in (0, 1)$, ω is a positive solution of the first-order delay differential inequality

$$\omega'(t)+\left(\frac{\mu}{6}\right)^{p_2-1}\frac{q(t)t^{3(p_2-1)}\exp\left(\int_{t_0}^{t}(p(s)/r(s))ds\right)A^{p_2-1}(\sigma(t))}{r^{(p_2-1)/(p_1-1)}(\tau^{-1}(\sigma(t)))\exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\tau^{-1}(\sigma(t))}(p(s)/r(s))ds\right)}\omega^{(p_2-1)/(p_1-1)}\left(\tau^{-1}(\sigma(t))\right)\leq 0. \tag{29}$$

It is well known (see [33] and Theorem 7) that the corresponding equation (20) also has a positive solution, which is a contradiction. The theorem is proved.

Corollary 8. *Assume that (H_1) and (H_2) hold, and $p_1 \geq p_2$. If*

$$\liminf_{t\longrightarrow\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\left(\frac{\mu}{6}\right)^{p_2-1}\frac{q(s)s^{3(p_2-1)}\exp\left(\int_{t_0}^{s}(p(\xi)/r(\xi))d\xi\right)A^{p_2-1}(\sigma(s))}{r^{(p_2-1)/(p_1-1)}(\tau^{-1}(\sigma(s)))\exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\tau^{-1}(\sigma(s))}(p(\xi)/r(\xi))d\xi\right)}ds>\frac{1}{e}, \tag{30}$$

for some $\mu\in(0,1)$, then the differential equation (1) is oscillatory.

Proof. From Lemma 6, we know that (30) implies the oscillatory of (20).

Theorem 9. *Assume that (H_1) and (H_3) $a\in C([t_0,\infty))$, $0\leq a(t)<a_0<1$ hold. If the differential equation*

$$\omega'(t)+\left(\frac{\mu}{6}\right)^{p_2-1}\frac{q(t)(\sigma(t))^{3(p_2-1)}\exp\left(\int_{t_0}^{t}(p(s)/r(s))ds\right)(1-a(\sigma(t)))^{p_2-1}}{r^{(p_2-1)/(p_1-1)}(\sigma(t))\exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\sigma(t)}(p(s)/r(s))ds\right)}\omega^{(p_2-1)/(p_1-1)}(\sigma(t))=0, \tag{31}$$

is oscillatory, then every solution of equation (1) is oscillatory.

Proof. Assume the contrary that x is an eventually positive solution of equation (1). Thus, we may suppose that $x(t)$, $x(\tau(t))$, and $x(\sigma(t))$ are positive for all $t\geq t_1$ that are sufficiently large. From $z(t)=x(t)+a(t)x(\tau(t))$ and $z'(t)>0$ (by Lemma 5), we obtain by $x(t)\leq z(t)$ and $\tau(t)\leq t$ that

$$\begin{aligned}x(t)&=z(t)-a(t)x(\tau(t))\geq z(t)-a(t)z(\tau(t))\\&\geq(1-a(t))z(t).\end{aligned} \tag{32}$$

Combining (21) and (32), one has

$$\omega'(t)+q(t)\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)(1-a(\sigma(t)))^{p_2-1}z(\sigma(t))^{p_2-1}\leq 0. \tag{33}$$

By using (28) and (33), we get that

$$\begin{aligned}\omega'(t)+\left(\frac{\mu}{6}\right)^{p_2-1}&q(t)(\sigma(t))^{3(p_2-1)}\exp\\&\cdot\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)(1-a(\sigma(t)))^{p_2-1}\left(z'''(\sigma(t))\right)^{p_2-1}\leq 0.\end{aligned} \tag{34}$$

From (21), (34) implies that

$$\omega'(t)+\left(\frac{\mu}{6}\right)^{p_2-1}\frac{q(t)(\sigma(t))^{3(p_2-1)}\exp\left(\int_{t_0}^{t}(p(s)/r(s))ds\right)(1-a(\sigma(t)))^{p_2-1}}{r^{(p_2-1)/(p_1-1)}(\sigma(t))\exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\sigma(t)}(p(s)/r(s))ds\right)}\omega^{(p_2-1)/(p_1-1)}(\sigma(t))\leq 0. \tag{35}$$

It is well known (see [33] and Theorem 7) that the corresponding equation (31) also has a positive solution, which is a contradiction. The theorem is proved.

Corollary 10. *Assume that (H_1) and (H_3) hold, and $p_1 \geq p_2$. If*

$$\lim_{t\longrightarrow\infty}\inf\int_{\sigma(t)}^{t}\left(\frac{\mu}{6}\right)^{p_2-1}\frac{q(s)(\sigma(s))^{3(p_2-1)}\exp\left(\int_{t_0}^{s}p(\xi)/r(\xi)d\xi\right)(1-a(\sigma(s)))^{p_2-1}}{r^{(p_2-1)/(p_1-1)}(\sigma(s))\exp\left((p_2-1)/(p_1-1)\int_{t_0}^{\sigma(s)}(p(\xi)/r(\xi))d\xi\right)}ds > \frac{1}{e}, \tag{36}$$

for some $\mu \in (0,1)$, then the differential equation (1) is oscillatory.

Proof. From Lemma 6, we know that (36) implies the oscillatory of (31).

Lemma 11. *Assume that (H_1) holds, x is an eventually positive solution of (1), and*

$$\int_{t_2}^{\infty}\left(h(s)q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\frac{\sigma(s)^{3(p_1-1)}}{s^{3(p_1-1)}} - \frac{2^{p_1-1}}{p_1^{p_1}}\frac{r(s)\left(h'(s)\right)^{p_1}}{(\varepsilon_1 s^2 h(s))^{p_1-1}\exp\left(-\int_{t_0}^{s}(p(\xi)/r(\xi))d\xi\right)}\right)ds = \infty, \tag{37}$$

for some constants $\varepsilon_1 \in (0,1)$. Then, we have $z'' < 0$.

Proof. Our proof by reduction to the absurd. Assume that $z''(t) > 0$. From Lemma 2, we obtain

$$\frac{z'(t)}{z(t)} \leq \frac{3}{t}. \tag{38}$$

Integrating the above equality from $\sigma(t)$ to t, one find that

$$\frac{z(\sigma(t))}{z(t)} \geq \frac{(\sigma(t))^3}{t^3}. \tag{39}$$

Let $h(t) = z'(t)$ in Lemma 3, then

$$z'(t) \geq \frac{\varepsilon_1}{2}t^2 z'''(t), \tag{40}$$

for all $\varepsilon_1 \in (0,1)$ and every sufficiently large t. Now, we define a function ϕ by

$$\phi(t) := h(t)\frac{\exp\left(\int_{t_0}^{t}(p(s)/r(s))ds\right)r(t)\left(z'''(t)\right)^{p_1-1}}{z^{p_1-1}(t)} > 0. \tag{41}$$

By differentiating (41) and using the inequalities (39) and (40), we get

$$\begin{aligned}\phi'(t) \leq{}& \frac{h'(t)}{h(t)}\phi(t) - h(t)q(t)\exp \\ &\cdot\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)\frac{\sigma(t)^{3(p_1-1)}}{t^{3(p_1-1)}}z^{p_2-p_1}(\sigma(t)) \\ &- \frac{(p_1-1)\varepsilon_1}{2}\frac{t^2\exp\left(-1/(p_1-1)\int_{t_0}^{t}(p(s)/r(s))ds\right)}{h^{1/(p_1-1)}(t)r^{1/(p_1-1)}(t)}\phi^{1+1/(p_1-1)}.\end{aligned} \tag{42}$$

Since $z'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that $z(t) > M$, for all $t \geq t_2$. Without loss of generality, we may let $M \geq 1$. By using Lemma 4 with

$$\begin{aligned}U &= \frac{h'}{h}, \\ V &= \frac{(p_1-1)\varepsilon_1}{2}\frac{t^2\exp\left(-(1/(p_1-1))\int_{t_0}^{t}(p(s)/r(s))ds\right)}{h^{1/(p_1-1)}(t)r^{1/(p_1-1)}(t)},\end{aligned} \tag{43}$$

we obtain

$$\begin{aligned}\phi'(t) \leq{}& -M^{p_2-p_1}h(t)q(t)\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)\frac{\sigma(t)^{3(p_1-1)}}{t^{3(p_1-1)}} \\ &+ \frac{2^{p_1-1}}{p_1^{p_1}}\frac{r(t)\left(h'(t)\right)^{p_1}}{(\varepsilon_1 t^2 h(t))^{p_1-1}\exp\left(-\int_{t_0}^{t}(p(s)/r(s))ds\right)}.\end{aligned} \tag{44}$$

This implies that

$$\begin{aligned}&\int_{t_2}^{t}\left(h(s)q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\frac{\sigma(s)^{3(p_1-1)}}{s^{3(p_1-1)}}\right.\\&\left.-\frac{2^{p_1-1}}{p_1^{p_1}}\frac{r(s)\left(h'(s)\right)^{p_1}}{(\varepsilon_1 s^2 h(s))^{p_1-1}\exp\left(-\int_{t_0}^{s}(p(\xi)/r(\xi))d\xi\right)}\right)ds\le\phi(t_2),\\&\le\int_{t_2}^{t}\left(M^{p_2-p_1}h(s)q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\frac{\sigma(s)^{3(p_1-1)}}{s^{3(p_1-1)}}\right.\\&\left.-\frac{2^{p_1-1}}{p_1^{p_1}}\frac{r(s)\left(h'(s)\right)^{p_1}}{(\varepsilon_1 s^2 h(s))^{p_1-1}\exp\left(-\int_{t_0}^{s}(p(\xi)/r(\xi))d\xi\right)}\right)ds\le\phi(t_2),\end{aligned}\tag{45}$$

which contradicts (37). The proof is completed.

For convenience, let

$$\begin{aligned}\theta(t)=\lambda^{(p_2-1)/(p_1-1)}\int_t^{\infty}&\left(\frac{1}{r(\xi)}\int_{\xi}^{\infty}q(s)\exp\right.\\&\left.\cdot\left(\int_t^s\frac{p(\xi)}{r(\xi)}d\xi\right)\left(\frac{\sigma(s)}{s}\right)^{p_2-1}ds\right)^{1/(p_1-1)}d\xi.\end{aligned}\tag{46}$$

Theorem 12. *Assume that (H_1) and (37) hold for some $\varepsilon_1\in(0,1)$ and $p_2\ge p_1$. If*

$$y''(t)+\theta(t)y(t)=0,\tag{47}$$

is oscillatory, then (1) is also oscillatory.

Proof. We use the reduction to the absurd arguments. Assume that (1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we only need to be concerned with positive solutions of equation (1). Then, there exists a $t_1\ge t_0$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t\ge t_1$. From Lemmas 4 and 11, one has that

$$\begin{aligned}z'(t)&>0,\\z''(t)&<0,\\z'''(t)&>0,\end{aligned}\tag{48}$$

for $t\ge t_2$, where t_2 is sufficiently large. Integrating (8) from t to ρ, we obtain

$$\begin{aligned}&\exp\left(\int_{t_0}^{\rho}\frac{p(s)}{r(s)}ds\right)r(\rho)\left(z'''(\rho)\right)^{p_1-1}\\&=\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)r(t)\left(z'''(t)\right)^{p_1-1}\\&\quad-\int_t^{\rho}q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)z^{p_2-1}(\sigma(s))ds.\end{aligned}\tag{49}$$

By using Lemma 3 in [26] together with (48), we have

$$\frac{z(\sigma(t))}{z(t)}\ge\lambda\frac{\sigma(t)}{t},\tag{50}$$

for all $\lambda\in(0,1)$. This coupled with (49); we can arrive at

$$\begin{aligned}&\exp\left(\int_{t_0}^{\rho}\frac{p(s)}{r(s)}ds\right)r(\rho)\left(z'''(\rho)\right)^{p_1-1}\\&-\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)r(t)\left(z'''(t)\right)^{p_1-1}\\&+\lambda^{p_2-1}\int_t^{\rho}q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\left(\frac{\sigma(s)}{s}\right)^{p_2-1}z^{p_2-1}(s)ds\le0.\end{aligned}\tag{51}$$

Since $z'>0$, then (51) reduced to the following inequality

$$\begin{aligned}&\exp\left(\int_{t_0}^{\rho}\frac{p(s)}{r(s)}ds\right)r(\rho)\left(z'''(\rho)\right)^{p_1-1}\\&-\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)r(t)\left(z'''(t)\right)^{p_1-1}\\&+\lambda^{p_2-1}z^{p_2-1}(t)\int_t^{\rho}q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\left(\frac{\sigma(s)}{s}\right)^{p_2-1}ds\le0.\end{aligned}\tag{52}$$

Taking $\rho\longrightarrow\infty$ in (52), one has

$$\begin{aligned}&-\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)r(t)\left(z'''(t)\right)^{p_1-1}\\&+\lambda^{p_2-1}z^{p_2-1}(t)\int_t^{\infty}q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\left(\frac{\sigma(s)}{s}\right)^{p_2-1}ds\le0,\end{aligned}\tag{53}$$

from which we readily obtain

$$\begin{aligned}z'''(t)\ge&\frac{\lambda^{(p_2-1)/(p_1-1)}}{r^{1/(p_1-1)}(t)}\exp\left(-\frac{1}{p_1-1}\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)z^{(p_2-1)/(p_1-1)}(t)\\&\times\left(\int_t^{\infty}q(s)\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)\left(\frac{\sigma(s)}{s}\right)^{p_2-1}ds\right)^{1/(p_1-1)}.\end{aligned}\tag{54}$$

Integrating the above inequality from t to ∞, we obtain

$$\begin{aligned}-z''(t)\ge&\lambda^{(p_2-1)/(p_1-1)}z^{(p_2-1)/(p_1-1)}(t)\\&\times\int_t^{\infty}\left(\frac{1}{r(\xi)}\int_{\xi}^{\infty}q(s)\exp\left(\int_t^s\frac{p(\xi)}{r(\xi)}d\xi\right)\right.\\&\left.\cdot\left(\frac{\sigma(s)}{s}\right)^{p_2-1}ds\right)^{1/(p_1-1)}d\xi,\end{aligned}\tag{55}$$

which implies that

$$z''(t) \le -\theta(t) z^{(p_2-1)/(p_1-1)}(t). \tag{56}$$

Now, if we define $\gamma(t) = z'(t)/z(t)$, then $\gamma(t) > 0$ for $t \ge t_1$, and

$$\gamma'(t) = \frac{z''(t)}{z(t)} - \left(\frac{z'(t)}{z(t)}\right)^2. \tag{57}$$

By using (56) and the definition of $\gamma(t)$, we see that

$$\gamma'(t) \le -\theta(t) \frac{z^{(p_2-1)/(p_1-1)}(t)}{z(t)} - \gamma^2(t). \tag{58}$$

Since $z'(t) > 0$, there exists a constant $M > 0$ such that $z(t) \ge M^{p_2-1}$, for all $t \ge t_2$, where t_2 is sufficiently large. Without loss of generality, we may let $M \ge 1$. Then, by (56), one has

$$\gamma'(t) + \gamma^2(t) + \theta(t) \le \gamma'(t) + \gamma^2(t) + M^{p_2-p_1}\theta(t) \le 0. \tag{59}$$

It is well known (see [34]) that the differential equation (47) is oscillatory if and only if there exists a $t_3 > \max\{t_1, t_2\}$ such that (59) holds, which is a contradiction. The theorem is proved.

For convenience, let

$$\beta(t) = \int_t^\infty \left(\frac{1}{r(\xi)} \int_\xi^\infty q(s) \exp\left(\int_t^s \frac{p(\xi)}{r(\xi)} d\xi\right) ds\right)^{1/(p_1-1)} d\xi. \tag{60}$$

Theorem 13. *Assume that (H_1) and (37) hold for some $\varepsilon_1 \in (0, 1)$, $p_2 \ge p_1$, and $\sigma'(t) \ge 1$. If*

$$\left(\frac{1}{\sigma'(t)} y'(t)\right)' + \beta(t) y(t) = 0, \tag{61}$$

is oscillatory, then (1) is also oscillatory.

Proof. As in the proof of Theorem 12, one has (49). Hence, it follows between $\sigma'(t) \ge 0$ and $z'(t) \ge 0$ that

$$\begin{aligned} &\exp\left(\int_{t_0}^\rho \frac{p(s)}{r(s)} ds\right) r(\rho) \left(z'''(\rho)\right)^{p_1-1} \\ &- \exp\left(\int_{t_0}^t \frac{p(s)}{r(s)} ds\right) r(t) \left(z'''(t)\right)^{p_1-1} \\ &+ z^{p_2-1}(\sigma(t)) \int_t^\rho q(s) \exp\left(\int_t^s \frac{p(\xi)}{r(\xi)} d\xi\right) ds \le 0. \end{aligned} \tag{62}$$

By (62), similar to (56), we obtain

$$z''(t) \le -\beta(t) z^{(p_2-1)/(p_1-1)}(\sigma(t)). \tag{63}$$

We now define κ by

$$\kappa(t) = \frac{z'(t)}{z(\sigma(t))}, \tag{64}$$

then $\kappa(t) > 0$ for $t \ge t_1$, and

$$\begin{aligned} \kappa'(t) &= \frac{z''(t)}{z(\sigma(t))} - \frac{z'(t)}{z^2(\sigma(t))} z'(\sigma(t))\sigma'(t) \\ &\le \frac{z''(t)}{z(\sigma(t))} - \left(\frac{z'(t)}{z(\sigma(t))}\right)^2 \sigma'(t). \end{aligned} \tag{65}$$

By using (63) and the definition of $\kappa(t)$, similar to (59), we see that

$$\kappa'(t) + \beta(t) + \sigma'(t)\kappa^2(t) \le 0. \tag{66}$$

It is well known (see [34]) that the differential equation in (61) is nonoscillatory if and only if there exists a $t_3 > \max\{t_1, t_2\}$ such that (66) holds, which is a contradiction. So, the theorem is proved.

In the following, we employ the integral averaging technique to establish a Philos-type oscillation criterion for (1).

Let

$$\begin{aligned} D &= \{(t, s) \in R^2 : t \ge s \ge t_0\}, \\ D_0 &= \{(t, s) \in R^2 : t > s \ge t_0\}. \end{aligned} \tag{67}$$

Corollary 14. *Assume that (H_1) and (37) hold for some $\varepsilon_1 \in (0, 1)$ and $p_1 = p_2$. Let there exists a continuous function $H : D \longrightarrow R$ such that $H(t, s) = 0$ for $t \ge t_0$, $H(t, s) > 0$, $(t, s) \in D_0$, and H has a nonpositive continuous partial derivative with respect to the second variable in D. Suppose that there exists function $h \in C(D, R)$ such that*

$$-\frac{\partial}{\partial s} H(t, s) = h(t, s)\sqrt{H(t, s)} \text{ for all } (t, s) \in D. \tag{68}$$

If

$$\limsup_{t \longrightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s)\theta(s) - \frac{1}{4} h^2(t, s)\right] ds = \infty, \tag{69}$$

or

$$\liminf_{t \longrightarrow \infty} t \int_t^\infty \theta(s) ds > \frac{1}{4}, \tag{70}$$

then (1) is oscillatory.

Example 15. Consider the equation

$$z^{(4)}(t) + \frac{1}{t} z'''(t) + \frac{q_0}{t^4} x(\sigma_0 t) = 0, \tag{71}$$

where $t \geq 1$, $z(t) = x(t) + a_0 x(\tau_0 t)$, $q_0 > 0$, $\tau_0 \in (a_0^{-1/3}, 1)$, and $\sigma_0 \in (0, \tau_0)$. Let $p_1 = p_2 = 2$, $r(t) = 1$, $a(t) = a_0$, $\tau(t) = \tau_0 t$, $\sigma(t) = \sigma_0 t$, $p(t) = 1/t$, and $q(t) = q_0/t^4$. It is easy to check that (H_1) and (H_2) are satisfied. Moreover, we have

$$A(t) = \frac{1}{a_0}\left(1 - \frac{1}{a_0 \tau_0^3}\right). \tag{72}$$

If $q_0 > 0$ satisfies inequality (30), that is

$$\begin{aligned} &\liminf_{t\longrightarrow\infty} \int_{\sigma_0/\tau_0 t}^{t} \frac{\mu}{6}\frac{q_0}{s^4} s^3 A(\sigma_0 s) \exp\left(\int_{\sigma_0/\tau_0 s}^{s} \frac{1}{\xi} d\xi\right) ds \\ &= q_0 \frac{\mu}{6}\frac{1}{a_0}\left(1 - \frac{1}{a_0\tau_0^3}\right)\frac{\tau_0}{\sigma_0} \ln \frac{\tau_0}{\sigma_0} > \frac{1}{e}, \end{aligned} \tag{73}$$

then equation (71) is oscillatory by Corollary 8.

Example 16. Consider the equation

$$\left(t \mid z'''(t) \mid z'''(t)\right)' + \left|z'''(t)\right| z'''(t) + \frac{q_0}{t^4}\left|x\left(\frac{t}{2}\right)\right| x\left(\frac{t}{2}\right) = 0, \tag{74}$$

where $t \geq t_0 = 1$, $z(t) = x(t) + (1/3)x(t/4)$, and $q_0 > 0$. Let $p_1 = 3$, $p_2 = 2$, $r(t) = t$, $\tau(t) = t/4$, $\sigma(t) = t/2$, $a(t) = 1/3$, $p(t) = 1$, and $q(t) = q_0/t^4$. It is easy to check that (H_1) and (H_3) are satisfied. Moreover, we find

$$\begin{aligned} &\liminf_{t\longrightarrow\infty} \int_{\sigma(t)}^{t} \left(\frac{\mu}{6}\right)^{p_2-1} \frac{q(s)(\sigma(s))^{3(p_2-1)} \exp\left(\int_{t_0}^{s} (p(\xi)/r(\xi)) d\xi\right)(1 - a(\sigma(s)))^{p_2-1}}{r^{(p_2-1)/(p_1-1)}(\sigma(s)) \exp\left(((p_2-1)/(p_1-1))\int_{t_0}^{\sigma(s)} (p(\xi)/r(\xi)) d\xi\right)} ds \\ &= \liminf_{t\longrightarrow\infty} \int_{t/2}^{t} \frac{\mu}{6} \frac{\left(q_0/s^4\right)(s/2)^3 (2/3)s}{(s/2)^{1/2} \exp\left((1/2)\int_1^{s/2} (1/\xi) d\xi\right)} ds = \frac{\mu q_0}{36} \ln 2. \end{aligned} \tag{75}$$

Thus, by Corollary 10, all solutions of equation (74) are oscillatory if $q_0 > 36/\mu e \ln 2$.

Example 17. Consider the equation

$$\left(t^2 \left|z'''(t)\right|^2 z'''(t)\right)' + t\left|z'''(t)\right|^2 z'''(t) + \frac{q_0}{t^7}|x(ct)|^2 x(ct) = 0, \tag{76}$$

where $t \geq t_0 = 1$, $z(t) = x(t) + a_0 x(\tau_0 t)$, $q_0 > 0$, and $c \in (0, 1)$. Let $p_1 = p_2 = 4$, $r(t) = t^2$, $\sigma(t) = ct$, $p(t) = t$, and $q(t) = q_0/t^8$. It is easy to verify that (H_1) holds. Set $h(s) = s^6$, for any $q_0 > 0$; then, it is easy to verify that

$$\begin{aligned} &\int_{t_2}^{\infty} \left(h(s) q(s) \exp\left(\int_{t_0}^{s} \frac{p(\xi)}{r(\xi)} d\xi\right) \frac{\sigma(s)^{3(p_1-1)}}{s^{3(p_1-1)}} \right. \\ &\left. - \frac{2^{p_1-1}}{p_1^{p_1}} \frac{r(s)\left(h'(s)\right)^{p_1}}{(\varepsilon_1 s^2 h(s))^{p_1-1} \exp\left(-\int_{t_0}^{s} (p(\xi)/r(\xi)) d\xi\right)}\right) ds \\ &= \int_{t_2}^{\infty} \left(\frac{q_0 c^9}{t_0}\frac{1}{s} - \frac{81}{2\varepsilon_1^3 t_0}\frac{1}{s^2}\right) ds = \infty, \end{aligned} \tag{77}$$

that is, (37) holds. Moreover, one has

$$\begin{aligned} \theta(t) &= \lambda^{p_2-1/p_1-1} \int_t^{\infty} \left(\frac{1}{r(\xi)} \int_{\xi}^{\infty} q(s) \exp\left(\int_t^s \frac{p(\xi)}{r(\xi)} d\xi\right) \right. \\ &\left. \cdot \left(\frac{\sigma(s)}{s}\right)^{p_2-1} ds\right)^{1/(p_1-1)} d\xi \\ &= \lambda \int_t^{\infty} \left(\frac{c^3}{\xi^2} \int_{\xi}^{\infty} \frac{q_0}{s^7}\frac{s}{t} ds\right)^{1/3} d\xi = \frac{3}{5}\lambda c \sqrt[3]{\frac{q_0}{6}}\frac{1}{t^2}. \end{aligned} \tag{78}$$

Thus, by Corollary 14, we obtain that equation (76) is oscillatory if $q_0 > \max\{6(5/12\lambda c)^3, 81(2\varepsilon_1^3 c^9)\}$.

4. Conclusions

With the help of the comparison strategies involving first- and second-order differential equations and a refinement of the Riccati transformations, some new criteria for oscillation of fourth-order p-Laplacian differential equations with middle term are established. The results obtained here complement and extend some known results in [26].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors have contributed equally and significantly in this paper. All authors have read and agreed to the published version of the manuscript.

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