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Research Article

Products of Toeplitz Operators on the 2-Analytic Bergman Space

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Let f and g be bounded functions, and let T_f and T_g be Toeplitz operators on $A_2^2(\mathbb{D})$. We show that if the product $T_f T_g$ equals zero and one of f and g is a radial function satisfying a Mellin transform condition, then the other function must be zero.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ equipped with the normalized Lebesgue area measure $dA(z) = (1/\pi)dxdy$, and let $L^2 = L^2(\mathbb{D}, dA)$ denote the Lebesgue space on $\mathbb{D}$. For $n \in \mathbb{Z}^+$, let A_n^2 denote the n-analytic Bergman space, that is, the subspaces of L^2 consisting of n-differentiable functions such that $\partial_{\bar{z}}^n f = 0$, where

$$\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \quad (1)$$

As we know, A_n^2 is a Hilbert subspace with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(\omega)\bar{g(\omega)}dA(\omega), \quad (2)$$

where $f, g \in A_n^2$.

The planar Beurling transform is the singular integral operator given by

$$Sf(z) = -\int_{\mathbb{C}} \frac{f(\omega)}{(\omega - z)^2} dA(\omega), \quad z \in \mathbb{C}. \quad (3)$$

It is well known that the Beurling transform is a unitary operator acting on $L^2(\mathbb{C}, dA)$ (see [1], p. 364). For $\mathbb{D} \subset \mathbb{C}$, the compression of the Beurling transform to L^2 is a bounded linear operator acting on L^2 defined by

$$S_{\mathbb{D}}f(z) = -\int_{\mathbb{D}} \frac{f(\omega)}{(\omega - z)^2} dA(\omega), \quad f(z) \in L^2. \quad (4)$$

The n-analytic Bergman projection P_n is defined to be the orthogonal projection of L^2 onto A_n^2. The singular integral operator $S_{\mathbb{D}}$ is related to P_n, and it is known (see [2]) that

$$P_n = I - (S_{\mathbb{D}})^n (S_{\mathbb{D}}^*)^n, \quad n \in \mathbb{Z}^+. \quad (5)$$

For a function $u \in L^\infty$, the Toeplitz operator T_u with symbol u on A_n^2 is defined by

$$T_u f = P_n(uf), \quad f \in A_n^2. \quad (6)$$

n-analytic functions play an important role in mathematical, and the space A_n^2 has been intensively studied. More details about the structure of these spaces can be found in paper [3–5] and Balk's book [6].

Zero-product problem is a very important question in the operator theory. For Toeplitz operators, we have the general zero-product problem. Namely, if f and g are bounded functions such that $T_f T_g = 0$, then must one of the functions be zero? Ahern and Čučković (see [7]) obtained an affirmative answer for Toeplitz operators on A_1^2 when one of the functions is radial. Le (see [8, 9]) generalized this result to more than two functions. Čučković and Le (see [10]) gave a positive answer when both functions are harmonic. While the general zero-product problem (even on A_1^2) is still far from being solved, it is known that Toeplitz operators with radial symbols are diagonal with respect to the standard orthonormal basis of A_1^2. However, this is not the case on A_n^2 when $n \geq 2$. Then, Čučković and Le (see [10]) raised the following open question:

Question 1. Let f and g be bounded functions, one of which is radial. If $T_f T_g = 0$ on A_n^2 (or more generally, $T_f T_g$ has finite rank), must one of these functions be zero?

In this paper, we give a partial answer to this question on the 2-analytic Bergman space A_2^2. We show that if g is a radial function satisfying a Mellin transform condition, then $T_f T_g = 0$ if and only if f is a zero function.

2. Some Preliminary Results

We adopt the following boundary conditions for the binomial coefficients:

$$\begin{aligned} &\binom{n}{-m} = 0, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{and } m = 1, 2, \cdots, \\ &\binom{n}{n+m} = 0, \text{ where } n = 0, 1, 2, \cdots \text{and } m = 1, 2, \cdots. \end{aligned} \tag{7}$$

An orthogonal basis in the space A_n^2 is given by (see [3, 11])

$$\phi_{j,k} = \sqrt{k+j-1}\,\frac{1}{(k+j-2)!}\frac{\partial^{k+j-2}}{\partial \bar{z}^{k-1}\partial z^{j-1}}\left(|z|^2 - 1\right)^{k+j-2}, \tag{8}$$

where $k = 1, 2, \cdots$and $j = 1, 2, \cdots, n$. The orthogonal basis can also be written as

$$\phi_{j,k} = \sqrt{k+j-1}\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}\binom{j+k-i-2}{j-1}z^{k-i-1}\bar{z}^{j-i-1}, \tag{9}$$

where $k = 1, 2, \cdots$and $j = 1, 2, \cdots, n$. For $n = 2$, we have the following lemma.

Lemma 2. *An orthogonal basis in A_2^2 is given by*

$$\begin{aligned} \phi_{1,k} &= \sqrt{k}\, z^{k-1}, \\ \phi_{2,k} &= \sqrt{k+1}\left(k z^{k-1}\bar{z} - (k-1)z^{k-2}\right), \end{aligned} \tag{10}$$

where $k = 1, 2, \cdots$.

For each $z \in \mathbb{D}$, since the point evaluation at z is a bounded linear functional on A_n^2, there exists a unique reproducing kernel function $K(z,\omega) \in A_n^2$ such that

$$g(z) = \int_{\mathbb{D}} g(\omega)K(z,\omega)dA(\omega), \quad z \in \mathbb{D}, \tag{11}$$

for every $g \in A_n^2$. On 2-analytic Bergman space A_2^2,

$$K(z,\omega) = \sum_{k=1}^{+\infty}\phi_{1,k}(z)\bar{\phi}_{1,k}(\omega) + \sum_{k=1}^{+\infty}\phi_{2,k}(z)\bar{\phi}_{2,k}(\omega). \tag{12}$$

The Mellin transform $\widehat{g}$ of a function $g \in L^1([0,1], rdr)$ is defined by

$$\widehat{g}(z) = \int_0^1 g(s)s^{z-1}ds. \tag{13}$$

It is easy to see that $\widehat{g}$ is well defined and analytic on the right half-plane $\{z : \text{Re } z \geq 2\}$. Čučković and Rao (see [12]) first used the Mellin transform to study Toeplitz operators on the classical Bergman space.

For notational convenience, we define $\phi_{1,0} = \phi_{2,0} = 0$ and $a_0 = b_0 = c_0 = d_0 = 0$. For some Toeplitz operators on 2-analytic Bergman space $A_2^2(\mathbb{D})$, we obtain the following lemmas.

Lemma 3. *Let g be a bounded radial function. Then, for each $p = 1, 2, \cdots$, we have*

$$\begin{aligned} &\int_{\mathbb{D}} g(r)\omega^{p-1}K(z,w)dA(\omega) = a_p\phi_{1,p}(z) + b_p\phi_{2,p+1}(z), \\ &\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}K(z,w)dA(\omega) = c_p\phi_{1,p-1}(z) + d_p\phi_{2,p}(z), \end{aligned} \tag{14}$$

where $a_p = 2\sqrt{p}\widehat{g}(2p)$, $b_p = 2\sqrt{p+2}[(p+1)\widehat{g}(2p+2) - p\widehat{g}(2p)]$, $c_p = 2\sqrt{p-1}\widehat{g}(2p)$, and $d_p = 2\sqrt{p+1}[p\widehat{g}(2p+2) - (p-1)\widehat{g}(2p)]$.

Proof. For each $p = 1, 2, \cdots$, since g is a bounded radial function, thus

$$\begin{aligned} \int_{\mathbb{D}} g(r)\omega^{p-1}K(z,w)dA(\omega) &= \int_{\mathbb{D}} g(r)\omega^{p-1}\left[\sum_{k=1}^{+\infty}\phi_{1,k}(z)\bar{\phi}_{1,k}(\omega) + \sum_{k=1}^{+\infty}\phi_{2,k}(z)\bar{\phi}_{2,k}(\omega)\right]dA(\omega) \\ &= \sum_{k=1}^{+\infty}\phi_{1,k}(z)\int_{\mathbb{D}} g(r)\omega^{p-1}\sqrt{k}\bar{\omega}^{k-1}dA(\omega) \\ &\quad + \sum_{k=1}^{+\infty}\phi_{2,k}(z)\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\phi}_{2,k}(\omega)dA(\omega) \\ &= 2\sqrt{p}\widehat{g}(2p)\phi_{1,p}(z) + 2\sqrt{p+2} \\ &\quad \cdot [(p+1)\widehat{g}(2p+2) - p\widehat{g}(2p)]\phi_{2,p+1}(z), \end{aligned}$$

$$\begin{aligned}\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}K(z,w)dA(\omega) &= \int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}\left[\sum_{k=1}^{+\infty}\phi_{1,k}(z)\bar{\phi}_{1,k}(\omega)\right.\\ &\quad\left.+\sum_{k=1}^{+\infty}\phi_{2,k}(z)\bar{\phi}_{2,k}(\omega)\right]dA(\omega)\\ &= \sum_{k=1}^{+\infty}\phi_{1,k}(z)\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}\sqrt{k}\bar{\omega}^{k-1}dA(\omega)\\ &\quad+\sum_{k=1}^{+\infty}\phi_{2,k}(z)\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}\bar{\phi}_{2,k}(\omega)dA(\omega)\\ &= 2\sqrt{p-1}\widehat{g}(2p)\phi_{1,p-1}(z)+2\sqrt{p+1}\\ &\quad\cdot\left[p\widehat{g}(2p+2)-(p-1)\widehat{g}(2p)\right]\phi_{2,p}(z).\end{aligned} \tag{15}$$

□

It is well known that radial Toeplitz operators acting on A_1^2 are diagonal, and radial Toeplitz operators acting on A_n^2 can be represented as matrix sequences (see [13]). In the following, we give the exact expression of radial Toeplitz operators on A_2^2.

Lemma 4. *Let g be a bounded radial function. Then, for each $p\in\mathbb{Z}^+$,*

$$T_g\left(\phi_{1,p}\right)=\sqrt{p}\left[a_p\phi_{1,p}(z)+b_p\phi_{2,p+1}(z)\right],$$

$$\begin{aligned}T_g\left(\phi_{2,p}\right) &= \sqrt{p+1}\left\{\left[pc_p-(p-1)a_{p-1}\right]\phi_{1,p-1}(z)\right.\\ &\quad\left.+\left[pd_p-(p-1)b_{p-1}\right]\phi_{2,p}(z)\right\}.\end{aligned} \tag{16}$$

Proof. Since g is a bounded radial function, for each $p\in\mathbb{Z}^+$, using Lemma 3, we get

$$\begin{aligned}T_g\left(\phi_{1,p}\right) &= P_n\left(g\sqrt{p}\omega^{p-1}\right)(z)\\ &= \int_{\mathbb{D}} g(r)\sqrt{p}\omega^{p-1}K(z,w)dA(\omega)\\ &= \sqrt{p}\left[a_p\phi_{1,p}(z)+b_p\phi_{2,p+1}(z)\right],\end{aligned}$$

$$\begin{aligned}T_g\left(\phi_{2,p}\right) &= P_n\left(g\sqrt{p+1}\left(p\,\omega^{p-1}\bar{\omega}-(p-1)\omega^{p-2}\right)\right)(z)\\ &= \sqrt{p+1}\int_{\mathbb{D}} g(r)\left(p\,\omega^{p-1}\bar{\omega}-(p-1)\omega^{p-2}\right)K(z,w)dA(\omega)\\ &= \sqrt{p+1}\left[p\int_{\mathbb{D}} g(r)\omega^{p-1}\bar{\omega}K(z,w)dA(\omega)\right.\\ &\quad\left.-(p-1)\int_{\mathbb{D}} g(r)\omega^{p-2}K(z,w)dA(\omega)\right]\\ &= \sqrt{p+1}\left\{p\left[c_p\phi_{1,p-1}(z)+d_p\phi_{2,p}(z)\right]\right.\\ &\quad\left.-(p-1)\left[a_{p-1}\phi_{1,p-1}(z)+b_{p-1}\phi_{2,p}(z)\right]\right\}\\ &= \sqrt{p+1}\left\{\left[pc_p-(p-1)a_{p-1}\right]\phi_{1,p-1}(z)\right.\\ &\quad\left.+\left[pd_p-(p-1)b_{p-1}\right]\phi_{2,p}(z)\right\}.\end{aligned} \tag{17}$$

□

Applying Lemma 4, we conclude that radial Toeplitz operators on A_2^2 are not diagonal. The following corollary is an immediate consequence of Lemma 4.

Corollary 5. *Let g be a bounded radial function. Then, for each $p,q\in\mathbb{Z}^+$,*

$$\begin{aligned}\left\langle T_g\phi_{1,p},\phi_{1,q}\right\rangle &= \begin{cases}\sqrt{p}a_p, & \text{if } q=p,\\ 0, & \text{if } q\neq p,\end{cases}\\ \left\langle T_g\phi_{1,p},\phi_{2,q}\right\rangle &= \begin{cases}\sqrt{p}b_p, & \text{if } q=p+1,\\ 0, & \text{if } q\neq p+1,\end{cases}\\ \left\langle T_g\phi_{2,p},\phi_{1,q}\right\rangle &= \begin{cases}\sqrt{p+1}\left[pc_p-(p-1)a_{p-1}\right], & \text{if } q=p-1,\\ 0, & \text{if } q\neq p-1,\end{cases}\\ \left\langle T_g\phi_{2,p},\phi_{2,q}\right\rangle &= \begin{cases}\sqrt{p+1}\left[pd_p-(p-1)b_{p-1}\right], & \text{if } q=p,\\ 0, & \text{if } q\neq p.\end{cases}\end{aligned} \tag{18}$$

3. Products of Two Toeplitz Operators

A bounded function f is said to be quasihomogeneous of degree $k\in\mathbb{Z}$ if

$$f\left(re^{i\theta}\right)=e^{ik\theta}g(r), \tag{19}$$

where $g(r)$ is a radial function (see [14]). For any function $f\in L^2(\mathbb{D},dA)$, it has the polar decomposition, i.e.,

$$f\left(re^{i\theta}\right)=\sum_{k\in\mathbb{Z}} e^{ik\theta}f_k(r), \tag{20}$$

where $f_k(r)$ are radial functions in $L^2([0,1],rdr)$ (see [12]). A direct calculation gives the following lemma.

Lemma 6. *Let f be a bounded function. Then, for each $p,q\in\mathbb{Z}^+$,*

$$\begin{aligned}\left\langle f\phi_{1,p},\phi_{1,q}\right\rangle &= 2\sqrt{pq}\widehat{f}_{q-p}(p+q),\\ \left\langle f\phi_{1,p},\phi_{2,q}\right\rangle &= 2\sqrt{p(q+1)}\left[q\widehat{f}_{q-p-1}(p+q+1)\right.\\ &\quad\left.-(q-1)\widehat{f}_{q-p-1}(p+q-1)\right],\\ \left\langle f\phi_{2,p},\phi_{1,q}\right\rangle &= 2\sqrt{(p+1)q}\left[p\widehat{f}_{q-p-1}(p+q+1)\right.\\ &\quad\left.-(p-1)\widehat{f}_{q-p+1}(p+q-1)\right],\\ \left\langle f\phi_{2,p},\phi_{2,q}\right\rangle &= 2\sqrt{(p+1)(q+1)}\left[pq\widehat{f}_{q-p}(p+q+2)\right.\\ &\quad+(p+q-2pq)\widehat{f}_{q-p}(p+q)\\ &\quad\left.+(p-1)(q-1)\widehat{f}_{q-p}(p+q-2)\right].\end{aligned} \tag{21}$$

Proof. For all $p, q \in \mathbb{Z}^+$, it is easy to verify that

$$\begin{aligned}\left\langle f\phi_{1,p}, \phi_{1,q}\right\rangle &= \sum_{k\in\mathbb{Z}}\left\langle e^{ik\theta}f_k(r)\sqrt{p}z^{p-1}, \sqrt{q}z^{q-1}\right\rangle \\ &= \sum_{k\in\mathbb{Z}}\sqrt{pq}\left\langle e^{ik\theta}f_k(r)z^{p-1}, z^{q-1}\right\rangle \\ &= 2\sqrt{pq}\widehat{f}_{q-p}(p+q).\end{aligned} \tag{22}$$

Similarly, the rest of the lemma can be proved. □

When considering the product of two Toeplitz operators, we often use the Mellin convolution. If $f, g \in L^1([0,1], rdr)$, then their Mellin convolution is given by

$$(f * g)(r) = \int_r^1 f\left(\frac{r}{t}\right)g(t)\frac{dt}{t}, 0 \le t < 1. \tag{23}$$

The Mellin convolution theorem (see [15]) states that

$$\widehat{f * g}(s) = \widehat{f}(s)\widehat{g}(s), \tag{24}$$

and if f and g are bounded, then so is $f * g$.

It is well known that the Mellin transform is uniquely determined by its value on an arithmetic sequence of integers. The following results (see [15], p. 102, [16]) will be needed later.

Theorem 7. *Suppose f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes at the pairwise distinct points z_1, z_2, $\cdots$, where*

$$\begin{aligned}&\inf\{|z_n|\} > 0, \\ &\sum_{n\ge1}\operatorname{Re}\left(\frac{1}{z_n}\right) = \infty.\end{aligned} \tag{25}$$

Then, f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 8. Using this theorem, we can see that if $g \in L^1([0,1], rdr)$ and if there exists a sequence $\{n_k\}_{k\ge0} \subset \mathbb{N}$ such that

$$\begin{aligned}&\widehat{g}(n_k) = 0, \\ &\sum_{k\ge0}\frac{1}{n_k} = \infty,\end{aligned} \tag{26}$$

then, $\widehat{g}(z) = 0$ for all $z \in \{z : \operatorname{Re} z > 2\}$, by the Müntz-Szasz theorem (see [17], p. 312), $g = 0$.

For $p \in \mathbb{Z}^+$, we obtain

$$\widehat{g}(p) = \int_0^1 g(s)s^{p-1}ds. \tag{27}$$

the numbers $\widehat{g}(p)$ can also be called the moment Mellin sequence of g. Let

$$A(p) = \begin{pmatrix} a_{p-1} & b_{p-1} \\ pc_p - (p-1)a_{p-1} & pd_p - (p-1)b_{p-1}\end{pmatrix}. \tag{28}$$

$A(p)$ is closed related to the moment Mellin sequence of g, and we have the following lemma.

Lemma 9. *Let p be a fixed positive integer. Then, the following statements hold:*

(i) $a_p = c_p = 0$ if and only if $\widehat{g}(2p) = 0$

(ii) $b_p = 0$ if and only if $(p+1)\widehat{g}(2p+2) - p\widehat{g}(2p) = 0$

(iii) $d_p = 0$ if and only if $p\widehat{g}(2p+2) - (p-1)\widehat{g}(2p) = 0$

(iv) $|A(p+1)| = 0$ if and only if $(\widehat{r^2g * r^2g})(2p) - (\widehat{r^4g * g})(2p) = 0$

Proof. From Lemma 3, it is easy to check that (i), (ii), and (iii) hold.

To prove (iv), in fact, for a fixed $p \in \mathbb{Z}^+$,

$$\begin{aligned}|A(p+1)| &= \left|\begin{pmatrix} a_p & b_p \\ (p+1)c_{p+1} - pa_p & (p+1)d_{p+1} - pb_p\end{pmatrix}\right| \\ &= (p+1)a_pd_{p+1} - (p+1)b_pc_{p+1} \\ &= 4(p+1)^2\sqrt{p(p+2)}\left\{\widehat{g}(2p+4)\widehat{g}(2p) - [g\wedge(2p+2)]^2\right\}.\end{aligned} \tag{29}$$

It follows that $|A(p+1)| = 0$ if and only if

$$[g\wedge(2p+2)]^2 - \widehat{g}(2p+4)\widehat{g}(2p) = 0. \tag{30}$$

Using $\widehat{g}(2p+2) = \widehat{r^2g}(2p)$, $\widehat{g}(2p+4) = \widehat{r^4g}(2p)$, and Mellin convolution (24), we get the above equality is equivalent to

$$(\widehat{r^2g * r^2g})(2p) - (\widehat{r^4g * g})(2p) = 0. \tag{31}$$

□

Lemma 10. *Let g be a bounded radial function. The function $g = 0$ if and only if there exists a sequence $\{p_k\}_{k\ge0} \subset \mathbb{Z}^+$,*

$$\sum_{k\ge0}\frac{1}{p_k} = \infty, \text{such that } (\widehat{r^2g * r^2g})(2p_k) = (\widehat{r^4g * g})(2p_k). \tag{32}$$

Proof. If the function $g = 0$, then $r^2g * r^2g = r^4g * g = 0$, for each $p \in \mathbb{Z}^+$,

$$(\widehat{r^2g * r^2g})(2p) = (\widehat{r^4g * g})(2p) = 0. \tag{33}$$

This proves the sufficient condition.

Next, we prove the necessary condition. Suppose $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$,

$$\sum_{k\geq 0} \frac{1}{p_k} = \infty, \qquad (r^2 \widehat{g * r^2} g)(2p_k) = (r^4 \widehat{g * g})(2p_k). \tag{34}$$

Using Remark 8, we have

$$(r^2 \widehat{g * r^2} g)(z) = (r^4 \widehat{g * g})(z), \tag{35}$$

for all $z \in \{z : \operatorname{Re} z > 2\}$. Therefore, for each $p \in \mathbb{Z}^+$,

$$[g\wedge(2p+2)]^2 = \widehat{g}(2p+4)\widehat{g}(2p). \tag{36}$$

That is, $\{\widehat{g}(2p)\}_{p=1}^{\infty}$ is a geometric sequence. There exists a constant a such that

$$\widehat{g}(2p+2) = a \cdot \widehat{g}(2p). \tag{37}$$

Then,

$$(r^2 \widehat{g - a} g)(2p) = 0. \tag{38}$$

Since $\{2p\}_{p=1}^{\infty} \subset \mathbb{Z}^+$ is a sequence and $\sum_{p=1}^{\infty}(1/2p) = \infty$, by Remark 8, $(r^2 - a)g = 0$, which implies $g = 0$. □

For each $p, q \in \mathbb{Z}^+$, let $b_{11}(p,q) = \langle f\phi_{1,p}, \phi_{1,q}\rangle$, $b_{12}(p,q) = \langle f\phi_{1,p}, \phi_{2,q}\rangle$, $b_{21}(p,q) = \langle f\phi_{2,p}, \phi_{1,q}\rangle$, and $b_{22}(p,q) = \langle f\phi_{2,p}, \phi_{2,q}\rangle$. Let

$$B(p,q) = \begin{pmatrix} b_{11}(p,q) & b_{12}(p,q) \\ b_{21}(p+1,q) & b_{22}(p+1,q) \end{pmatrix}. \tag{39}$$

The first main result of this paper is the following theorem.

Theorem 11. *Let f be a bounded function and g be a bounded radial function. Then, $T_f T_g = 0$ on A_2^2 if and only if for each $p, q \in \mathbb{Z}^+$, $A(p)B(p-1,q) = 0$.*

Proof. Using the fact that f is a bounded function, we have

$$f\left(re^{i\theta}\right) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r). \tag{40}$$

If $T_f T_g = 0$, then for each $p, q \in \mathbb{Z}^+$,

$$\left\langle T_f T_g \phi_{1,p}, \phi_{1,q} \right\rangle = 0, \qquad \left\langle T_f T_g \phi_{1,p}, \phi_{2,q} \right\rangle = 0. \tag{41}$$

By Lemma 4,

$$a_p\left\langle f\phi_{1,p}, \phi_{1,q}\right\rangle + b_p\left\langle f\phi_{2,p+1}, \phi_{1,q}\right\rangle = 0, \qquad a_p\left\langle f\phi_{1,p}, \phi_{2,q}\right\rangle + b_p\left\langle f\phi_{2,p+1}, \phi_{2,q}\right\rangle = 0, \tag{42}$$

from which we conclude that

$$\left(a_p, b_p\right) B(p,q) = 0. \tag{43}$$

Since p is arbitrary, it follows that

$$\left(a_{p-1}, b_{p-1}\right) B(p-1,q) = 0. \tag{44}$$

Analogously, for each $p, q \in \mathbb{Z}^+$, it is easily verified that

$$\left\langle T_f T_g \phi_{2,p}, \phi_{1,q} \right\rangle = 0, \qquad \left\langle T_f T_g \phi_{2,p}, \phi_{2,q} \right\rangle = 0, \tag{45}$$

thus, we get

$$\left(pc_p - (p-1)a_{p-1}, pd_p - (p-1)b_{p-1}\right) B(p-1,q) = 0. \tag{46}$$

The above equations are equivalent to

$$A(p)B(p-1,q) = 0. \tag{47}$$

This completes the proof of the theorem. □

For $p = 1, 2, \cdots$, firstly if $a_p = c_p = 0$, then $\widehat{g}(2p) = 0$, using Remark 8, we get $g = 0$. Now, if $b_p = 0$, then

$$(2p+2)\widehat{g}(2p+2) - 2p\widehat{g}(2p) = 0. \tag{48}$$

Letting $\zeta = 2p$, we have

$$\zeta\widehat{g}(\zeta) = (\zeta+2)\widehat{g}(\zeta+2). \tag{49}$$

It is easy to see that the function $\zeta\widehat{g}(\zeta)$ is a periodic function with a period 2. Using the same argument as the one at the end of Section 2 in [12], we conclude that $\zeta\widehat{g}(\zeta)$ must be a constant function. Hence,

$$\widehat{g}(\zeta) = \frac{C}{\zeta}, \tag{50}$$

where C is a constant and it is clear that g is also a constant. Finally, if $d_p = 0$, then

$$2p\widehat{g}(2p+2) - (2p-2)\widehat{g}(2p) = 0\,; \tag{51}$$

that is,

$$2p\,\widehat{g \cdot r^2}(2p) - (2p-2)\,\widehat{g \cdot r^2}(2p-2) = 0. \tag{52}$$

Similarly, we can also conclude that $r^2 \cdot g$ is a constant. Thus, if g is a bounded radial function, it must be zero. Finally, we obtain the following lemma.

Lemma 12. *Let g be a bounded radial function. Then, the following statements hold:*

(i) $a_p = c_p = 0$ for all $p \in \mathbb{Z}^+$ if and only if $g = 0$

(ii) $b_p = 0$ for all $p \in \mathbb{Z}^+$ if and only if g is a constant

(iii) $d_p = 0$ for all $p \in \mathbb{Z}^+$ if and only if $g = 0$

Remark 13. In Lemma 12, the condition "for all $p \in \mathbb{Z}^+$" can also be replaced by "a sequence $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$ satisfying $\sum_{k\geq 0}(1/p_k) = \infty$."

In Theorem 11, if $a_p = 0$ or $c_p = 0$, or $d_p = 0$, then $g = 0$, so it is clear that $T_f T_g = 0$. If $b_p = 0$, then g is a constant; it is also easy to see that if g is not zero and $T_f T_g = 0$, then f must be zero. If $|A(p+1)| = 0$, $A(p+1)$ is not invertible. On the other hand, when $|A(p+1)| \neq 0$, then $A(p+1)$ is an invertible matrix. For a bounded radial function g such that $|A(p+1)| \neq 0$, if $T_f T_g = 0$, is it necessary that $f = 0$? The second main theorem of this paper answers this question by giving a sufficient and necessary condition.

Theorem 14. *Let g and f be bounded functions and g be a bounded radial function satisfying*

$$(r^2\widehat{g * r^2}g)(2p) \neq (r^4\widehat{g *} g)(2p), \tag{53}$$

for each $p \in \mathbb{Z}^+$. Then, $T_f T_g = 0$ on A_2^2 if and only if $f = 0$.

Proof. If f is a zero function, it is obvious that $T_f T_g = 0$.

Now, we assume $T_f T_g = 0$ and we shall prove $f = 0$. If g is a bounded radial function and for each $p \in \mathbb{Z}^+$,

$$(r^2\widehat{g * r^2}g)(2p) \neq (r^4\widehat{g *} g)(2p), \tag{54}$$

then, by the Mellin convolution theorem (24), it follows that

$$[g\wedge(2p+2)]^2 \neq \widehat{g}(2p+4)\widehat{g}(2p). \tag{55}$$

For each $p \in \mathbb{Z}^+$,

$$|A(p+1)| = 4(p+1)^2\sqrt{p(p+2)}\left\{\widehat{g}(2p+4)\widehat{g}(2p) - [g\wedge(2p+2)]^2\right\}. \tag{56}$$

Applying (55), we get $|A(p+1)| \neq 0$, that is, $A(p+1)$ is an invertible matrix. If $T_f T_g = 0$ and for each $q \in \mathbb{Z}^+$, we get

$$A(p+1)B(p,q) = 0. \tag{57}$$

Since $A(p+1)$ is invertible,

$$B(p,q) = \begin{pmatrix} b_{11}(p,q) & b_{12}(p,q) \\ b_{21}(p+1,q) & b_{22}(p+1,q) \end{pmatrix} = 0. \tag{58}$$

Thus, $b_{11}(p,q) = 0$, by Lemma 6, we have

$$\widehat{f}_{q-p}(p+q) = 0. \tag{59}$$

That is,

$$\widehat{f}_k(k+2p) = 0, \tag{60}$$

where $k = q - p$. Since p and q are arbitrary elements in $\mathbb{Z}^+$, by Remark 8, we obtain $f_k = 0$ for all k in $\mathbb{Z}$. It follows that $f = 0$. This completes the proof of the theorem. □

Example 1. Let $g = r^m$, where $m \in \mathbb{Z}^+$. Then, for each $p \in \mathbb{Z}^+$,

$$\begin{aligned} (r^2\widehat{g * r^2}g)(2p) &= \left(\frac{1}{2p+m+2}\right)^2, \\ (r^4\widehat{g *} g)(2p) &= \frac{1}{(2p+m)(2p+m+4)}. \end{aligned} \tag{61}$$

Obviously, $(r^2\widehat{g * r^2}g)(2p) \neq (r^4\widehat{g *} g)(2p)$. It is easy to see that $T_f T_{r^m} = 0$ on A_2^2 if and only if $f = 0$.

In the following, we discuss when condition (53) is not satisfied.

Case 1. If g is a bounded radial function and for each $p \in \mathbb{Z}^+$,

$$(r^2\widehat{g * r^2}g)(2p) = (r^4\widehat{g *} g)(2p). \tag{62}$$

Then, using the Mellin convolution theorem (24), we have

$$[g\wedge(2p+2)]^2 = \widehat{g}(2p+4)\widehat{g}(2p). \tag{63}$$

That is, $\{\widehat{g}(2p)\}_{p=1}^{\infty}$ is a geometric sequence. Using Lemma 10, we get g must be zero. It is clear that $T_f T_g = 0$.

Case 2. If g is a bounded radial function and for some $p \in \mathbb{Z}^+$,

$$(r^2\widehat{g * r^2}g)(2p) = (r^4\widehat{g *} g)(2p). \tag{64}$$

(1) If there exists a sequence $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$ satisfying $\sum_{k\geq 0}(1/p_k) = \infty$ such that

$$(r^2\widehat{g * r^2}g)(2p_k) = (r^4\widehat{g *} g)(2p_k), \tag{65}$$

then, by using Lemma 10, we get that g must be zero function.

(2) If there exists a finite sequence $\{p_k\} \subset \mathbb{Z}^+$, or an infinite sequence $\{p_k\} \subset \mathbb{Z}^+$ satisfying $\sum_{k\geq 0}(1/p_k) < \infty$, such that

$$(r^2\widehat{g * r^2}g)(2p_k) = (r^4\widehat{g * g})(2p_k), \tag{66}$$

then, the radial function g may not be zero function. For example, if $\{p_k\} = \{p_1\}$ is finite sequence and $p_1 = 1$, there exist some nonzero bounded radial functions g such that

$$(r^2\widehat{g * r^2}g)(2) = (r^4\widehat{g * g})(2). \tag{67}$$

Let $g = ar^2 + br^4$, where $a, b \in \mathbb{R}$. Then,

$$\begin{aligned} \widehat{g}(4) &= \frac{a}{6} + \frac{b}{8}; \\ \widehat{g}(6) &= \frac{a}{8} + \frac{b}{10}; \\ \widehat{g}(2) &= \frac{a}{4} + \frac{b}{6}. \end{aligned} \tag{68}$$

When $a = 360$, $b = -720 + 120\sqrt{6}$, a direct calculation shows that condition (67) is satisfied. In this case, we can prove that $A(2)$ is not invertible. As

$$\begin{aligned} a_1 &= 2\widehat{g}(2); \\ b_1 &= 2\sqrt{3}[2\widehat{g}(4) - \widehat{g}(2)]; \\ c_1 &= 4\widehat{g}(4); \\ d_1 &= 4\sqrt{3}[2\widehat{g}(6) - \widehat{g}(4)], \end{aligned} \tag{69}$$

then

$$\begin{aligned} A(2) &= \begin{pmatrix} a_1 & b_1 \\ 2c_2 - a_1 & 2d_2 - b_1 \end{pmatrix} \\ &= \begin{pmatrix} 2\widehat{g}(2) & 2\sqrt{3}[2\widehat{g}(4) - \widehat{g}(2)] \\ 4\widehat{g}(4) - 2\widehat{g}(2) & 2\sqrt{3}[4\widehat{g}(6) - 4\widehat{g}(4) + \widehat{g}(2)] \end{pmatrix}. \end{aligned} \tag{70}$$

Since $g = ar^2 + br^4$, it follows from (68) and (70) that

$$A(2) = \begin{pmatrix} \dfrac{a}{2} + \dfrac{b}{3} & \dfrac{\sqrt{3}}{6}(a+b) \\ \dfrac{1}{6}(a+b) & 2\sqrt{3}\left(\dfrac{a}{12} + \dfrac{b}{15}\right) \end{pmatrix}. \tag{71}$$

When $a = 360$, $b = -720 + 120\sqrt{6}$, a direct calculation shows that $|A(2)| = 0$ and $A(2)$ is not invertible.

Remark 15. For a nonzero function g whose related matrices are $A(p)$, $p \in \mathbb{Z}^+$, if there exist matrices $B(p)$, $p \in \mathbb{Z}^+$ such that

(i) $B(p)$ are not all zero

(ii) For each $p \in \mathbb{Z}^+$, $A(p)B(p) = 0$

then, we can construct a nonzero function f, such that $T_f T_g = 0$. The following example solves (i) and (ii) for a fixed p. However, it is still unknown if (i) and (ii) hold for all $p \in \mathbb{Z}^+$, and we will study this question in the future work.

Example 2. Suppose $g = 360r^2 + (-720 + 120\sqrt{6})r^4$. Then

$$A(2) = \begin{pmatrix} -60 + 40\sqrt{6} & -60\sqrt{3} + 60\sqrt{2} \\ -60 + 20\sqrt{6} & -36\sqrt{3} + 48\sqrt{2} \end{pmatrix}. \tag{72}$$

As $A(2)$ is not invertible, there exist some nonzero matrix B such that $A(2)B = 0$. For example,

$$B = \begin{pmatrix} \sqrt{6} & 2\sqrt{3} \\ \sqrt{2} + 2\sqrt{3} & 2 + 2\sqrt{6} \end{pmatrix}. \tag{73}$$

For each $p = 1, 2, \cdots$, analogous to Lemma 3, we define

$$\begin{aligned} &a'_p = 2\sqrt{p}\widehat{f}(2p), b'_p = 2\sqrt{p+2}\left[(p+1)\widehat{f}(2p+2) - p\widehat{f}(2p)\right], \\ &c'_p = 2\sqrt{p-1}\widehat{f}(2p), d'_p = 2\sqrt{p+1}\left[p\widehat{f}(2p+2) - (p-1)\widehat{f}(2p)\right], \end{aligned} \tag{74}$$

and $a'_0 = b'_0 = c'_0 = d'_0 = 0$.

In Theorem 14, if f and g are all bounded radial function and there exists a sequence $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$,

$$\sum_{k\geq 0} \frac{1}{p_k} = \infty, \text{such that } (r^2\widehat{g * r^2}g)(2p_k) \neq (r^4\widehat{g * g})(2p_k), \tag{75}$$

the conclusion is still valid; then, we have the following corollary.

Corollary 16. *Let f and g be bounded radial functions. Suppose there exists a sequence $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$,*

$$\sum_{k\geq 0} \frac{1}{p_k} = \infty, \text{such that } (r^2\widehat{g * r^2}g)(2p_k) \neq (r^4\widehat{g * g})(2p_k). \tag{76}$$

If $T_f T_g = 0$ on A_2^2, then $f = 0$.

Proof. For $p \in \mathbb{Z}^+$, define

$$B_p = \begin{pmatrix} \sqrt{p}a'_p & \sqrt{p}b'_p \\ \sqrt{p+2}\left[(p+1)c'_{p+1} - pa'_p\right] & \sqrt{p+2}\left[(p+1)d'_{p+1} - pb'_p\right] \end{pmatrix}. \quad (77)$$

By the hypothesis, f is a bounded radial function, it follows from Lemma 4 and Theorem 11 that $T_f T_g = 0$ if and only if for each $p \in \mathbb{Z}^+$,

$$A_p B_{p-1} = 0. \quad (78)$$

Let $g \neq 0$ and there exists a sequence $\{p_k\}_{k\geq 1} \subset \mathbb{Z}^+$,

$$\sum_{k\geq 1} \frac{1}{p_k} = \infty, \text{such that } (r^2 \widehat{g * r^2} g)(2p_k) \neq (r^4 \widehat{g * g})(2p_k). \quad (79)$$

Then, it follows that $A(p_k + 1)$ is an invertible matrix. Combining this with $A_{p_k+1} B_{p_k} = 0$, we get $B_{p_k} = 0$. It follows that

$$a'_{p_k} = 2\sqrt{p_k}\widehat{f}(2p_k) = 0. \quad (80)$$

This implies that $\widehat{f}(2p_k) = 0$, combing with

$$\sum_{k\geq 1} \frac{1}{2p_k} = \infty, \quad (81)$$

and using Remark 8, we get $f = 0$.

For $p \in \mathbb{Z}^+$, if f and g are bounded radial functions, it follows from Lemma 4 that

$$\begin{aligned} T_f T_g\left(\phi_{1,p}\right) &= \lambda_{11}(p)\phi_{1,p} + \lambda_{12}(p)\phi_{2,p+1}, \\ T_f T_g\left(\phi_{2,p}\right) &= \lambda_{21}(p)\phi_{1,p-1} + \lambda_{22}(p)\phi_{2,p}, \end{aligned} \quad (82)$$

where

$$\begin{aligned} \lambda_{11}(p) &= pa_p a'_p + \sqrt{p}b_p \cdot \sqrt{p+2}\left[(p+1)c'_{p+1} - pa'_p\right], \\ \lambda_{12}(p) &= pa_p b'_p + \sqrt{p}b_p \cdot \sqrt{p+2}\left[(p+1)d'_{p+1} - pb'_p\right], \\ \lambda_{21}(p) &= \sqrt{p+1}\left\{\left[pc_p - (p-1)a_{p-1}\right]\sqrt{p-1}a'_{p-1} + \left[pd_p - (p-1)b_{p-1}\right]\sqrt{p+1}\left[pc'_p - (p-1)a'_{p-1}\right]\right\}, \\ \lambda_{22}(p) &= \sqrt{p+1}\left\{\left[pc_p - (p-1)a_{p-1}\right]\sqrt{p-1}b'_{p-1} + \left[pd_p - (p-1)b_{p-1}\right]\sqrt{p+1}\left[pd'_p - (p-1)b'_{p-1}\right]\right\}. \end{aligned} \quad (83)$$

If $T_f T_g$ has a finite rank, there exists $N \in \mathbb{Z}^+$, for all $p > N$, such that

$$\begin{aligned} T_f T_g\left(\phi_{1,p}\right) &= 0, \\ T_f T_g\left(\phi_{2,p}\right) &= 0. \end{aligned} \quad (84)$$

As in Corollary 16, there exists a sequence $\{p_k\}_{k\geq 0}$ meet the conditions, where $\{p_k\} \subset \mathbb{Z}^+$ and $p_k > N$; using properties of Mellin transform, we can obtain that $T_f T_g$ has a finite rank if and only if $f = 0$. □

Remark 17. As in Corollary 16, let f and g are bounded radial functions and there exists a sequence $\{p_k\}_{k\geq 0} \subset \mathbb{Z}^+$,

$$\sum_{k\geq 0} \frac{1}{p_k} = \infty, \text{such that } (r^2 \widehat{g * r^2} g)(2p_k) \neq (r^4 \widehat{g * g})(2p_k). \quad (85)$$

Then, $T_f T_g$ has finite rank if and only if $f = 0$.

The following question is the general zero-product problem on A_n^2 when $n \geq 3$.

Question 18. Let f be a bounded function and g be a bounded radial function. Suppose that $T_f T_g = 0$ on A_n^2 when $n \geq 3$, can we obtain any similar conclusions?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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