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Research Article

Resolvent Positive Operators and Positive Fractional Resolvent Families

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This paper is concerned with positive α-times resolvent families on an ordered Banach space E (with normal and generating cone), where $0 < \alpha \le 2$. We show that a closed and densely defined operator A on E generates a positive exponentially bounded α-times resolvent family for some $0 < \alpha < 1$ if and only if, for some $\omega \in \mathbb{R}$, when $\lambda > \omega$, $\lambda \in \rho(A)$, $R(\lambda, A) \ge 0$ and $\sup \{\|\lambda R(\lambda, A)\|: \lambda \ge \omega\} < \infty$. Moreover, we obtain that when $0 < \alpha < 1$, a positive exponentially bounded α-times resolvent family is always analytic. While A generates a positive α-times resolvent family for some $1 < \alpha \le 2$ if and only if the operator $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ is completely monotonic. By using such characterizations of positivity, we investigate the positivity-preserving of positive fractional resolvent family under positive perturbations. Some examples of positive solutions to fractional differential equations are presented to illustrate our results.

1. Introduction

Many linear dynamical systems can be modelled as an abstract Cauchy problem:

$$\begin{aligned} u'(t) &= Au(t), \quad t > 0, \\ u(0) &= x_0 \end{aligned} \tag{1}$$

on a Banach space and then be treated by the theory of C_0-semigroups [1–3] in a unified way. Motivated by problems in probability theory and PDEs, it is necessary to investigate Cauchy problems having positive solutions to each positive initial value. For this purpose, the theory of semigroups on a Banach space with an order structure was established. Feller [4] and Phillips [5] gave the first characteristics of generators of some special positive semigroups. After that, the theory of positive operators on ordered Banach spaces has been developed systematically during the 60s and 70s [6, 7]. This led to further progress in positive semigroups during the 80s, and these developments were recorded in the first monograph on positive semigroups [8]. For more recent work, we refer to [9].

Positive C_0-semigroups are closely related to resolvent positive operators. If $T(t)$ is a positive C_0-semigroup on an ordered Banach space, then its generator A is resolvent positive, i.e., $R(\lambda, A) \ge 0$ when $\lambda > \omega$ for some $\omega \in \mathbb{R}$. However, a densely defined resolvent positive operator does not always generate a C_0-semigroup. The first example was given in [10]. Later, Arendt [11] proved that a densely defined resolvent positive operator generates a once integrated semigroup (see Lemma 16).

During the last decades, there are considerable interests on the abstract fractional Cauchy problem of order α:

$$\begin{aligned} D_t^\alpha u(t) &= Au(t), \quad t > 0, \\ u^{(k)}(0) &= x_k, \quad x_k \in X, k = 0, 1, \cdots, m-1, \end{aligned} \tag{2}$$

where A is a closed, densely defined linear operator on X, D_t^α is the Caputo derivative, and $m = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. It is known that (2) is well posed if and only if A generates an α-times resolvent family [12]. The class of fractional resolvent families is a special class of resolvent families introduced by Prüss [13]. A once resolvent family is actually a C_0-semigroup, and a twice resolvent family is a cosine operator function. Bajlekova has proved that the generator of an exponentially bounded α-times resolvent family is bounded if $\alpha > 2$ [12] (Theorem 2.6). For this reason, we restrict ourselves only to the cases that $\alpha \in (0, 2]$. For the existence of solutions to the abstract Cauchy problem of fractional order, see for examples [12, 14–16]. There are also literatures devoted to abstract semilinear fractional Cauchy problems, see, e.g., [17, 18]. We refer to the survey paper [19] and the references therein for the basic theory of abstract fractional differential equations.

It is natural to consider positive solutions to fractional differential equations. For example, the positive solutions to the time fractional diffusion equation:

$$D_t^\alpha u(t,x) = \Delta u(t,x), \quad t > 0, x \in \mathbb{R}^N, \tag{3}$$

were discussed in [20–22], and more general equation with fractional Laplacian

$$D_t^\alpha u(t,x) = -(-\Delta)^\beta u(t,x), \quad t > 0, x \in \mathbb{R}^N, \tag{4}$$

was studied in [23, 24]. The positivity of the fundamental solutions was derived from detailed analysis on some special functions including Mittag-Leffler functions, Mainardi functions, Bessel functions, and Fox H-functions. The investigation of positive solutions to these concrete fractional differential equations inspires us to study positive fractional resolvent families in a unified way. To our best knowledge, the positive solutions for abstract fractional Cauchy problems were discussed only in [25] on Banach lattices, under the assumption that the operator A generates a C_0-semigroup.

By the theory of fractional resolvent families and subordination principles developed in [12, 26, 27], we are able to obtain the positivity of solutions to fractional Cauchy problems via an operator theoretic approach. Our main result in this paper establishes the relations between positive fractional resolvent families and resolvent positive operators on an ordered Banach space E with generating and normal cone. More precisely, we show that (Theorem 15) if A is a closed densely defined operator on E, then A generates a positive exponentially bounded α-times resolvent family for some $\alpha \in (0, 1)$ if and only if

$$\begin{gathered} A \text{ is resolvent positive}, \\ \sup \{\|\lambda R(\lambda, A)\| : \lambda \geq \omega\} < \infty, \end{gathered} \tag{5}$$

for some $\omega \in \mathbb{R}$. Condition (5) is not enough to guarantee that A generates a C_0-semigroup, see Example 7, but it is sufficient for the generator of an α-times resolvent family for $0 < \alpha < 1$. Our proof is based on a result given by Arendt mentioned above. As a byproduct, we derive that a positive exponentially bounded α-times resolvent family with $\alpha \in (0, 1)$ is "automatically" analytic (Theorem 17). While for $\alpha \in (1, 2]$, A generates a positive α-times resolvent family if and only if $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ is completely monotonic. Based on these characterizations, we obtain the positivity-preserving of a positive fractional resolvent family under positive perturbations of relatively bounded operators, which generalizes those for positive semigroups [28, 29]. We would like to mention that the stability of positive fractional resolvent families was studied by one of the authors recently in [30].

The paper is organized as follows. In Section 2, we give some introductions on fractional calculus, fractional resolvent families, and ordered Banach spaces. Several examples of fractional differential equations with positive solutions are given in Section 3. Our main results on the relations between resolvent positive operators and positive resolvent families are presented in Section 4, and the positivity of fractional resolvent families is characterized there. We consider in Section 5 the positive perturbations of positive fractional resolvent families and apply our results to two examples related to Schrödinger operators.

2. Preliminaries

Let us first recall the basic definitions of fractional calculus (see [12, 31]). Let $\alpha > 0$, $m = \lceil \alpha \rceil$, and $I = (0, T)$ for some $T > 0$. For $f \in L^1(I)$, the fractional integral of order $\alpha > 0$ is defined by

$$(J_t^\alpha f)(t) := (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s) f(s)\, ds (t > 0), \tag{6}$$

where

$$g_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{7}$$

and $\Gamma(\alpha)$ is the gamma function. Set moreover $g_0(t) := \delta(t)$, the Dirac delta-function. The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D_t^\alpha f(t) := J^{m-\alpha} \left(\frac{d}{dt}\right)^m f(t), \tag{8}$$

if $f \in L^1(I) \cap C^{m-1}(I)$ and $g_{m-\alpha} * f \in W^{m,1}(I)$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{H_a} \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad z \in \mathbb{C}, \tag{9}$$

where $\alpha, \beta > 0$, H_a is the Hankel contour which starts and ends at $-\infty$ and encircles the disc $|t| \leq |z|^{1/\alpha}$ counterclockwise. We use $E_\alpha(t) := E_{\alpha,1}(t)$ for short. The Mittag-Leffler function $E_\alpha(t)$ satisfies the fractional differential equation:

$$D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha). \tag{10}$$

And the most important properties of these functions are associated with their Laplace integral:

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(st^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - s}, \quad \text{Re}\,(\lambda) > |s|^{1/\alpha}, \tag{11}$$

and their asymptotic expansion as $z \longrightarrow \infty$. For $0 < \alpha < 2$ and $\beta > 0$,

$$\begin{aligned} E_{\alpha,\beta}(z) &= \frac{1}{\alpha} z^{1-\beta/\alpha} \exp\left(z^{1/\alpha}\right) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| \le \frac{1}{2}\alpha\pi, \\ E_{\alpha,\beta}(z) &= \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < \left(1 - \frac{1}{2}\alpha\right)\pi, \end{aligned} \tag{12}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right), \tag{13}$$

as $z \longrightarrow \infty$ and $2 \le N \in \mathbb{N}$.

Another function is the scaled Wright-type function $\psi_{\alpha,\beta}$ with indexes $0 < \alpha < 1, \beta \ge 0$:

$$\psi_{\alpha,\beta}(t,s) := \sum_{n=0}^{\infty} \frac{(-1)^n s^n t^{\beta-\alpha n-1}}{n!\Gamma((-\alpha)n + \beta)} = \frac{1}{2\pi i}\int_{H_a} z^{-\beta} e^{tz - sz^\alpha} dz, \quad t > 0, s \in \mathbb{C}. \tag{14}$$

The Wright function $\psi_{\alpha,\beta}$ satisfies $\psi_{\alpha,\beta}(t,s) \ge 0$ for $t, s > 0$,

$$\int_0^\infty e^{-\lambda t}\psi_{\alpha,\beta}(t,s)dt = \lambda^{-\beta} e^{-\lambda^\alpha s}, \quad s, \lambda > 0, \tag{15}$$

$$\int_0^\infty e^{\lambda s}\psi_{\alpha,\beta}(t,s)ds = t^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(\lambda t^\alpha), \quad t > 0, \lambda \in \mathbb{C}. \tag{16}$$

Moreover,

$$\psi_{\alpha,\beta+\gamma}(t,s) = \left(g_\gamma * \psi_{\alpha,\beta}(\cdot, s)\right)(t), \quad t, s, \gamma > 0. \tag{17}$$

More details about positivity and integral representation of these two functions can be found in [32–34].

Now, we turn to the abstract fractional Cauchy problem on a Banach space X. For a Banach space X, we denote by $L(X)$ the Banach algebra of all bounded linear operators on X. Throughout this paper, we assume that A is a densely defined closed operator on X. We denote by $R(\lambda, A) := (\lambda - A)^{-1}$ the resolvent of A at λ if $\lambda \in \rho(A)$, the resolvent set of A. And we denote by $s(A) := \sup\{\text{Re}\,\lambda : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

Consider the Cauchy problem on X of order $\alpha \in (0, 2]$:

$$\begin{aligned} &D_t^\alpha u(t) = Au(t), \quad t > 0, \\ &u(0) = x_0 \left(\text{in addition } u'(0) = 0 \text{ if } \alpha > 1\right). \end{aligned} \tag{18}$$

The problem (18) is well posed if and only if the corresponding Volterra integral equation

$$u(t) = x_0 + \int_0^t g_\alpha(t-s)Au(s)ds \tag{19}$$

is well posed in the sense of [13] (Definition 1.2). This leads to the definition of the α-times resolvent family [12].

Definition 1. Let $\alpha \in (0, 2]$. A function $S_\alpha(\cdot)$: $[0, +\infty) \longrightarrow L(X)$ is called an α-times resolvent family generated by A if the following conditions are satisfied:

(i) $S_\alpha(t)$ is strongly continuous for $t \ge 0$ and $S_\alpha(0) = I$

(ii) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \ge 0$

(iii) For $x \in D(A)$, the resolvent equation

$$S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)Ax, \tag{20}$$

holds for all $t \ge 0$

Suppose that A generates an α-times resolvent family $S_\alpha(t)$. Then the Cauchy problem (18) is solvable, and its unique mild solution is given by $u(t) = S_\alpha(t)x_0$. And for $\alpha \in (1, 2]$, the unique mild solution to the Cauchy problem

$$\begin{aligned} &D_t^\alpha u(t) = Au(t), \quad t > 0, \\ &u(0) = x_0, \quad u'(0) = x_1, \end{aligned} \tag{21}$$

is given by

$$u(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(\tau)x_1 d\tau. \tag{22}$$

We call an α-times resolvent family $S_\alpha(t)$ exponentially bounded if there exist constants $M \ge 1$ and $\omega \ge 0$ such that $\|S_\alpha(t)\| \le Me^{\omega t}$ for all $t \ge 0$. If A generates an exponentially bounded α-times resolvent family, we will write for short $A \in \mathscr{C}^\alpha$.

$S_\alpha(t)$ is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} := \{z \in \mathbb{C} : z \ne 0 \text{ and } |\arg z| < \theta_0\}$ for some $\theta_0 \in (0, \pi/2]$ and for each $\theta \in (0, \theta_0)$ there exist constants $M_\theta \ge 1$, $\omega_\theta \ge 0$ such that $\|S_\alpha(z)\| \le M_\theta e^{\omega_\theta \text{Re}\, z}$ $(z \in \Sigma_\theta)$. If A generates an analytic exponentially bounded α-times resolvent family, we will write for short $A \in \mathscr{A}^\alpha$.

Lemma 2 ([12], Theorem 2.9). *Let $0 < \alpha \le 2$. Then $A \in \mathscr{C}^\alpha$ if and only if there are constants $M \ge 1$ and $\omega \ge 0$ such that $(\omega^\alpha, +\infty) \subset \rho(A)$, and there is a strongly continuous family $\{S_\alpha(t)\} \subset L(X)$ satisfying $\|S_\alpha(t)\| \le Me^{\omega t}$ for all $t \ge 0$ and*

$$\lambda^{\alpha-1} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad \lambda > \omega, x \in X. \tag{23}$$

In [35] (Lemma 3), it was shown that if A generates an exponentially bounded α-times resolvent family, then A is densely defined.

It is well known that the analyticity of a cosine operator function implies the boundedness of its generator. Therefore, we will restrict ourselves to the analyticity only when $0<\alpha<2$. We need the following result on generating bounded analytic α-times resolvent families.

Lemma 3 ([12], Corollary 2.17). *If $\rho(A)\subset\{\lambda : \mathrm{Re}\ \lambda>0\}$ and there exists a constant C such that*

$$\|R(\lambda,A)\|\le\frac{C}{\mathrm{Re}\ \lambda},\qquad \mathrm{Re}\ \lambda>0, \tag{24}$$

then $A\in\mathscr{A}^{\alpha}$ for any $\alpha\in(0,1)$.

The next lemma is concerned with the perturbation of fractional resolvent families [12] (Theorems 2.25 and 2.26).

Lemma 4.

(i) *If $A\in\mathscr{C}^{\alpha}$ for some $1\le\alpha\le2$ and $B\in L(X)$, then $A+B\in\mathscr{C}^{\alpha}$*

(ii) *If $A\in\mathscr{A}^{\alpha}$ for some $0<\alpha<2$ and B is a closed linear operator satisfying $D(B)\supset D(A)$ and*

$$\|Bx\|\le a\|Ax\|+b\|x\|,\quad x\in D(A) \tag{25}$$

Then there exists $\delta>0$ such that if $0\le a\le\delta$ then $A+B\in\mathscr{A}^{\alpha}$.

The following subordination principles are very important for C_0-semigroups and fractional resolvent families. The first one comes from [36]; the second one comes from [12] (Theorems 3.1 and 3.3).

Lemma 5 (subordination principle).

(i) *If A generates a bounded C_0-semigroup $T(t)$, then for $0<\alpha<1$, $-(-A)^{\alpha}$ generates the analytic C_0-semigroup $T_\alpha(t)$ given by*

$$T_\alpha(t)=\int_0^\infty\Psi_{\alpha,0}(s,t)T(s)ds,\quad t>0 \tag{26}$$

(ii) *Let $0<\beta<\alpha\le2$ and A generate an exponentially bounded α-times resolvent family $S_\alpha(t)$. Then A generates the analytic exponentially bounded β-times resolvent family $S_\beta(t)$, which is given by*

$$S_\beta(t)=\int_0^\infty\Psi_{\beta\alpha,1-\beta\alpha}(t,s)S_\alpha(s)ds,\quad t>0 \tag{27}$$

Finally, we give a brief introduction on ordered Banach spaces, as well as the positive operators, completely monotonic functions, and Bernstein functions on ordered Banach spaces. For details, we refer to [1, 8, 11, 37].

Definition 6. Let E be a real Banach space. By a positive cone in E, we understand a closed subset E_+ of E satisfying $E_+ + E_+\subset E_+$ and $\lambda E_+\subset E_+$ for all $\lambda>0$. The order relations on E are defined by $x\ge y$ iff $x-y\in E_+$. And moreover, the cone E_+ is said to be generating if $E=E_+-E_+$; the cone E_+ is said to be normal if there is some constant c such that $z\le y\le x$ always implies $\|y\|\le c\ \max\{\|z\|,\|x\|\}$.

Associated to each ordered Banach space E, there is an ordering on its dual space E' with the dual cone E'_+ defined by

$$E'_+=\left\{\omega\in E' : \omega(x)\ge0 \text{ for all } x\in E_+\right\}. \tag{28}$$

Definition 7. An operator $T\in L(E)$ is called positive if $T(E_+)\subset E_+$ and will be denoted by $T\ge0$. A function $f:\Lambda\longrightarrow E$ is called positive if $f(\lambda)\in E_+$ for every $\lambda\in\Lambda$ and will be denoted by $f\ge0$.

A function $f\in C^\infty((0,+\infty),E)$ is called completely monotonic if

$$(-1)^n f^{(n)}(\lambda)\ge0,\quad \forall\lambda>0. \tag{29}$$

A function $\varphi\in C^\infty((0,+\infty),E)$ is called a Bernstein function if $\varphi(\lambda)\ge0$ for any $\lambda>0$ and φ' is completely monotonic.

The exponential function $e^{-ax}(a>0)$ and the Mittag-Leffler function $E_{\alpha,\beta}(-ax)(a>0)$ with $0<\alpha<1,\alpha\le\beta$, are scalar completely monotonic functions, and $x^\alpha(0<\alpha<1)$ is a Bernstein function.

Lemma 8. *Let f,g be completely monotonic functions on E and φ be a scalar Bernstein function. Then*

(i) *$f+g, f\cdot g$ are completely monotonic*

(ii) *$f\circ\varphi$ is completely monotonic*

3. Positive Solutions to Fractional Differential Equations

In this section, we give several examples of fractional differential equations with positive solutions. These are motivations for us to consider positive fractional resolvent families on an ordered Banach space and then investigate the positive solutions to fractional differential equations in a unified way.

Example 1. If $0 < \alpha < 1$, the solution to the following scalar fractional differential equation

$$\begin{aligned} D_t^{\alpha} u(t) &= \lambda u(t), \quad t > 0, \\ u(0) &= u_0 \end{aligned} \tag{30}$$

is given by $u(t) = u_0 E_{\alpha}(\lambda t^{\alpha})$. It is obvious that $u(t) \geq 0$ for $t > 0$ if $\lambda > 0$ and $u_0 \geq 0$. In case that $\lambda < 0$, since the function $E_{\alpha}(-t)$ is completely monotonic for $t \geq 0$ when $0 < \alpha < 1$, we also have the solution $u(t)$ positive if the initial data $u_0 \geq 0$.

Now, let $1 < \alpha \leq 2$. The solution to

$$\begin{aligned} D_t^{\alpha} u(t) &= \lambda u(t), \quad t > 0, \\ u(0) &= u_0, u'(0) = u_1, \end{aligned} \tag{31}$$

is given by

$$u(t) = u_0 E_{\alpha}(\lambda t^{\alpha}) + u_1 E_{\alpha,2}(\lambda t^{\alpha}). \tag{32}$$

As in the case of $0 < \alpha < 1$, it is obvious that $u(t) \geq 0$ if $\lambda > 0$ and $u_0, u_1 \geq 0$. However, if $\lambda < 0$, the Mittag-Leffler function $E_{\alpha}(\lambda t)(1 < \alpha < 2)$ has a finite number of zeros on the positive real axis and thus is not positive [33, 38].

Example 2. Let E be an ordered Banach space with normal and generating cone and let A be a linear bounded operator on E. For $0 < \alpha \leq 2$, consider the fractional differential equation

$$\begin{aligned} D_t^{\alpha} u(t) &= A u(t), \quad t > 0, \\ u(0) &= u_0 \left(\text{in addition } u'(0) = u_1 \text{ if } \alpha > 1\right). \end{aligned} \tag{33}$$

The solution is given by

$$u(t) = E_{\alpha}(t^{\alpha} A) u_0 = \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)} A^n u_0, \tag{34}$$

if $0 < \alpha \leq 1$ or

$$\begin{aligned} u(t) = E_{\alpha}(t^{\alpha} A) u_0 + E_{\alpha,2}(t^{\alpha} A) u_1 &= \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)} A^n u_0 \\ &+ \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n + 2)} A^n u_1, \end{aligned} \tag{35}$$

if $1 < \alpha \leq 2$. If $A \geq 0$, the solution $u(t)$ remains positive if both u_0 and u_1 are positive. If $A = \lambda I$, a constant operator on E, the solution $u(t)$ is positive if $u_0, u_1 \geq 0$ when either $0 < \alpha \leq 1$ and $\lambda \in \mathbb{R}$ or $1 < \alpha \leq 2$ and $\lambda > 0$.

Example 3. Let $0 < \alpha < 1$. Consider the following fractional transport equation

$$\begin{aligned} D_t^{\alpha} u(t,x) &= -\frac{\partial}{\partial x} u(t,x), \quad t > 0, \\ u(0,x) &= f(x). \end{aligned} \tag{36}$$

By the subordination principle (Lemma 5(ii)), on $X = L^1(\mathbb{R})$, the solution is given by

$$u(t,x) = \int_0^{\infty} \psi_{\alpha,1-\alpha}(t,s) f(x-s)\, ds; \tag{37}$$

while on $X = L^1(\mathbb{R}_+)$, the solution is given by

$$u(t,x) = \left(\psi_{\alpha,1-\alpha}(t,\cdot) * f\right)(x) = \int_0^{x} \psi_{\alpha,1-\alpha}(t,s) f(x-s)\, ds. \tag{38}$$

See also [32] (Example 12). Since $\psi_{\alpha,\beta}(t,s) \geq 0$ when $\alpha \in (0,1)$ and $\beta > 0$, the solution $u(t,x)$ is positive if $f(x) \geq 0$.

Example 4. Let $0 < \alpha < 1$; consider the following fractional diffusion equation:

$$\begin{aligned} D_t^{\alpha} u(t,x) &= \Delta u(t,x), \quad t > 0, x \in \mathbb{R}^N, \\ u(0,x) &= f(x) \end{aligned} \tag{39}$$

on $L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$. When $\alpha = 1$, this is the classical heat equation. It is well known that the Laplacian operator Δ generates a bounded C_0-semigroup $T(t)$ on $L^p(\mathbb{R}^N)$ given by

$$(T(t)f)(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} f(y)\, dy. \tag{40}$$

Thus again by the subordination principle, the solution to the above fractional diffusion equation (38) is given by

$$u(t,x) = \int_0^{\infty} \frac{\psi_{\alpha,1-\alpha}(t,s)}{(4\pi s)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4s} f(y)\, dy\, ds, \tag{41}$$

which is positive if $f \geq 0$.

Example 5. Let $1 < \alpha \leq 2$. Consider the following fractional wave equation:

$$\begin{aligned} D_t^{\alpha} u(t,x) &= \Delta u(t,x), \quad t > 0, x \in \mathbb{R}, \\ u(0,x) &= f(x), u_t(0,x) = g(x) \end{aligned} \tag{42}$$

on $L^p(\mathbb{R})$ $(1 \leq p < \infty)$. This is exactly the wave equation on the real line; its solution is given by d'Alembert's formula

$$u_2(t,x) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s)\, ds, \tag{43}$$

which is positive if both f and g are positive. Thus, by the subordination principle and (21), the solution to (41) is given by

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_0^\infty \psi_{\alpha 2,1-\alpha 2}(t,s)[f(x+s)+f(x-s)]ds \\ &\quad + \frac{1}{2}\int_0^t\int_0^\infty \psi_{\alpha 2,1-\alpha 2}(\tau,s)[g(x+s)+g(x-s)]dsd\tau \\ &= \frac{1}{2}\int_0^\infty \psi_{\alpha 2,1-\alpha 2}(t,s)[f(x+s)+f(x-s)]ds \\ &\quad + \frac{1}{2}\int_0^\infty \psi_{\alpha 2,2-\alpha 2}(t,s)[g(x+s)+g(x-s)]ds, \end{aligned} \tag{44}$$

where we have used (16). So, the solution to (41) is positive if both f and g are positive.

Remark 9. The fundamental solution for fractional diffusion equations (38) and (41) on $\mathbb{R}^N$ was first given by Schneider and Wyss in [22], where it was also shown that the fundamental solution changes sign in dimension $N \geq 2$ *if* $\alpha > 1$. And in the case $\alpha > 1$ and $N = 1$, the positivity of the fundamental solution was proved in [21].

Example 6. Consider the following fractional differential equation for fractional Laplacian:

$$\begin{aligned} D_t^\alpha u(t,x) &= -(-\Delta)^\beta u(t,x), \quad t>0, x\in\mathbb{R}^N, \\ u(0,x) &= f(x) \end{aligned} \tag{45}$$

on $L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$, where $\alpha, \beta \in (0,1)$. Let $T(t)$ be the C_0-semigroup generated by the Laplacian Δ as in (39), by Lemma 5(i), $-(-\Delta)^\beta$ generates an analytic semigroup $T_\beta(t)$ for every $\beta \in (0,1)$, which is given by

$$T_\beta(t)f(x) = \int_0^\infty \psi_{\beta,0}(\tau,t)(T(\tau)f)(x)d\tau. \tag{46}$$

Therefore, the solution for (44) can be represented by

$$\begin{aligned} u(t,x) &= \int_0^\infty \psi_{\alpha,1-\alpha}(t,s)T_\beta(s)f(x)ds \\ &= \int_0^\infty\int_0^\infty \psi_{\alpha,1-\alpha}(t,s)\psi_{\beta,0}(\tau,s)(T(\tau)f)(x)d\tau ds, \end{aligned} \tag{47}$$

which is positive if $f \geq 0$.

Remark 10. Kemppainen gave the fundamental solution for (44) in terms of the Fox H-functions in [23] and its positivity. It turns out that the fundamental solution is positive only if $\alpha \in (0,1]$ *and* $\beta \in (0,1]$, *or* $\alpha \in (1,2)$, $\alpha \leq 2\beta \leq 2$ *and* $N = 1$.

4. Resolvent Positive Operators and Positive Fractional Resolvent Families

Throughout the rest of this paper, we assume that E is an ordered Banach space with normal and generating cone E_+ . We will investigate in this section the relations between resolvent positive operators and the generators of positive fractional resolvent families.

Definition 11. An operator A on E is called resolvent positive if there exists $\omega \in \mathbb{C}$ such that $(\omega,\infty) \subset \rho(A)$ *and* $R(\lambda, A) \geq 0$ (i.e., $R(\lambda, A)E_+ \subseteq E_+$) for all $\lambda > \omega$.

An α-times resolvent family $S_\alpha(t)$ is called positive if $S_\alpha(t) \geq 0$ for $t \geq 0$.

We denote by $A \in \mathscr{C}_+^\alpha$ to mean that A generates a positive exponentially bounded α-times resolvent family. And by $A \in \mathscr{A}_+^\alpha$, we mean that A generates an analytic exponentially bounded α-times resolvent family which is positive on $[0,\infty)$.

Remark 12.

(i) Let A be a resolvent positive operator. Then

$$s(A) = \inf\{w \in \mathbb{R} : (w,\infty) \subset \rho(A), R(\lambda,A) \geq 0 \text{ for all } \lambda > w\}, \tag{48}$$

and the function $R(\cdot, A)$ is decreasing on $(s(A),\infty)$ [1, 11]

(ii) If A generates a positive α-times resolvent family $S_\alpha(t)$, then (17) has a positive mild solution $S_\alpha(t)x_0$ for every initial value $x_0 \in E_+$, and by (21), the mild solutions to (20) are also positive if x_0, $x_1 \in E_+$

(iii) By the subordination principle (Lemma 5), if $A \in \mathscr{C}_+^\alpha$, then $A \in \mathscr{A}_+^\beta$ for every $0 < \beta < \alpha$. More general results for the positivity-preserving concerning the fractional powers will be given in Proposition 18

The positivity of an α-times resolvent family is characterized in the following result.

Theorem 13. *Let $\alpha \in (0,2]$ and A be the generator of an exponentially bounded α -times resolvent family $S_\alpha(t)$ on E.*

(i) If $\alpha \in (1,2]$, then $S_\alpha(t)$ is positive for $t \geq 0$ if and only if

$$(-1)^k \frac{d^k}{d\lambda^k}\left[\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\right] \geq 0, \quad \forall k \in \mathbb{N}\cup\{0\}, \tag{49}$$

for λ large enough

(ii) If $\alpha \in (0,1]$, then $S_\alpha(t)$ is positive for $t \geq 0$ if and only if $R(\lambda, A) \geq 0$ for λ large enough

Proof.

(i) Suppose that $\|S_\alpha(t)\| \le M e^{\omega t}$ for some constants M and ω, then by Lemma 2 $\lambda^\alpha \in \rho(A)$ for $\lambda > \omega$ and (22) holds; it then follows that

$$(-1)^k \frac{d^k}{d\lambda^k}\left[\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\right] = \int_0^\infty e^{-\lambda t} t^k S_\alpha(t)x\,dt. \quad (50)$$

The necessity of our assertion follows immediately from the above identity. The sufficiency follows from the Post-Widder inversion formula:

$$\begin{aligned} S_\alpha(t)x &= \lim_{k\longrightarrow\infty} (-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} S_\alpha \wedge^{(k)}\left(\frac{k}{t}\right) \\ &= \lim_{k\longrightarrow\infty} \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} (-1)^k \frac{d^k}{d\lambda^k}\left[\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\right]\Bigg|_{\lambda=k/t} x \end{aligned} \quad (51)$$

(ii) The necessity follows from (22) and the positivity of $S_\alpha(t)$. It remains to show the sufficiency in the case $\alpha \in (0,1)$. This follows from the asymptotic formula below (see [13] (Proposition 2.11)):

$$S_\alpha(t) = \lim_{n\longrightarrow\infty} \frac{1}{n!}\sum_{k=1}^{n+1} b^\alpha_{k,n+1}\left(I - \left(\frac{t}{n}\right)^\alpha A\right)^{-k} x, \quad (52)$$

where $b^\alpha_{j,k}$ are given by the recurrence relations

$$\begin{aligned} b^\alpha_{1,1} &= 1, \\ b^\alpha_{j,k} &= (k-1-j\alpha) b^\alpha_{j,k-1} + \alpha(j-1) b^\alpha_{j-1,k-1}, \quad 1 \le j \le k, k = 2, 3, \cdots, \\ b^\alpha_{j,k} &= 0, \quad j > k, k = 1, 2, \cdots, \end{aligned} \quad (53)$$

and the fact that the coefficients $b^\alpha_{j,k} \ge 0$ when $\alpha \in (0,1)$ and A is resolvent positive. □

Remark 14. When $1 < \alpha \le 2$, $R(\lambda, A) \ge 0$ for λ large enough does not imply that $S_\alpha(t) \ge 0$ for all $t \ge 0$. Since the scalar function cos (ax) oscillates for every $a \in \mathbb{R}$, it is known that for $1 < \alpha < 2$, the function $E_\alpha(-ax)$ has finite zeros on the positive real axis if $a > 0$ (see also Example 1) while $(\lambda + a)^{-1}$ remains positive if λ is large enough.

Some examples are given in [11] to illustrate that a densely defined resolvent positive operator A may not generate a C_0-semigroup. To guarantee such operator generates a C_0-semigroup, some additional conditions are needed. Among them are the interior of E_+ which is nonempty, $D(A)_+$ is cofinal in E_+, the resolvent of A is bounded below, and so on. However, we will show in our main theorem that a densely defined resolvent positive operator always generates a fractional resolvent family if a uniform boundedness condition of its resolvents is satisfied, and such condition is also necessary for the operator to generate a fractional resolvent family.

Theorem 15. *Let A be a closed densely defined operator on E. Then $A \in \mathscr{C}^\alpha_+$ for some $\alpha \in (0, 1)$ if and only if A is resolvent positive and there exists some $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, and*

$$\sup\{\|\lambda R(\lambda, A)\| : \lambda \ge \omega\} < \infty. \quad (54)$$

To prove the theorem, we need the following result from [11], which says that a densely defined resolvent positive operator can always generate a once integrated semigroup.

Lemma 16. *Let A be a resolvent positive operator on an ordered Banach space X. If $D(A)$ is dense, then A generates a once integrated semigroup $S(t)$ such that*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t), \quad \lambda > \omega, \quad (55)$$

when $\omega > s(A)$. Moreover, $S(t)$ satisfies $S(0) = 0$, $0 \le S(s) \le S(t)$ when $0 \le s \le t$ and $\lim_{t\longrightarrow\infty} S(t) = -A^{-1}$ if $s(A) < 0$.

Proof of Theorem 15. First, we show the sufficiency. Suppose that A is a densely defined resolvent positive operator satisfying (53). Set $B = A - \omega I$, then B is resolvent positive with $s(B) < 0$. For $\lambda > 0$, we have

$$\lambda R(\lambda, B) = \lambda R(\lambda + \omega, A) = (\lambda + \omega) R(\lambda + \omega, A) - \omega R(\lambda + \omega, A); \quad (56)$$

it then follows

$$\|\lambda R(\lambda, B)\| \le \sup\{\|\mu R(\mu, A)\| : \mu \ge \omega\} + C|\omega| \cdot \|R(\omega, A)\| \quad (57)$$

for some constant C since $R(\lambda + \omega, A) \le R(\omega, A)$ (Remark 12(i)) and E_+ is normal. Therefore, we obtain

$$\sup\{\|\lambda R(\lambda, B)\| : \lambda > 0\} < \infty. \quad (58)$$

By Lemma 16, B generates integrated semigroup $S(t)$ such that

$$R(\lambda, B) = \int_0^\infty e^{-\lambda t} dS(t), \quad \mathrm{Re}\,\lambda > 0, \quad (59)$$

and $S(t)$ satisfies $S(0) = 0$, $0 \le S(s) \le S(t)$ when $0 \le s \le t$ and $\lim_{t\longrightarrow\infty} S(t) = R(0, B)$. It follows from (58) that for λ satisfying $\mathrm{Re}\lambda > 0$, $x \in E_+$ and $x' \in {E_+}'$

$$\left|\left\langle R(\lambda, B)x, x'\right\rangle\right| \le \left|\left\langle \int_0^\infty e^{-\lambda t}\, dS(t)x, x'\right\rangle\right| = \left|\int_0^\infty e^{-\lambda t}\, d\left\langle S(t)x, x'\right\rangle\right| \le \int_0^\infty e^{-\mathrm{Re}\,\lambda t}\, d\left\langle S(t)x, x'\right\rangle = \left\langle R(\mathrm{Re}\,\lambda, B)x, x'\right\rangle. \tag{60}$$

Since E_+ is generating and normal, so is $E_+{}'$ [39]. For every $x' \in E'$, there is a decomposition $x' = x_1{}' - x_2{}'$ with $x_1{}', x_2{}' \in E_+{}'$ satisfying $\max\{\|x_{1'}\|, \|x_{2'}\|\} \le M'\|x'\|$ for some constant $M' \ge 1$. Thus, for $x \in E_+$,

$$\|R(\lambda, B)x\| = \sup_{x'\in E', \|x'\|\le 1}\left|\left\langle R(\lambda, B)x, x'\right\rangle\right| \le 2 \sup_{y'\in E'_+, \|y'\|\le M'}\left|\left\langle R(\lambda, B)x, y'\right\rangle\right| \le 2 \sup_{y'\in E'_+, \|y'\|\le M'}\left|\left\langle R(\mathrm{Re}\,\lambda, B)x, y'\right\rangle\right| \le 2M'\|R(\mathrm{Re}\,\lambda, B)x\|. \tag{61}$$

Again, since E_+ is generating, there exists another constant $M > 1$ such that for every $x \in E$, there is a decomposition $x = x_1 - x_2$ with $x_1, x_2 \in E_+$ satisfying $\max\{\|x_1\|, \|x_2\|\} \le M\|x\|$. Thus, for $x \in E$,

$$\|R(\lambda, B)x\| \le 2M'(\|R(\mathrm{Re}\,\lambda, B)x_1\| + \|R(\mathrm{Re}\,\lambda, B)x_2\|) \le 4MM'\|R(\mathrm{Re}\lambda, B)\|\cdot\|x\|. \tag{62}$$

Therefore, we obtain

$$\sup\{\|\mathrm{Re}\,\lambda\, R(\lambda, B)\| : \mathrm{Re}\,\lambda > 0\} \le 4MM' \sup\{\|\mu R(\mu, B)\| : \mu > 0\} < \infty. \tag{63}$$

Thanks to this relation, we conclude by Lemma 3 that $B \in \mathscr{A}^\alpha$ for any $\alpha \in (0, 1)$. Since $A = B + \omega I$, by Lemma 4, A also generates analytic α-times resolvent family $S_\alpha(t)$. The positivity of $S_\alpha(t)$ follows from Theorem 13 since A is resolvent positive.

Conversely, if A generates a positive exponentially bounded α-times resolvent family $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \le Me^{\omega_0 t}$ for all $t \ge 0$, by Theorem 13, A is resolvent positive and by Lemma 2,

$$\lambda^{\alpha-1}R(\lambda^\alpha, A) = \int_0^\infty e^{-\lambda t}S_\alpha(t)dt, \quad \lambda > \omega_0. \tag{64}$$

It then follows

$$\left\|\lambda^{\alpha-1}R(\lambda^\alpha, A)\right\| \le \frac{M}{\lambda - \omega_0}, \quad \lambda > \omega_0. \tag{65}$$

Choose $\omega > s(A)$ such that $\omega^{1/\alpha} > \omega_0$, then for $\lambda \ge \omega^{1/\alpha}$, we have $\lambda^\alpha \in \rho(A)$ with

$$\|\lambda^\alpha R(\lambda^\alpha, A)\| = \left\|\lambda^{\alpha-1}(\lambda - \omega_0)R(\lambda^\alpha, A) + \omega_0\lambda^{\alpha-1}R(\lambda^\alpha, A)\right\| \le M + \frac{\omega_0 M}{\omega^{1/\alpha} - \omega_0}, \tag{66}$$

and thus, (53) follows. □

It is remarkable that condition (53) is independent of $\alpha \in (0, 1)$. So, if $A \in \mathscr{C}_+^\alpha$ for some $0 < \alpha < 1$, then $A \in \mathscr{C}_+^\beta$ for all $0 < \beta < 1$. On the other hand, by the subordination principle (Lemma 5(ii)), $A \in \mathscr{C}^\beta$ implies that $A \in \mathscr{A}^\alpha$ for all $0 < \alpha < \beta \le 2$. Therefore, from the "positivity" of an exponentially bounded α-times resolvent family with $\alpha \in (0, 1)$, one can derive its "analyticity."

Theorem 17. *Let $\alpha \in (0, 1)$. If $A \in \mathscr{C}_+^\alpha$, then $A \in \mathscr{A}_+^\beta$ for all $\beta \in (0, 1)$. In particular, $A \in \mathscr{C}_+^\alpha$ implies $A \in \mathscr{A}_+^\alpha$.*

The next example provides an example of a resolvent positive operator satisfying (53). By Theorem 15, such operator can generate α-times resolvent family for every $\alpha \in (0, 1)$; however, it is not the generator of a C_0-semigroup.

Example 7. Let $E = C_0[0, 1] = \{f \in C[0, 1]: f(0) = 0\}$, $0 < \theta < 1$ and $A : D(A) \longrightarrow E$ defined by

$$(Af)(x) = \begin{cases} -f'(x) + \dfrac{\theta}{x}f(x), & x \in (0, 1], \\ 0, & x = 0, \end{cases} \tag{67}$$

with $D(A) = \{f \in C^1[0, 1]: f'(0) = f(0) = 0\}$. Then the following assertions hold:

(i) A is resolvent positive and satisfies

$$\sup\{\|\mu R(\mu, A)\|: \mu \ge 0\} \le \frac{1}{1-\theta} \tag{68}$$

(ii) A generates an α-times resolvent family for every $\alpha \in (0, 1)$ but not a C_0-semigroup

(iii) A generates a β-times integrated semigroup $T_\beta(t)$ for every $\beta > 0$ which is given by

$$\left(T_\beta(t)f\right)(x) = \int_0^{x\wedge t}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}x^\theta(x-s)^{1-\theta}f(x-s)ds. \tag{69}$$

The proof of (i) was given in [11]. By Theorem 15, we obtain that A generates an α-times resolvent family for any $\alpha \in (0, 1)$. But as showed in [11], A is not a generator of a C_0-semigroup. This is (ii).

Finally, if we define $T_\beta(t)$ by (68), it is not hard to show that $T_\beta(t)$ is bounded on E. By a routine calculation, one can verify that $T_\beta(t)$ is the β-times integrated semigroup generated by A.

By the theory of fractional powers developed by Martínez and Sanz [40], we are able to show that the fractional power of a resolvent positive operator is also resolvent positive.

Proposition 18. *Suppose that A is a resolvent positive operator satisfying $(0,\infty) \subset \rho(A)$ and $\sup\{\lambda\|R(\lambda, A)\|: \lambda > 0\} < \infty$. Then for $0 < \beta \le 1$, the operator $-(-A)^\beta$ is resolvent positive and*

$$\sup\left\{\lambda\left\|R\left(\lambda,-(-A)^\beta\right)\right\|: \lambda > 0\right\} \le \sup\{\lambda\|R(\lambda, A)\|: \lambda > 0\}. \tag{70}$$

If in addition $D(A)$ is dense, then so is $D(-(-A)^\beta)$ and $-(-A)^\beta \in \mathscr{A}_+^\alpha$ for all $\alpha \in (0, 1)$.

Proof. By [40] (Proposition 5.3.2), we have the uniform boundedness of the resolvent of the operator $-x(-A)^\beta$ and the representation of its resolvents:

$$R\left(\lambda,-(-A)^\beta\right) = \frac{\sin\beta\pi}{\pi}\int_0^\infty \frac{\mu^\beta}{\lambda^2 + 2\lambda\mu^\beta\cos\beta\pi + \mu^{2\beta}} R(\mu, A)d\mu, \quad \lambda > 0; \tag{71}$$

from this formula, one can derive the positivity of $R(\lambda,-(-A)^\beta)$ from the fact that A is resolvent positive. The last assertion follows immediately from Theorems 15 and 17. □

Remark 19.

(i) From Proposition 18, one immediately has the positivity of the solution to (44) with positive initial data

(ii) The subordination relations between the fractional resolvent families generated by A and $-(-A)^\beta$ were established in [27], while the positivity of the subordination function was discussed in [41]

At the end of this section, we consider the following fractional inhomogeneous equation on E with $\alpha \in (0, 1)$:

$$\begin{gathered} D_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,\tau), \\ u(0) = x_0, \end{gathered} \tag{72}$$

where $x_0 \in E$ and $f \in C([0,\tau), E)$. Recall that u is a mild solution of (71) if $u \in C([0,\tau), E)$, $(g_\alpha * u)(t) \in D(A)$ for $t \in [0,\tau)$ and

$$u(t) = x_0 + (g_\alpha * f)(t) + A(g_\alpha * u)(t), \quad t \in [0,\tau). \tag{73}$$

Proposition 20. *Let A be a densely defined resolvent positive operator satisfying (53), $x_0 \in E$ and $f \in C([0,\tau), E)$.*

(i) If $u(t)$ is a mild solution of (71), then

$$u(t) = S_\alpha(t)x_0 + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \in (0,\tau), \tag{74}$$

where $S_\alpha(t)$ is the α-times resolvent family generated by A

(ii) If $f = f(0) + \int_0^t f'(s)ds$ with $f' \in L^1([0,\tau), E)$, then (71) has a unique mild solution

$$u(t) = S_\alpha(t)x_0 + (g_\alpha * S_\alpha)(t)f(0) + \left(g_\alpha * S_\alpha * f'\right)(t), \quad t \in (0,\tau). \tag{75}$$

If in addition $x_0, f_0 \in E_+$ and $f' \in L^1([0,\tau), E_+)$, then $u(t) \ge 0$ for all $t \in [0,\tau)$.

(iii) If $f = f(0) + (g_{1-\alpha} * h)(t)$ with $h \in L^1([0,\tau), E)$, then (71) has a unique mild solution

$$u(t) = S_\alpha(t)x_0 + (g_\alpha * S_\alpha)(t)f(0) + (S_\alpha * h)(t), t \in (0,\tau). \tag{76}$$

If in addition $x_0, f_0 \in E_+$ and $h \in L^1([0,\tau), E_+)$, then $u(t) \ge 0$ for all $t \in [0,\tau)$.

Proof. Our assumptions imply that A generates a positive α-times resolvent family $S_\alpha(t)$ for every $\alpha \in (0, 1)$ by Theorem 15. The assertion (i) was proved in [27]. The first half of (ii) follows from [13] (Proposition 1.2); the second half follows from the positivity of $S_\alpha(t)$, x_0, $f(0)$, and f'.

Now, we prove (iii). If $f = f(0) + (g_{1-\alpha} * h)(t)$ and $u(t)$ is a mild solution to (71), we have by (i)

$$\begin{aligned} u(t) &= S_\alpha(t)x_0 + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t) = S_\alpha(t)x_0 \\ &\quad + \frac{d}{dt}[(g_{1+\alpha} * S_\alpha)(t)f(0) + (g_\alpha * S_\alpha * g_{1-\alpha} * h)(t)] \\ &= S_\alpha(t)x_0 + (g_\alpha * S_\alpha)(t)f(0) + (S_\alpha * h)(t); \end{aligned} \tag{77}$$

this is (75). Conversely, if $u(t)$ is given by the above formula, then $u(t)$ is continuous by [1] (Proposition 1.3.4), and it is obvious that $(g_\alpha * u)(t) \in D(A)$ with

$$\begin{aligned} A(g_\alpha * u)(t) &= A(g_\alpha * S_\alpha)(t)x_0 + A(g_\alpha * g_\alpha * S_\alpha)(t)f(0) \\ &\quad + A(g_\alpha * S_\alpha * h)(t) = S_\alpha(t)x_0 - x_0 \\ &\quad + (g_\alpha * S_\alpha)(t)f(0) - g_{1+\alpha}(t)f(0) + (S_\alpha * h)(t) \\ &\quad - (g_1 * h)(t) = u(t) - x_0 - g_{1+\alpha}(t)f(0) \\ &\quad - (g_{1-\alpha} * g_\alpha * h)(t) = u(t) - x_0 - (g_\alpha * f)(t). \end{aligned} \tag{78}$$

Thus, $u(t)$ is a mild solution of (71). □

5. Positive Perturbations of Positive Fractional Resolvent Families

In this section, we consider the positive perturbations of positive fractional resolvent families. Let us begin with bounded perturbations. If $A \in \mathscr{C}^\alpha$ for some $\alpha \in (1, 2]$ on a Banach space and B is a bounded operator, we know by Lemma 4 that $A + B \in \mathscr{C}^\alpha$. We show in the following result that the positivity can be preserved if the perturbation is positive.

Theorem 21. *Let $\alpha \in (1, 2]$ and $A \in \mathscr{C}_+^\alpha$. If B is a bounded positive operator on E, then $A + B \in \mathscr{C}_+^\alpha$.*

Proof. Let $S_\alpha(t)$ be the α-times resolvent family generated by A. By Lemma 4, $A + B$ generates an α-times resolvent family $S_\alpha(t\,; A + B)$, and it is given by a modified Dyson-Phillips series:

$$S_\alpha(t\,; A + B) = \sum_{n=0}^{\infty} S_{\alpha,n}(t), \tag{79}$$

with $S_{\alpha,0}(t) = S_\alpha(t)$, and

$$S_{\alpha,n}(t) = \int_0^t (g_{\alpha-1} * S_\alpha)(t - s) B S_{\alpha,n-1}(s) ds, \quad n = 1, 2, \cdots. \tag{80}$$

From the above representation formula, one can conclude that $S_\alpha(t\,; A + B) \geq 0$ by induction. □

For unbounded perturbations, we first consider a perturbation of Miyadera-Voigt-type [42].

Theorem 22. *Let $\alpha \in (1, 2]$. Suppose that A generates a positive α -times resolvent family $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \leq M e^{\omega t}$, and $B : D(A) \longrightarrow E$ is a positive operator. If there are constants $\mu > \omega$ and $\gamma \in [0, 1)$ such that*

$$\int_0^\infty e^{-\mu r} \left\| B \int_0^r g_{\alpha-1}(r - s) S_\alpha(s) x \, ds \right\| dr \leq \gamma \|x\|, \quad x \in D(A), \tag{81}$$

then $A + B \in \mathscr{C}_+^\alpha$.

Proof. It follows from [42] (Theorem 3.1) that $A + B$ generates an α-times resolvent family $S_\alpha(t\,; A + B)$. It remains to show the positivity of $S_\alpha(t\,; A + B)$. By Theorem 13, it is enough to show that

$$(-1)^k \frac{d^k}{d\lambda^k} \left[\lambda^{\alpha-1} R(\lambda^\alpha, A + B) \right] \geq 0, \quad k = 0, 1, 2, \cdots, \tag{82}$$

for λ large enough.

We first show that our assumptions imply $\|BR(\lambda^\alpha, A)\| \leq \gamma < 1$ for $\lambda \geq \mu$. Indeed, since $(\omega^\alpha, \infty) \subset \rho(A)$ by Lemma 2, the operator $BR(\lambda^\alpha, A)$: $E \longrightarrow E$ is bounded and positive for $\lambda \geq \mu$. For $x \in D(A)$, using (80), we get

$$\begin{aligned} \|BR(\lambda^\alpha, A)x\| &= \left\| B \int_0^\infty e^{-\lambda t} (g_{\alpha-1} * S_\alpha)(t) x \, dt \right\| \\ &\leq \int_0^\infty e^{-\lambda t} \|B(g_{\alpha-1} * S_\alpha)(t) x\| dt \\ &\leq \int_0^\infty e^{-\mu t} \|B(g_{\alpha-1} * S_\alpha)(t) x\| dt \leq \gamma \|x\|. \end{aligned} \tag{83}$$

The positivity of $S_\alpha(t)$ implies that $\lambda^{\alpha-1} R(\lambda^\alpha, A)$ is completely monotonic by Theorem 13. Since $R(\lambda^\alpha, A) = \lambda^{1-\alpha} \cdot \lambda^{\alpha-1} R(\lambda^\alpha, A)$, by Lemma 8, $R(\lambda^\alpha, A)$ is completely monotonic and so is $BR(\lambda^\alpha, A)$ since $B : D(A) \longrightarrow E$ is positive. Thus, (81) follows from Lemma 8, the identity

$$\begin{aligned} \lambda^{\alpha-1} R(\lambda^\alpha, A + B) &= \lambda^{\alpha-1} R(\lambda^\alpha, A)(I - BR(\lambda^\alpha, A))^{-1} \\ &= \lambda^{\alpha-1} R(\lambda^\alpha, A) \sum_{n=0}^{\infty} (BR(\lambda^\alpha, A))^n, \end{aligned} \tag{84}$$

the complete monotonicity of $\lambda^{\alpha-1} R(\lambda^\alpha, A)$ and $R(\lambda^\alpha, A)$, and the absolute convergence of the series $\sum_{n=0}^{\infty} (BR(\lambda^\alpha, A))^n$ since $\|BR(\lambda^\alpha, A)\| < 1$ for $\lambda \geq \mu$. □

Next, we consider the relative bounded perturbations. The following lemma is needed.

Lemma 23 (see [11]). *Let A be a resolvent positive operator and $B : D(A) \longrightarrow E$ a positive operator. If $r(B(R(\lambda, A)) < 1$ for some $\lambda > s(A)$, then $A + B$ with domain $D(A)$ is a resolvent positive operator and $s(A + B) < \lambda$. Moreover, if $\sup \{\|\mu R(\mu, A)\|: \mu \geq \lambda\} < \infty$, then also $\sup \{\|\mu R(\mu, A + B)\|: \mu \geq \lambda\} < \infty$.*

Theorem 24. *Let $\alpha \in (0, 2)$. Suppose that $A \in \mathscr{C}_+^\alpha$, $B : D(B) \longrightarrow E$ is a positive and closed linear operator such that $D(B) \supset D(A)$ and there are constants $a, b \geq 0$ such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in D(A). \tag{85}$$

If any of the following conditions is satisfied:

(i) If $\alpha \in (0, 1)$

(ii) If $\alpha \in [1, 2)$ and $S_\alpha(t)$ is analytic

then there exists $\delta > 0$, such that if $0 \leq a \leq \delta$, then $A + B \in \mathscr{A}_+^\alpha$.

Proof. Since $A \in \mathscr{C}_+^\alpha$, it follows from Theorem 15 and its proof that there exists $\omega > 0$ such that

$$M \coloneqq \sup \{\lambda R(\lambda, A): \lambda \geq \omega\} < \infty. \tag{86}$$

By (84), for $x \in X$ and $\lambda > \omega$,

$$\begin{aligned} \|BR(\lambda, A)x\| &\le a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \\ &= a\|\lambda R(\lambda, A)x - x\| + b\|R(\lambda, A)x\| \\ &\le a(M+1)\|x\| + \frac{bM}{\lambda}\|x\|. \end{aligned} \tag{87}$$

If we choose $0 < \delta_1 < 1/(1+M)$, then $\|BR(\lambda, A)\| < 1$ when $\lambda \ge \omega_1$ for some $\omega_1 \ge \omega$.

Now, suppose that $\alpha \in (0,1)$. Using Lemma 23, we also have

$$\sup\{\lambda R(\lambda, A+B) : \lambda \ge \omega_1\} < \infty. \tag{88}$$

Applying Theorem 15 to the operator $A+B$, we obtain that $A+B \in \mathscr{C}_+^\alpha$, which implies that $A+B \in \mathscr{A}_+^\alpha$ by Theorem 17.

If (ii) holds, then $S_\alpha(t)$ is analytic. By Lemma 4, there is some $\delta_2 > 0$ such that if $0 \le a \le \delta_2$, then $A+B \in \mathscr{A}^\alpha$. Choose $\delta = \min\{\delta_1, \delta_2\}$, then when $0 \le a \le \delta_2$, $A+B \in \mathscr{A}^\alpha$ and $\|B R(\lambda, A)\| < 1$ when $\lambda \ge \omega_1$. The positivity can be proved as in the proof of Theorem 22. □

Remark 25.

(i) (84) can be replaced by the following condition: the operator B is closed with $D(B) \supset D(A)$ and

$$\|Bx\| \le a\|Ax\| + b\|x\|, \quad x \in K, \tag{89}$$

where K is a core of the generator A, i.e., $\overline{\{(x, Ax); x \in K\}} = \{(x, Ax); x \in D(A)\}$. In fact, for any $x \in D(A)$, there exists a sequence of elements $\{x_n\} \in K$, such that $(x_n, Ax_n) \longrightarrow (x, Ax)$, $n \longrightarrow \infty$. Thus, $x_n \longrightarrow x$ and $Ax_n \longrightarrow Ax$. Since both $\{x_n\}$ and $\{Ax_n\}$ are Cauchy sequences, by (88), $\{Bx_n\}$ is also a Cauchy sequence. By the closedness of B, $Bx_n \longrightarrow Bx$. For any $x_n \in K$, we have by (88) that $\|Bx_n\| \le a\|Ax_n\| + b\|x_n\|$. By letting $n \longrightarrow \infty$, one gets (84) for every $x \in D(A)$

(ii) It is known that if A is closed and B satisfies (84) with $a < 1$, then $(A+B, D(A))$ is a closed operator [2] (Ch. III, Lemma 2.4)

(iii) Assume that $A \in \mathscr{C}_+^\alpha$ for some $\alpha \in (0,1)$ and $B : D(A) \longrightarrow E$ is a positive operator. If for any $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that

$$\|Bx\| \le \varepsilon\|Ax\| + b_\varepsilon\|x\|, \quad \forall x \in D(A) \tag{90}$$

Then, it follows from Theorems 15 and 17 that $A+B \in \mathscr{A}_+^\alpha$.

We end this paper with two examples related to Schrödinger operators [43, 44] to illustrate our perturbation theory.

Example 8. Let $E = L^1(\mathbb{R}^N)$. Consider the following fractional Schrödinger equation:

$$\begin{aligned} D_t^\alpha u(t,x) &= \Delta u(t,x) + V(x)u(t,x), \quad t > 0, x \in \mathbb{R}^N, \\ u(0,x) &= u_0(x), \end{aligned} \tag{91}$$

where Δ is the Laplacian and V is a potential function such that $V(x) \ge 0$. Let A be the closure of the Laplacian Δ defined on $C_0^\infty(\mathbb{R}^N)$. As a perturbation, we choose

$$Bf(x) \coloneqq V(x)f(x) \quad \text{for } f \in D(B) \coloneqq \{f \in E : V(\cdot)f(\cdot) \in E\}. \tag{92}$$

It is clear that B is positive and closed on E. Moreover, we assume that the potential function $V \in K_N$, the Kato class, that is,

$$\begin{aligned} &\lim_{\delta \longrightarrow 0} \sup_{x \in \mathbb{R}^N} \int_{|x-y| \le \delta} |x-y|^{2-N} |V(y)| dy = 0, \quad N \ge 3, \\ &\lim_{\delta \longrightarrow 0} \sup_{x \in \mathbb{R}^2} \int_{|x-y| \le \delta} \ln\{|x-y|^{-1}\} |V(y)| dy = 0, \quad N = 2, \\ &\sup_{x \in \mathbb{R}} \int_{|x-y| \le 1} |V(y)| dy < \infty, \quad N = 1. \end{aligned} \tag{93}$$

It was proved in [43] (Theorem 4.14) that for every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\|Vf\|_1 \le \varepsilon\|\Delta f\|_1 + C(\varepsilon)\|f\|_1, \quad \forall f \in D(A), \tag{94}$$

if and only if $V \in K_N$. By applying our Theorem 24, we conclude that if $V \in K_N$, then $A + B = \Delta + V \in \mathscr{A}_+^\alpha$ for $\alpha \in (0,1)$ on E. Therefore, for $u_0(x) \ge 0$, there is a positive mild solution in $E = L^1(\mathbb{R}^N)$ to the fractional Schrödinger equation (90) if $\alpha \in (0,1)$.

This is also true if $E = L^p(\mathbb{R}^N)$ for $1 < p < \infty$, since the Schrödinger operator $\Delta + V$ also generates C_0-semigroup on $L^p(\mathbb{R}^N)$.

Next, we consider the positive solution for (90) in the case of $1 < \alpha < 2$. By the analysis in Example 5, this is possible only if $N = 1$. Since the Laplacian Δ generates a cosine operator function,

$$(C(t)u)(x) = \frac{u(x+t) + u(x-t)}{2}, \tag{95}$$

on $E = L^1(\mathbb{R})$, $\Delta \in \mathscr{C}_+^\alpha$ for $0 < \alpha < 2$ by the subordination principle. Provided the potential function $V \ge 0$ and $V \in K_1$, then it follows from (93) and Theorem 24 that $\Delta + V \in \mathscr{A}_+^\alpha$.

Remark 26. By using the higher-order Kato class K_β introduced by Davies and Hinz [45], similar results can be obtained for the fractional Laplacian operator with positive potentials in $K_{2\beta}(0 < \beta < 1)$:

$$D_t^\alpha u(t,x) = -(-\Delta)^\beta u(t,x) + V(x)u(t,x), \quad t>0, x\in\mathbb{R}^N,$$
$$u(0,x) = u_0(x). \tag{96}$$

While the higher-order Kato class $K_{2\beta}$ is defined by

$$K_\beta := \left\{ V : \lim_{\delta\longrightarrow 0} \sup_{x\in\mathbb{R}^N} \int_{|x-y|\le\delta} |x-y|^{2\beta-N} |V(y)| dy = 0 \right\}, \tag{97}$$

when $0<2\beta<N$. Proposition 18 provides one way to show $-(-\Delta)^\beta \in \mathscr{A}_+^\alpha$ for $\alpha\in(0,1)$. To apply our perturbation theory, we can use the following fact from [46]: $V\in K_{2\beta}$ if and only if for every $\varepsilon>0$ there exists $C(\varepsilon)>0$ such that

$$\|Vf\|_1 \le \varepsilon \left\| (-\Delta)^\beta f \right\|_1 + C(\varepsilon)\|f\|_1, \quad \forall f\in D\left((-\Delta)^\beta\right). \tag{98}$$

It is known [43] that for $N\ge 2$,

$$\left\{ V : \sup_{x\in\mathbb{R}^N} \int_{|x-y|\le 1} |V(y)|^p dy<\infty \right\} \subset K_N \subset \cdot \left\{ V : \sup_{x\in\mathbb{R}^N} \int_{|x-y|\le 1} |V(y)| dy<\infty \right\}, \tag{99}$$

if $p>N/2$. And if $V(x)=f(|x|)$ is a spherically symmetric function when $N\ge 3$, then $V\in K_N$ if and only if

$$\sup_{|x|\ge 2} \int_{|x-y|\le 1} |f(y)|\, dy < \infty,$$
$$\int_0^1 r|f(r)|\, dr < \infty. \tag{100}$$

For examples, the solution for (90) is positivity-preserving if the potential V is of the following form:

(i) $|x|^{-a}$ with $0\le a<2$ if $N\ge 3$, in particular the Coulomb potential $|x|^{-1}$ when $N=3$

(ii) $|x|^{-2}|\log|x||^{-b}$ with $b>1$ if $N\ge 3$

(iii) $\sum_{i<j} 1/|x_i - x_j|$ on $\mathbb{R}^{3N}$ with $x_i\in\mathbb{R}^3$ for $i=1,\cdots,N$, the N-body Hamiltonian

At the borderline $L^{N/2}\not\subset K_N$, however, by using the Sobolev estimates, we can obtain the positivity-preserving of (90) with potentials in $L^{N/2}$.

Example 9. We assume $E=L^p(\mathbb{R}^N)$, $3\le N\in\mathbb{Z}_+$, $1\le p<N/2$ *and* $\alpha\in(0,1)$. Let A and B be defined as in Example 8 with $V\in L^{N/2}(\mathbb{R}^N)$ and $V(x)\ge 0$ for all $x\in\mathbb{R}^N$. We will show that B satisfies (89). Let V_n be the truncation of V:

$$V_n(x) = \begin{cases} V(x), & \text{if } |V(x)|\le n, \\ 0, & \text{otherwise.} \end{cases} \tag{101}$$

By Hölder's inequality, we have for every $f\in C_0^\infty(\mathbb{R}^N)$,

$$\|Vf\|_p \le \|(V-V_n)f\|_p + \|V_n f\|_p \le \|V-V_n\|_{N/2}\|f\|_r + \|V_n\|_\infty \|f\|_p, \tag{102}$$

where $2/N + 1/r = 1/p$ and $p<r$; furthermore, by Sobolev's inequality,

$$\|f\|_r \le C\|\Delta f\|_p, \quad \forall f\in C_0^\infty\left(\mathbb{R}^N\right). \tag{103}$$

Thus, we obtain

$$\|Vf\|_p \le C\|V-V_n\|_{N/2}\|\Delta f\|_p + \|V_n\|_\infty\|f\|_p, \quad \forall f\in C_0^\infty\left(\mathbb{R}^N\right). \tag{104}$$

Taking n large enough, one gets (89). Thus, $A+B\in\mathscr{A}_+^\alpha$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

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