





## Research Article

# An Analytical Survey on the Solutions of the Generalized Double-Order $\varphi$ -Integrodifferential Equation

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We study the existence of solutions for a newly configured model of a double-order integrodifferential equation including  $\varphi$ -Caputo double-order  $\varphi$ -integral boundary conditions. In this way, we use the Krasnoselskii and Leray-Schauder fixed point results. Also, we invoke the Banach contraction principle to confirm the uniqueness of the existing solutions. Finally, we provide three examples to illustrate our analytical findings.

## 1. Introduction

An arbitrary order calculus, specifically the fractional order calculus, has been one of the most important subbranches of mathematics in other existing computational and applied sciences. This amount of applicability is for the sake of the high ability of the relations and operators formulated in the mentioned theory. For this reason, most researchers have utilized numerous applied fractional operators in the past years to model various forms of natural processes that occurred in the world. The applicability and diversity of mentioned fractional operators in modeling can be observed in many literatures including [1–16]. In addition, since by making the model on the basis of fractional operators, we obtain more accurate computational results than existing usual models on the basis of integer order operators; thus, this subject motivates all researchers to construct new generalizations of these fractional operators. In other words, some generalized formulations of these operators have been defined to com-

bine the previous operators effectively to avoid confusion when using existing fractional operators (see [17–20]).

In 2017, an extension of the usual Caputo operator entitled  $\varphi$ -Caputo derivative ( $\varphi$ -CapFr) is introduced by Almeida ([21]) in which the kernel of mentioned operator depends on an increasing function  $\varphi$ . The most applied advantage of the  $\varphi$ -CapFr derivative is its flexibility to combine all fractional derivatives introduced before. This extended operator possesses the semigroup property which is vital to obtain the structure of solutions. Hence,  $\varphi$ -CapFr derivative is regarded as a generalized construction of arbitrary order derivatives. By invoking the newly defined  $\varphi$ -CapFr operator and its other generalizations, several limited studies have been done which we refer the reader to those including [22–27].

To review several previous research notes implemented in terms of  $\varphi$ -CapFr operators, we point out a study discussed by Belmor et al. ([28]). In the mentioned article, the authors configured a  $\varphi$ -fractional differential inclusion endowed with  $\varphi$ -integral conditions as

$$\begin{cases} \mathcal{I}_{0^+}^{\sigma^*; \varphi} \omega^*(z) \in \tilde{\mathfrak{B}}(z, \omega^*(z)), \\ \omega^*(0) - \rho_\varphi^* \omega^*(0) = m_1^* \mathcal{R} \mathcal{L} \mathcal{I}_{0^+}^{\zeta_1^*; \varphi} \tilde{h}_*^{-1}(c_1, \omega^*(c_1)), \\ \omega^*(k) - \rho_\varphi^* \omega^*(k) = m_2^* \mathcal{R} \mathcal{L} \mathcal{I}_{0^+}^{\zeta_2^*; \varphi} \tilde{h}_*^{-2}(c_2, \omega^*(c_2)), \end{cases} \quad (1)$$

where  $z \in [0, k]$ ,  $\sigma^* \in (1, 2)$ ,  $c_1, c_2 \in [0, k]$ , and  $m_1^*, m_2^*$  are chosen as arbitrary constants and  $\tilde{\mathfrak{B}} : [0, k] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  introduces a multifunction and  $\tilde{h}_*^{-1}, \tilde{h}_*^{-2} : [0, k] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous along with  $\rho_\varphi^* = (1/\varphi'(z))(d/dz)$ . The authors deduced the required criteria for the existence by the aid of an endpoint concept on the category of the  $\psi$ -weak contractions ([28]).

In the same year, Wahash et al. ([29]) turned to a structure involving the generalized  $\varphi$ -Caputo equation subject to integral conditions as

$$\begin{cases} \mathcal{I}_{0^+}^{\sigma^*; \varphi} \omega^*(z) = \tilde{h}_*(z, \omega^*(z)), \\ \omega^*(0) = m^* \int_0^1 \xi(q)u(q) dq + v, \end{cases} \quad \text{respectively.} \quad (2)$$

where  $z \in [0, 1]$ ,  $\sigma^* \in (0, 1)$ ,  $m^* \geq 0$ , and  $v \in \mathbb{R}_+$ , and further,  $\tilde{h}_* : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  illustrates a continuous function along with  $\xi \in L^1_{\mathbb{R}_+}([0, 1])$ . The upper-lower solution is the method utilized in this work by authors in which the authors invoke a fixed point result on cones. In addition to this, an upper-lower control maps are built with respect to the nonlinear term without a specific monotone condition ([29]).

We can find many works in the literature for describing different phenomena or physical meanings of fractional integrodifferential equations. But, some generalizations of fractional notions have no physical meaning at this time and maybe it will find some new physical interpretations for those in the future. Our results satisfy this second version. There are many works which apply different fractional derivatives such as Caputo-Fabrizio (see for example, [4, 30–33]) and discuss different views on applied mathematical modelings (see for example, [34–39]).

Inspired by the above-implemented works, we intend to formulate and generalize a new category of the nonlinear fractional double-order integrodifferential equation in the generalized fractional  $\varphi$ -operators settings given by

$$\begin{cases} \left[ m_1^* \mathcal{I}_{s_0}^{\sigma^*; \varphi} + (1 - m_1^*) \mathcal{I}_{s_0}^{\rho^*; \varphi} \right] \omega^*(z) = p_1 \int_{s_0}^z \frac{\varphi'(q)(\varphi(z) - \varphi(q))^{\delta_1^* - 1}}{\Gamma(\delta_1^*)} \tilde{h}_*(q, \omega^*(q)) dq + p_2 \int_{s_0}^z \frac{\varphi'(q)(\varphi(z) - \varphi(q))^{\delta_2^* - 1}}{\Gamma(\delta_2^*)} \tilde{f}_*(q, \omega^*(q)) dq, \end{cases} \quad (3)$$

furnished with double-order integral boundary conditions in the  $\varphi$ -Riemann-Liouville frame

$$\begin{cases} \omega^*(s_0) = 0, \\ m_2^* \int_{s_0}^T \frac{\varphi'(q)(\varphi(T) - \varphi(q))^{\theta_1^* - 1}}{\Gamma(\theta_1^*)} \omega^*(q) dq + (1 - m_2^*) \int_{s_0}^T \frac{\varphi'(q)(\varphi(T) - \varphi(q))^{\theta_2^* - 1}}{\Gamma(\theta_2^*)} \omega^*(q) dq = 0, \end{cases} \quad (4)$$

so that  $z \in [s_0, T]$  with  $s_0 \geq 0$  and  $T \in \mathbb{R}^+$ . Moreover, let  $\sigma^*, \rho^* \in (1, 2)$  so that  $\sigma^* - \rho^* > 1$  and  $\delta_1^*, \delta_2^* \in (0, 1)$ ,  $\theta_1^*, \theta_2^* > 0$ ,  $m_1^*, m_2^* \in (0, 1]$ , and  $p_1, p_2 \in \mathbb{R}^+$ . Both real-valued functions  $\tilde{h}_*, \tilde{f}_* : [s_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are supposed to be continuous. The notation  $\mathcal{I}_{s_0}^{\eta; \varphi}$  represents the  $\varphi$ -Caputo derivative ( $\varphi$ -CapFr derivative) of arbitrary order  $\eta \in \{\sigma^*, \rho^*\}$ , and  $\mathcal{R} \mathcal{L} \mathcal{I}_{s_0}^{\rho; \varphi}$  illustrates the  $\varphi$ -Riemann-Liouville integral ( $\varphi$ -RLFr integral) of arbitrary order  $\rho \in \{\delta_1^*, \delta_2^*, \theta_1^*, \theta_2^*\}$ .

It is necessary that all researchers pay attention to this subject that the proposed double-order double- $\varphi$ -CapFr-integrodifferential equation has a novel and unique structure. In other words, the formulated structure for the given fractional double- $\varphi$ -CapFr-integrodifferential problem (3)-(4)

includes two  $\varphi$ -CapFr-derivatives and also four  $\varphi$ -RLFr-integrals with different orders. This combined boundary value problem (BoVaPr) covers many different special cases of various nonlinear integrodifferential equations. Therefore, we emphasize that this kind of the nonlinear double- $\varphi$ -CapFr-integro-differential BoVaPr has not been investigated in any literature so far. In this direction, we apply well-known analytical techniques to derive desired criteria which guarantee the existence aspects of desired solutions for the proposed double-order  $\varphi$ -CapFr-integrodifferential BoVaPr (3)-(4).

The organization of the contents of the current manuscript is as follows. In the next section, some required notions in the context of the generalized  $\varphi$ -calculus are assembled. Section 3 is devoted to establish the main theorems in which

the existence criteria can be obtained under some required conditions. In Section 4, we present three simulative examples to confirm the validity of our analytical findings.

## 2. Fundamental Preliminaries

In the current section, we collect and review some fundamental and auxiliary notions in the framework of our analytical methods applied in this paper. As you know, the concept of the Riemann-Liouville integral of order  $\sigma^* > 0$  for a function  $\omega^* : [0, +\infty) \rightarrow \mathbb{R}$  is defined as

$${}_{\mathcal{RL}}\mathcal{I}_0^{\sigma^*} \omega^*(z) = \int_0^z \frac{(z-q)^{\sigma^*-1}}{\Gamma(\sigma^*)} \omega^*(q) dq, \quad (5)$$

provided that the value of the integral is finite [40, 41]. In this position, let us assume that  $\sigma^* \in (n-1, n)$  so that  $n = 1 + [\sigma^*]$ . For a continuous function  $\omega^* : [0, +\infty) \rightarrow \mathbb{R}$ , the Riemann-Liouville derivative of order  $\rho^*$  is given by

$${}_{\mathcal{RL}}\mathfrak{D}_0^{\sigma^*} \omega^*(z) = \left(\frac{d}{dz}\right)^n \int_0^z \frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)} \omega^*(q) dq, \quad (6)$$

provided that the value of the integral is finite [40, 41]. In the next step, for an absolutely continuous function  $\omega^* \in \mathcal{AC}_{\mathbb{R}}^{(n)}([0, +\infty))$ , the fractional derivative of Caputo type is given by

$${}_{\mathcal{C}}\mathfrak{D}_0^{\sigma^*} \omega^*(z) = \int_0^z \frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)} \omega^{*(n)}(q) dq, \quad (7)$$

provided that the integral is finite-valued [40, 41].

**Definition 1** ([41–43]). Let  $\varphi \in \mathcal{C}^n([s_0, b], \mathbb{R})$  be an increasing function so that  $\varphi'(z) > 0$  for any  $z \in [s_0, b]$ . Then, the  $\varphi$ -Riemann-Liouville integral of order  $\sigma^*$  for an integrable function  $\omega^* : [s_0, b] \rightarrow \mathbb{R}$  with respect to another increasing function  $\varphi$  is defined as

$${}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\sigma^*; \varphi} \omega^*(z) = \frac{1}{\Gamma(\sigma^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^*-1} \omega^*(q) dq, \quad (8)$$

provided that the R.H.S. of above equality is finite-valued.

It is to be noted that if  $\varphi(z) = z$  and  $s_0 = 0$ , then clearly the  $\varphi$ -RLFr integral (8) reduces to the standard Riemann-Liouville integral (5).

**Definition 2** ([41–43]). For  $\varphi \in \mathcal{C}^n([s_0, b], \mathbb{R})$  as above and for a continuous function  $\omega^* : [s_0, +\infty) \rightarrow \mathbb{R}$ , the  $\varphi$ -Riemann-Liouville derivative of order  $\sigma^*$  is given by

$${}_{\mathcal{RL}}\mathfrak{D}_{s_0}^{\sigma^*; \varphi} \omega^*(z) = \frac{1}{\Gamma(n-\sigma^*)} \left(\frac{1}{\varphi'(z)} \frac{d}{dz}\right)^n \int_{s_0}^z \varphi'(q) \cdot (\varphi(z) - \varphi(q))^{n-\sigma^*-1} \omega^*(q) dq, \quad (9)$$

provided that the R.H.S. of the above equality is finite-valued and  $n = 1 + [\sigma^*]$ .

In a manner, if  $\varphi(z) = z$  and  $s_0 = 0$ , then it is obvious that the  $\varphi$ -RLFr derivative (9) reduces to the standard Riemann-Liouville derivative (6). Inspired by these operators, Almeida presented a new  $\varphi$ -version of the Caputo derivative as the following formulation.

**Definition 3** ([21]). For  $\omega^* \in \mathcal{C}^n([s_0, b])$ ,  $\varphi$ -version of the Caputo derivative is illustrated as

$${}_{\mathcal{C}}\mathfrak{D}_{s_0}^{\sigma^*; \varphi} \omega^*(z) = \frac{1}{\Gamma(n-\sigma^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{n-\sigma^*-1} \left(\frac{1}{\varphi'(q)} \frac{d}{dq}\right)^n \omega^*(q) dq, \quad (10)$$

provided that the R.H.S. of the above equality is finite-valued and  $n = 1 + [\sigma^*]$ .

It is notable that if  $\varphi(z) = z$ , then it is obvious that the  $\varphi$ -CapFr derivative (10) reduces to the standard Caputo derivative (7). In the following, some important properties of the  $\varphi$ -Caputo and  $\varphi$ -Riemann-Liouville fractional operators can be seen in the following lemmas.

**Lemma 4** ([21, 41–43]). Assume that  $\sigma^* > 0$ ,  $\rho^* > 0$ , and  $\beta^* > 0$ , and  $\varphi \in \mathcal{C}^n([s_0, b], \mathbb{R})$  is an increasing function so that  $\varphi'(z) > 0$  for any  $z \in [s_0, b]$ . Then, the following statements hold:

- (i1)  ${}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\sigma^*; \varphi} ({}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\rho^*; \varphi} \omega^*)(z) = ({}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\sigma^* + \rho^*; \varphi} \omega^*)(z)$
- (i2)  ${}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\sigma^*; \varphi} (\varphi(z) - \varphi(s_0))^{\beta^*} (y) = ((\Gamma(\beta^* + 1))/(\Gamma(\sigma^* + \beta^* + 1))) (\varphi(y) - \varphi(s_0))^{\sigma^* + \beta^*}$
- (i3)  ${}_{\mathcal{C}}\mathfrak{D}_{s_0}^{\sigma^*; \varphi} (\varphi(z) - \varphi(s_0))^{\beta^*} (y) = ((\Gamma(\beta^* + 1))/(\Gamma(\beta^* - \sigma^* + 1))) (\varphi(y) - \varphi(s_0))^{\beta^* - \sigma^*}, (\beta^* > -1)$
- (i4)  ${}_{\mathcal{RL}}\mathfrak{D}_{s_0}^{\sigma^*; \varphi} ({}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\rho^*; \varphi} \omega^*)(z) = ({}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\rho^* - \sigma^*; \varphi} \omega^*)(z), (\sigma^* < \rho^*)$

For instance, in Figures 1 and 2, we plot the graphs of the  $\varphi$ -CapFr derivative of two arbitrary functions  $\omega(z) = z^6$  and  $\omega(z) = z^5/32$  for  $\varphi(z) = z^2 + 1$  and  $\varphi(z) = z/2$ , respectively.

**Lemma 5** ([21]). Let  $n-1 < \sigma^* < n$  and  $\varphi \in \mathcal{C}^n([s_0, b], \mathbb{R})$  be an increasing function so that  $\varphi'(z) > 0$  for any  $z \in [a, b]$ . Then, for every  $\omega^* \in \mathcal{C}^{n-1}([s_0, b], \mathbb{R})$ , we have

$${}_{\mathcal{RL}}\mathcal{I}_{s_0}^{\sigma^*; \varphi} \left( {}_{\mathcal{C}}\mathfrak{D}_{s_0}^{\sigma^*; \varphi} \omega^* \right) (z) = \omega^*(z) - \sum_{j=0}^{n-1} \frac{(\rho_\varphi)^j \omega^*(s_0)}{j!} (\varphi(z) - \varphi(s_0))^j, \left( \rho_\varphi = \frac{1}{\varphi'(z)} \frac{d}{dz} \right). \quad (11)$$

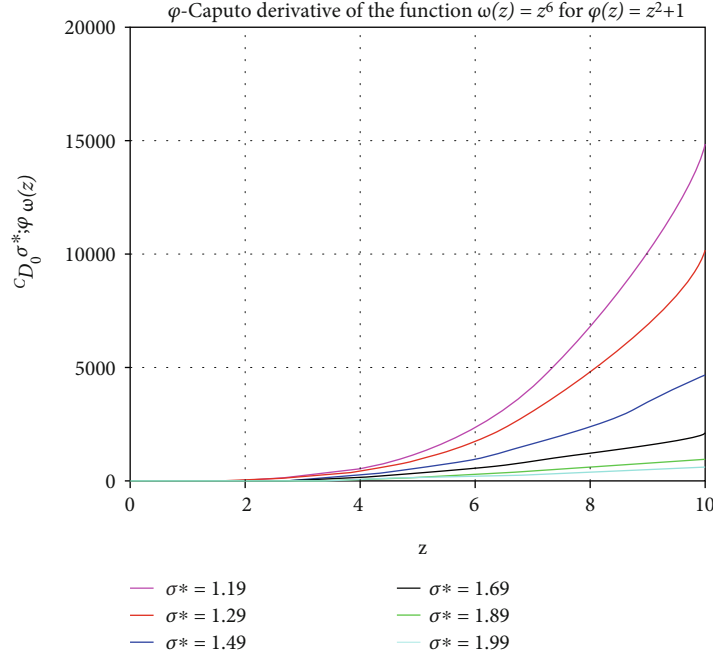


FIGURE 1: The CapFr-derivative of  $\hat{\omega}(z) = z^6$  for  $\varphi(z) = z^2 + 1$ .

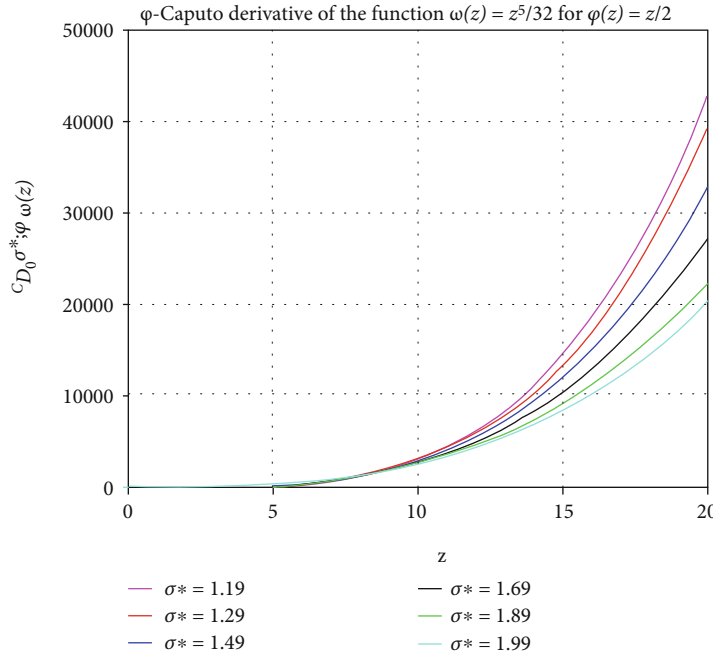


FIGURE 2: The CapFr-derivative of  $\hat{\omega}(z) = z^5/32$  for  $\varphi(z) = z/2$ .

In view of the above lemma, it is verified that the general solution of the homogeneous equation  $({}^{\mathcal{C}}\mathfrak{D}_{s_0}^{\sigma^*, \varphi} \hat{\omega}^*)(z) = 0$  is given by

$$\begin{aligned} \hat{\omega}^*(z) = & \sum_{j=0}^{n-1} c_j^* (\varphi(z) - \varphi(s_0))^j = c_0^* + c_1^* (\varphi(z) - \varphi(s_0)) \\ & + c_2^* (\varphi(z) - \varphi(s_0))^2 + \cdots + c_{n-1}^* (\varphi(z) - \varphi(s_0))^{n-1}, \end{aligned} \quad (12)$$

where  $n-1 < \sigma^* < n$  and  $c_0^*, c_1^*, \dots, c_{n-1}^* \in \mathbb{R}$  ([21]). Both next theorems required analytical tools to derive the desired criteria in the direction of our goals. The first theorem is attributed to Krasnosel'skii and the second one is due to Leray-Schauder.

**Theorem 6** [44]. *The set  $\mathbb{B}^*$  is assumed to be a nonempty, convex, bounded, and closed subset of a given Banach space  $\mathfrak{G}$ . Let  $\hat{h}_*, \hat{f}_* : \mathbb{B}^* \rightarrow \mathfrak{G}$  be so that  $\hat{h}_* \omega + \hat{f}_* \nu \in \mathbb{B}^*$  for any  $\omega, \nu \in \mathbb{B}^*$ ,*

where  $\widehat{h}_*$  is regarded as a compact and continuous mapping and  $\widehat{f}_*$  a contraction. Then, an element  $v \in \mathbb{B}^*$  exists so that  $v = \widehat{h}_* v + \widehat{f}_* v$ .

**Theorem 7** [45]. Assume that  $\mathbb{B}^*$  is a subset of a Banach space  $\mathfrak{C}$  with convexity and closeness property and  $\mathfrak{G}$  is an open subset of  $\mathbb{B}^*$  such that  $0 \in \mathfrak{G}$ . Then, a compact continuous function  $\widehat{A}_* : \mathfrak{G} \rightarrow \mathbb{B}^*$  possesses a fixed point in  $\mathfrak{G}$  or there is a  $g \in \partial\mathfrak{G}$  and  $0 < \sigma < 1$  such that  $g = \sigma\widehat{A}_*(g)$ , where  $\partial\mathfrak{G}$  stands for the boundary of  $\mathfrak{G}$  in  $\mathbb{B}^*$ .

### 3. Main Analytical Results

Suppose that  $\mathfrak{C} = \mathcal{C}([s_0, T], \mathbb{R})$  stands for the collection of all functions given on  $[s_0, T]$  with continuity property. In this case, one can simply verify that  $\mathfrak{C}$  is a Banach space together with

$$\|\widehat{\omega}^*\| = \sup_{z \in [s_0, T]} |\widehat{\omega}^*(z)|, \widehat{\omega}^* \in \mathfrak{C}. \quad (13)$$

We start with the next lemma which will be required in the subsequent sections.

**Lemma 8.** Take  $h^* \in \mathcal{C}([s_0, T], \mathbb{R})$  and

$$Y := \frac{m_2^*(\varphi(T) - \varphi(s_0))^{1+\theta_1^*}}{\Gamma(2 + \theta_1^*)} + \frac{(1 - m_2^*)(\varphi(T) - \varphi(s_0))^{1+\theta_2^*}}{\Gamma(2 + \theta_2^*)} \neq 0. \quad (14)$$

Then,  $\widehat{\omega}^*$  satisfies the nonlinear double-order differential boundary problem of the fractional type

$$\begin{cases} (m_1^* \mathfrak{D}_{s_0}^{\sigma^*; \varphi} + (1 - m_1^*) \mathfrak{C} \mathfrak{D}_{s_0}^{\rho^*; \varphi}) \widehat{\omega}^*(z) = h^*(z), \\ \widehat{\omega}^*(s_0) = 0, m_2^* \mathcal{I}_{s_0}^{\theta_1^*; \varphi} \widehat{\omega}^*(T) + (1 - m_2^*) \mathcal{I}_{s_0}^{\theta_2^*; \varphi} \widehat{\omega}^*(T) = 0, \end{cases} \quad (15)$$

iff  $\widehat{\omega}^*$  satisfies the fractional nonlinear double-order integral equation

$$\begin{aligned} \widehat{\omega}^*(z) = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \widehat{\omega}^*(q) dq \\ & + \frac{1}{m_1^* \Gamma(\sigma^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - 1} h^*(q) dq \\ & + \frac{\varphi(z) - \varphi(s_0)}{Y} \left[ -\frac{m_2^*(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \right. \\ & \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \widehat{\omega}^*(q) dq - \frac{m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^*)} \\ & \cdot \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* - 1} h^*(q) dq - \frac{(1 - m_2^*)(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \\ & \cdot \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} \widehat{\omega}^*(q) dq - \frac{(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^*)} \\ & \left. \cdot \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* - 1} h^*(q) dq \right]. \quad (16) \end{aligned}$$

*Proof.* To start the proof, assume that  $\widehat{\omega}^*$  satisfies the nonlinear double-order differential equation (15). We have

$$\mathfrak{C} \mathfrak{D}_{s_0}^{\sigma^*; \varphi} \widehat{\omega}^*(z) = \frac{(m_1^* - 1) \mathfrak{C}}{m_1^*} \mathfrak{D}_{s_0}^{\rho^*; \varphi} \widehat{\omega}^*(z) + \frac{1}{m_1^*} h^*(z). \quad (17)$$

By fractional integrating in the Riemann-Liouville setting of order  $\sigma^*$ , we get

$$\begin{aligned} \widehat{\omega}^*(z) = & \frac{(m_1^* - 1) \mathcal{I}_{s_0}^{\mathcal{R}\mathcal{L}}}{m_1^*} \mathcal{I}_{s_0}^{\sigma^*; \varphi} \mathfrak{C} \mathfrak{D}_{s_0}^{\rho^*; \varphi} \widehat{\omega}^*(z) \\ & + \frac{1}{m_1^*} \mathcal{I}_{s_0}^{\mathcal{R}\mathcal{L}} \mathcal{I}_{s_0}^{\sigma^*; \varphi} h^*(z) + c_0^* + c_1^* (\varphi(z) - \varphi(s_0)) \\ = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \widehat{\omega}^*(q) dq \\ & + \frac{1}{m_1^* \Gamma(\sigma^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - 1} h^*(q) dq \\ & + c_0^* + c_1^* (\varphi(z) - \varphi(s_0)), \quad (18) \end{aligned}$$

where  $c_0^*, c_1^* \in \mathbb{R}$  are arbitrary constants. By using the first boundary condition, we get  $c_0^* = 0$  and so

$$\begin{aligned} \widehat{\omega}^*(z) = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \widehat{\omega}^*(q) dq \\ & + \frac{1}{m_1^* \Gamma(\sigma^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - 1} h^*(q) dq \\ & + c_1^* (\varphi(z) - \varphi(s_0)). \quad (19) \end{aligned}$$

On the other hand, if we take  $\eta^* \in \{\theta_1^*, \theta_2^*\}$ , then we have

$$\begin{aligned} \mathcal{I}_{s_0}^{\mathcal{R}\mathcal{L}} \mathcal{I}_{s_0}^{\eta^*; \varphi} \widehat{\omega}^*(z) = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^* + \eta^*)} \int_{s_0}^z \varphi'(q) \\ & \cdot (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* + \eta^* - 1} \widehat{\omega}^*(q) dq \\ & + \frac{1}{m_1^* \Gamma(\sigma^* + \eta^*)} \int_{s_0}^z \varphi'(q) \\ & \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \eta^* - 1} h^*(q) dq \\ & + c_1^* \frac{1}{\Gamma(2 + \eta^*)} (\varphi(z) - \varphi(s_0))^{\eta^* + 1}. \quad (20) \end{aligned}$$

Now, by utilizing the second condition, we have

$$\begin{aligned} 0 = & \frac{m_2^*(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* - 1} \widehat{\omega}^*(q) dq \\ & + \frac{m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* - 1} h^*(q) dq \\ & + c_1^* \frac{m_2^*}{\Gamma(2 + \theta_1^*)} (\varphi(T) - \varphi(s_0))^{1 + \theta_1^*} + \frac{(1 - m_2^*)(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_{s_0}^T \varphi'(q)(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* - 1} \omega^*(q) \, dq + \frac{(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^*)} \\
& \cdot \int_{s_0}^T \varphi'(q)(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* - 1} h^*(q) \, dq \\
& + c_1 \frac{(1 - m_2^*)}{\Gamma(2 + \theta_2^*)} (\varphi(T) - \varphi(s_0))^{1 + \theta_2^*}. \tag{21}
\end{aligned}$$

By solving the above equation with respect to  $c_1$ , we get

$$\begin{aligned}
c_1^* = \frac{1}{Y} & \left[ -\frac{m_2^*(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \right. \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \omega^*(q) \, dq \\
& - \frac{m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* - 1} h^*(q) \, dq \\
& - \frac{(1 - m_2^*)(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} \omega^*(q) \, dq \\
& - \frac{(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \left. \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* - 1} h^*(q) \, dq \right], \tag{22}
\end{aligned}$$

where  $Y$  is illustrated in (14). Inserting the obtained value for  $c_1^*$  into (19), we obtain the double-order integral equation illustrated by (16). On the other hand, clearly,  $\omega^*$  satisfies the nonlinear fractional double-order differential equation (15) whenever it satisfies the double-order integral equation (16) and this ends the proof.

Keeping in view of the nonlinear double-integrodifferential problem (3) and (4) and by Lemma 8, we introduce a single-valued operator  $\mathfrak{Q} : \mathfrak{C} \rightarrow \mathfrak{C}$  as

$$\begin{aligned}
\mathfrak{Q}\omega^*(z) = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) \\
& \cdot (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \omega^*(q) \, dq \\
& + \frac{p_1}{m_1^* \Gamma(\sigma^* + \delta_1^*)} \int_{s_0}^z \varphi'(q) \\
& \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \, dq \\
& + \frac{p_2}{m_1^* \Gamma(\sigma^* + \delta_2^*)} \int_{s_0}^z \varphi'(q) \\
& \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \\
& + \frac{\varphi(z) - \varphi(s_0)}{Y} \left[ -\frac{m_2^*(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \right. \\
& \left. \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \omega^*(q) \, dq \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{p_1 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \, dq \\
& - \frac{p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \\
& - \frac{(1 - m_2^*)(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} \omega^*(q) \, dq \\
& - \frac{p_1(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \\
& \cdot dq - \frac{p_2(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \left. \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \right], \tag{23}
\end{aligned}$$

where  $\omega^* \in \mathfrak{C}$  and  $z \in [s_0, T]$ . From now on, we set

$$\begin{aligned}
\Lambda_0 = & \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^*} |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* + 1} m_2^* |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* + 1} (1 - m_2^*) |(m_1^* - 1)|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)}, \\
\Lambda_1 = & \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_1^*} p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_1^* + 1} p_1 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_1^* + 1} p_1 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)}, \\
\Lambda_2 = & \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)}. \tag{24}
\end{aligned}$$

We now derive the first criterion to confirm the existence of solutions for the proposed double-order integrodifferential problem (3)–(4). This purpose is achieved by utilizing Theorem 6 attributed to Krasnosel'skii.

**Theorem 9.** Both functions  $\widehat{h}_*, \widehat{f}_* : [s_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are supposed to be continuous having the following assertions:

( $\mathfrak{S}_1$ ) There is a constant  $\mathfrak{N}^* > 0$  so that we have the following inequality for any  $\omega_1^*, \omega_2^* \in \mathfrak{C}$ ,

$$\left| \widehat{h}_*(z, \omega_1^*(z)) - \widehat{h}_*(z, \omega_2^*(z)) \right| < \mathfrak{N}^* |\omega_1^*(z) - \omega_2^*(z)|, \quad (z \in [s_0, T]) \tag{25}$$

(S<sub>2</sub>) For any  $\omega^* \in \mathfrak{G}$ , a continuous function  $\mu^*$  exists on  $[s_0, T]$  such that

$$\left| \widehat{f}_*(z, \omega^*(z)) \right| < \mu^*(z), (z \in [s_0, T]). \quad (26)$$

Then, by assuming  $\Lambda_0 + \mathfrak{N}^* \Lambda_1 < 1$ , at least one solution exists on  $[s_0, T]$  for the nonlinear double-integrodifferential equation (3) furnished with double-order integral conditions (4), where  $\Lambda_0$  and  $\Lambda_1$  are illustrated by (24).

*Proof.* To start the proof, let  $\|\mu^*\| = \sup_{z \in [s_0, T]} |\mu^*(z)|$ . Define  $\mathbb{B}_\varepsilon^* := \{\omega^* \in \mathfrak{G} : \|\omega^*\| \leq \varepsilon\}$  with

$$\varepsilon \geq \frac{\|\mu^*\| \Lambda_2 + \mathfrak{K}^* \Lambda_1}{1 - (\Lambda_0 + \mathfrak{N}^* \Lambda_1)}, \quad (27)$$

so that  $\mathfrak{K}^* := \sup_{z \in [s_0, T]} |\widehat{h}_*(z, 0)|$  and  $\Lambda_0$  and  $\Lambda_1$  are given by (24). Obviously, nonempty ball  $\mathbb{B}_\varepsilon^*$  is a convex, bounded, and closed set contained in the Banach space  $\mathfrak{G}$ . Besides, let us assume the  $\mathfrak{Q} : \mathfrak{G} \rightarrow \mathfrak{G}$  is as in (23). It is an evident fact that all fixed points of  $\mathfrak{Q}$  will be all solutions of the proposed double-order double-integrodifferential BoVaPr (3)–(4) according to Lemma 8. Now, for each  $z \in [s_0, T]$ , we split  $\mathfrak{Q}$  into  $\mathfrak{Q}_1, \mathfrak{Q}_2 : \mathbb{B}_\varepsilon^* \rightarrow \mathfrak{G}$  by

$$\begin{aligned} \mathfrak{Q}_1 \omega^*(z) = & \frac{m_1^* - 1}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q)(\varphi(z) \\ & - \varphi(q))^{\sigma^* - \rho^* - 1} \omega^*(q) \, dq + \frac{p_1}{m_1^* \Gamma(\sigma^* + \delta_1^*)} \int_{s_0}^z \varphi'(q) \\ & \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \, dq \\ & + \frac{\varphi(z) - \varphi(s_0)}{Y} \left[ -\frac{m_2^*(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \right. \\ & \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \omega^*(q) \, dq \\ & - \frac{p_1 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^*)} \int_{s_0}^T \varphi'(q)(\varphi(T) \\ & - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \, dq \\ & - \frac{(1 - m_2^*)(m_1^* - 1)}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \int_{s_0}^T \varphi'(q)(\varphi(T) \\ & - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} \omega^*(q) \, dq \\ & - \frac{p_1(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^*)} \int_{s_0}^T \varphi'(q)(\varphi(T) \\ & \left. - \varphi(q))^{\sigma^* + \theta_2^* + \delta_1^* - 1} \widehat{h}_*(q, \omega^*(q)) \, dq \right], \end{aligned}$$

$$\begin{aligned} \mathfrak{Q}_2 \omega^*(z) = & \frac{p_2}{m_1^* \Gamma(\sigma^* + \delta_2^*)} \int_{s_0}^z \varphi'(q)(\varphi(z) \\ & - \varphi(q))^{\sigma^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq + \frac{\varphi(z) - \varphi(s_0)}{Y} \\ & \times \left[ -\frac{p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^*)} \int_{s_0}^T \varphi'(q)(\varphi(T) \right. \\ & - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \\ & - \frac{p_2(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^*)} \int_{s_0}^T \varphi'(q)(\varphi(T) \\ & \left. - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \right]. \quad (28) \end{aligned}$$

By the condition (S<sub>1</sub>), we have

$$\begin{aligned} \left| \widehat{h}_*(z, \omega^*(z)) \right| \leq & \left| \widehat{h}_*(z, \omega^*(z)) - \widehat{h}_*(z, 0) \right| + \left| \widehat{h}_*(z, 0) \right| \\ \leq & \mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^* \leq \mathfrak{N}^* \varepsilon + \mathfrak{K}^* \quad (29) \end{aligned}$$

for each  $\omega^* \in \mathfrak{G}$  and  $z \in [s_0, T]$ . In consequence, for both  $\omega_1^*, \omega_2^* \in \mathbb{B}_\varepsilon^*$  and  $z \in [s_0, T]$  and in view of (24) and (27), we get

$$\begin{aligned} |\mathfrak{Q}_1 \omega_1^*(z) + \mathfrak{Q}_2 \omega_2^*(z)| \leq & \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^*} |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} \|\omega^*\| \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_1^*} p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} (\mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^*) \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* + 1} m_2^* |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} \|\omega^*\| \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_1^* + 1} p_1 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} (\mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^*) \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* + 1} (1 - m_2^*) |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)} \|\omega^*\| \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_1^* + 1} p_1 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)} (\mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^*) \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \|\mu^*\| \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \|\mu^*\| \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \|\mu^*\| \\ = & \left[ \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^*} |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} \right. \\ & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* + 1} m_2^* |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} \\ & \left. + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* + 1} (1 - m_2^*) |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)} \right] \|\omega^*\| \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_1^*} p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_1^* + 1} p_1 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} \right. \\
& + \left. \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_1^* + 1} p_1 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)} \right] (\mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^*) \\
& + \left[ \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \right. \\
& + \left. \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \right] \|\mu^*\| \\
& = \Lambda_0 \|\omega^*\| + \Lambda_1 (\mathfrak{N}^* \|\omega^*\| + \mathfrak{K}^*) + \Lambda_2 \|\mu^*\| \\
& \leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1) \varepsilon + \Lambda_1 \mathfrak{K}^* + \Lambda_2 \|\mu^*\| \leq \varepsilon, \tag{30}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} |\widehat{f}_*(q, \omega^*(q))| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} |\widehat{f}_*(q, \omega^*(q))| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} |\widehat{f}_*(q, \omega^*(q))| \\
& \leq \|\mu^*\| \left[ \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \right. \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \\
& + \left. \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \right] = \|\mu^*\| \Lambda_2, \tag{32}
\end{aligned}$$

which implies  $\|\mathfrak{Q}_1 \omega_1^* + \mathfrak{Q}_2 \omega_2^*\| \leq \varepsilon$  and so  $(\mathfrak{Q}_1 \omega_1^*(z) + \mathfrak{Q}_2 \omega_2^*(z)) \in \mathbb{B}_\varepsilon^*$  for all  $\omega_1^*, \omega_2^* \in \mathbb{B}_\varepsilon^*$ . Now, we prove that  $\mathfrak{Q}_2$  is continuous. A sequence  $\{\omega_n^*\}$  in  $\mathbb{B}_\varepsilon^*$  is assumed to be convergent provided that  $\omega_n^* \rightarrow \omega^*$ . Then, for any element  $z \in [s_0, T]$ , one may write

$$\begin{aligned}
& |\mathfrak{Q}_2 \omega_n^*(z) - \mathfrak{Q}_2 \omega^*(z)| \\
& \leq \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \left| \widehat{f}_*(q, \omega_n^*(q)) - \widehat{f}_*(q, \omega^*(q)) \right| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \\
& \times \left| \widehat{f}_*(q, \omega_n^*(q)) - \widehat{f}_*(q, \omega^*(q)) \right| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \\
& \times \left| \widehat{f}_*(q, \omega_n^*(q)) - \widehat{f}_*(q, \omega^*(q)) \right|. \tag{31}
\end{aligned}$$

Since  $\widehat{f}_*$  is continuous, we get  $\|\mathfrak{Q}_2 \omega_n^* - \mathfrak{Q}_2 \omega^*\| \rightarrow 0$  as  $\omega_n^* \rightarrow \omega^*$ . From this, we realize that  $\mathfrak{Q}_2$  is continuous on  $\mathbb{B}_\varepsilon^*$ . In the sequel, to verify that  $\mathfrak{Q}_2$  is compact, we have to prove that  $\mathfrak{Q}_2$  has the uniform boundedness property. For every  $\omega^* \in \mathbb{B}_\varepsilon^*$  and all  $z \in [s_0, T]$ , the following estimate for  $\mathfrak{Q}_2$  holds

$$\begin{aligned}
|\mathfrak{Q}_2 \omega^*(z)| & = \left| \frac{p_2}{m_1^* \Gamma(\sigma^* + \delta_2^*)} \int_{s_0}^z \varphi'(q) \right. \\
& \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \\
& + \frac{\varphi(z) - \varphi(s_0)}{Y} \left[ -\frac{p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \right. \\
& \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \\
& - \frac{p_2 (1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \\
& \left. \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^* - 1} \widehat{f}_*(q, \omega^*(q)) \, dq \right] \Big|
\end{aligned}$$

which implies that  $\|\mathfrak{Q}_2 \omega^*\| \leq \|\mu^*\| \Lambda_2$ . In order to confirm the equicontinuity of  $\mathfrak{Q}_2$ , suppose that  $z_1, z_2 \in [s_0, T]$  such that  $z_2 > z_1$ . We investigate that  $\mathfrak{Q}_2$  corresponds bounded sets to equicontinuous sets. To guarantee this claim, for any  $\omega^*(z) \in \mathbb{B}_\varepsilon^*$ , we obtain

$$\begin{aligned}
& |\mathfrak{Q}_2 \omega^*(z_2) - \mathfrak{Q}_2 \omega^*(z_1)| \\
& \leq \frac{p_2 \|\mu^*\|}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \left[ (\varphi(z_1) - \varphi(s_0))^{\sigma^* + \delta_2^*} \right. \\
& - (\varphi(z_2) - \varphi(s_0))^{\sigma^* + \delta_2^*} \Big] + \frac{\|\mu^*\| p_2 (\varphi(z_2) - \varphi(z_1))}{|Y|} \\
& \times \left[ \frac{m_2^* (\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^*}}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \right. \\
& + \left. \frac{(1 - m_2^*) (\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^*}}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \right]. \tag{33}
\end{aligned}$$

It is to be notice that the R.H.S of (33) is not dependent to  $\omega^* \in \mathbb{B}_\varepsilon^*$  and also goes to 0 by assuming  $z_1 \rightarrow z_2$ . This describes that  $\mathfrak{Q}_2$  is equicontinuous. Hence,  $\mathfrak{Q}_2$  has the relative compactness property on  $\mathbb{B}_\varepsilon^*$ , and thus, application of the Arzelà-Ascoli result gives the complete continuity of  $\mathfrak{Q}_2$ , and eventually on  $\mathbb{B}_\varepsilon^*$ , we reach the compactness of  $\mathfrak{Q}_2$ . At last, we intend to conclude that  $\mathfrak{Q}_1$  is a contraction. For any  $\omega_1^*, \omega_2^* \in \mathbb{B}_\varepsilon^*$ , and  $z \in [s_0, T]$ , we get

$$\begin{aligned}
& |\mathfrak{Q}_1 \omega_1^*(z) - \mathfrak{Q}_1 \omega_2^*(z)| \\
& \leq \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^*} |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} |\omega_1^*(z) - \omega_2^*(z)| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_1^* + 1} p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} \mathfrak{N}^* |\omega_1^*(z) - \omega_2^*(z)| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* + 1} m_2^* |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} |\omega_1^*(z) - \omega_2^*(z)| \\
& + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_1^* + 1} p_1 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} \mathfrak{N}^* |\omega_1^*(z) - \omega_2^*(z)|
\end{aligned}$$



$$\begin{aligned}
 & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* + 1} (1 - m_2^*) (m_1^* - 1)}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)} |\omega_1^*(z) - \omega_2^*(z)| \\
 & + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_1^* + 1} p_1 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)} \mathfrak{N}^* |\omega_1^*(z) - \omega_2^*(z)| \\
 & \leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1) \|\omega_1^* - \omega_2^*\|. \tag{34}
 \end{aligned}$$

Hence,

$$\|\mathfrak{Q}_1 \omega_1^* - \mathfrak{Q}_1 \omega_2^*\| \leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1) \|\omega_1^* - \omega_2^*\|. \tag{35}$$

Since  $\Lambda_0 + \mathfrak{N}^* \Lambda_1 < 1$ ,  $\mathfrak{Q}_1$  is a contraction. Thus, all the conditions of Theorem 6 are verified. Thus, the double- $\varphi$ -integrodifferential-BoVaPr (3) with double-order  $\varphi$ -RLFr integral boundary conditions (4) possesses at least one solution in  $[s_0, T]$  and the proof is completed.

In the next result, we prove the existence of a solution for the double-order double- $\varphi$ -integrodifferential-BoVaPr (3)-(4) with the help of the Leray-Schauder nonlinear alternative.

**Theorem 10.** Consider  $\widehat{h}_*, \widehat{f}_* : [s_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  as continuous mappings having the following assertions:

( $\mathfrak{C}_3$ ) There exist continuous functions  $\eta_1, \eta_2 : [s_0, T] \rightarrow \mathbb{R}^+$  and functions  $\gamma_1, \gamma_2 \in \mathcal{C}([s_0, T], \mathbb{R}^+)$  such that  $|\widehat{h}_*(z, \omega^*)| \leq \gamma_1(z) \eta_1(|\omega^*|)$  and  $|\widehat{f}_*(z, \omega^*)| \leq \gamma_2(z) \eta_2(|\omega^*|)$  for each  $(z, \omega^*) \in [s_0, T] \times \mathbb{R}$

( $\mathfrak{C}_4$ ) There exists a constant  $\Omega^* > 0$  such that  $\Lambda_0 < 1$  and

$$\frac{(1 - \Lambda_0)\Omega^*}{\Lambda_1 \|\gamma_1\| \eta_1(\Omega^*) + \Lambda_2 \|\gamma_2\| \eta_2(\Omega^*)} > 1, \tag{36}$$

where  $\Lambda_0, \Lambda_1, \Lambda_2$  are given by (24).

Then, the nonlinear double- $\varphi$ -CapFr-integrodifferential equation (3) along with double-order  $\varphi$ -RLFr integral boundary conditions (4) possesses at least one solution in  $[s_0, T]$ .

*Proof.* We utilize the Leray-Schauder nonlinear alternative to prove the current result. First, we prove that  $\mathfrak{Q}$  formulated by (23) corresponds all bounded sets to all bounded sets in  $\mathfrak{C}$ . Corresponding to  $\varepsilon > 0$ , build a bounded ball  $\mathbb{B}_\varepsilon^* = \{\omega^* \in \mathfrak{C} : \|\omega^*\| < \varepsilon\}$  in  $\mathfrak{C}$ . Then, for  $z \in [s_0, T]$ , we have

$$\begin{aligned}
 |\mathfrak{Q}\omega^*(z)| & \leq \frac{|m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \|\omega^*\| dq \\
 & + \frac{p_1}{m_1^* \Gamma(\sigma^* + \delta_1^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* + \delta_1^* - 1} \\
 & \cdot \|\gamma_1\| \eta_1(\|\omega^*\|) dq + \frac{p_2}{m_1^* \Gamma(\sigma^* + \delta_2^*)} \int_{s_0}^z \varphi'(q) \\
 & \cdot (\varphi(z) - \varphi(q))^{\sigma^* + \delta_2^* - 1} \|\gamma_2\| \eta_2(\|\omega^*\|) dq \\
 & + \frac{\varphi(z) - \varphi(s_0)}{|Y|} \left[ \frac{m_2^* |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) \right. \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \|\omega^*\| dq
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{p_1 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^* - 1} \|\gamma_1\| \eta_1(\|\omega^*\|) dq \\
 & + \frac{p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^* - 1} \|\gamma_2\| \eta_2(\|\omega^*\|) dq \\
 & + \frac{(1 - m_2^*) |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \int_{s_0}^T \varphi'(q) \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} \|\omega^*\| dq \\
 & + \frac{p_1 (1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \delta_1^* + \theta_2^* - 1} \|\gamma_1\| \eta_1(\|\omega^*\|) dq \\
 & + \frac{p_2 (1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) \\
 & \cdot (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^* - 1} \|\gamma_2\| \eta_2(\|\omega^*\|) dq \\
 & \leq \Lambda_0 \|\omega^*\| + \Lambda_1 \|\gamma_1\| \eta_1(\|\omega^*\|) + \Lambda_2 \|\gamma_2\| \eta_2(\|\omega^*\|) \Big]. \tag{37}
 \end{aligned}$$

Consequently, we have

$$\|\mathfrak{Q}\omega^*\| \leq \Lambda_0 \|\omega^*\| + \Lambda_1 \|\gamma_1\| \eta_1(\|\omega^*\|) + \Lambda_2 \|\gamma_2\| \eta_2(\|\omega^*\|). \tag{38}$$

The obtained inequality indicates that  $\mathfrak{Q}$  is uniformly bounded. Now, let  $z_1, z_2 \in [s_0, T]$  with  $z_1 < z_2$  and  $\omega^* \in \mathbb{B}_\varepsilon^*$ . Then, we have

$$\begin{aligned}
 & |\mathfrak{Q}\omega^*(z_2) - \mathfrak{Q}\omega^*(z_1)| \\
 & \leq \frac{\varepsilon |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} \left[ (\varphi(z_1) - \varphi(s_0))^{\sigma^* - \rho^*} \right. \\
 & \left. - (\varphi(z_2) - \varphi(s_0))^{\sigma^* - \rho^*} \right] + \frac{\|\gamma_1\| \eta_1(\varepsilon) p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} \\
 & \times \left[ (\varphi(z_1) - \varphi(s_0))^{\sigma^* + \delta_1^*} - (\varphi(z_2) - \varphi(s_0))^{\sigma^* + \delta_1^*} \right] \\
 & + \frac{\|\gamma_2\| \eta_2(\varepsilon) p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \left[ (\varphi(z_1) - \varphi(s_0))^{\sigma^* + \delta_2^*} \right. \\
 & \left. - (\varphi(z_2) - \varphi(s_0))^{\sigma^* + \delta_2^*} \right] + \frac{\varphi(z_2) - \varphi(z_1)}{|Y|} \\
 & \times \left[ \frac{(\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^*} \varepsilon m_2^* |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} \right. \\
 & + \frac{\|\gamma_1\| \eta_1(\varepsilon) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^*} p_1 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} \\
 & + \frac{\|\gamma_2\| \eta_2(\varepsilon) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^*} p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \\
 & \left. + \frac{\varepsilon (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^*} (1 - m_2^*) |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)} \right]
 \end{aligned}$$

$$+ \left. \begin{aligned} & \frac{\|\gamma_1\|\eta_1(\varepsilon)(\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_1^*} p_1(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)} \\ & + \frac{\|\gamma_2\|\eta_2(\varepsilon)(\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^*} p_2(1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \end{aligned} \right]. \quad (39)$$

This illustrates that  $|\mathfrak{Q}\omega^*(z_2) - \mathfrak{Q}\omega^*(z_1)| \rightarrow 0$  when  $z_1 \rightarrow z_2$  independent of  $\omega^* \in \mathbb{B}_\varepsilon^*$ . Thus, the well-known result attributed to Arzelà-Ascoli confirms that  $\mathfrak{Q} : \mathfrak{C} \rightarrow \mathfrak{C}$  is completely continuous. In the final stage, we intend to check that the collection of all solutions for the equation  $\omega^*(z) = \vartheta \mathfrak{Q}\omega^*(z)$  is bounded for  $\vartheta \in [0, 1]$ . For that, let  $\omega^*$  be a solution of  $\omega^*(z) = \vartheta \mathfrak{Q}\omega^*(z)$  for  $\vartheta \in [0, 1]$ . Then, for  $z \in [s_0, T]$ , using the arguments given in the first step, we get that

$$\|\omega^*\| \leq \Lambda_0 \|\omega^*\| + \Lambda_1 \|\gamma_1\| \eta_1(\|\omega^*\|) + \Lambda_2 \|\gamma_2\| \eta_2(\|\omega^*\|). \quad (40)$$

Consequently, we have

$$\frac{(1 - \Lambda_0) \|\omega^*\|}{\Lambda_1 \|\gamma_1\| \eta_1(\|\omega^*\|) + \Lambda_2 \|\gamma_2\| \eta_2(\|\omega^*\|)} \leq 1. \quad (41)$$

From condition  $(\mathfrak{C}_4)$ , there is a number  $\Omega^* > 0$  so that  $\|\omega^*\| \neq \Omega^*$ . We construct a set

$$\mathbb{I} = \{\omega^* \in \mathfrak{C} : \|\omega^*\| < \Omega^*\}, \quad (42)$$

and notice that  $\mathfrak{Q} : \mathbb{I} \rightarrow \mathfrak{C}$  is continuous and completely continuous. For such set  $\mathbb{I}$ , there is not an element  $\omega^* \in \partial \mathbb{I}$  for which  $\omega^* = \vartheta \mathfrak{Q}\omega^*$  holds for some  $0 < \vartheta < 1$ . Hence, in virtue of Theorem 7, it is concluded that the operator  $\mathfrak{Q}$  possesses a fixed point in  $\mathbb{I}$  meaning that a solution exists on  $[s_0, T]$  for the nonlinear double- $\varphi$ -CapFr-integro-differential equation (3) along with double-order  $\varphi$ -RLFr integral boundary conditions (4), and in this case, the proof is ended.

In our conclusive outcome, the uniqueness aspect of the obtained solutions for the proposed double- $\varphi$ -CapFr-integro-differential BoVaPr (3)-(4) is checked by the aid of the standard contraction result attributed to Banach.

**Theorem 11.** *Assume that a function  $\hat{h}_* : [s_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  fulfills the presumption  $(\mathfrak{H}_1)$ . Additionally, presume that  $\hat{f}_* : [s_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz, i.e.,  $\mathfrak{M}^*$  exists so that for any  $\omega_1^*, \omega_2^* \in \mathbb{R}$ ,*

$$|\hat{f}_*(z, \omega_1^*) - \hat{f}_*(z, \omega_2^*)| \leq \mathfrak{M}^* |\omega_1^* - \omega_2^*|. \quad (43)$$

*Then, the nonlinear double- $\varphi$ -CapFr-integro-differential BoVaPr (3) with double-order  $\varphi$ -RLFr-integral conditions (4) possesses a unique solution on  $[s_0, T]$ , provided that  $\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2 < 1$ , where  $\Lambda_0, \Lambda_1, \Lambda_2$  are illustrated by (24).*

*Proof.* The argument is straightforward. In other words, by letting  $\sup_{z \in [s_0, T]} |\hat{h}_*(z, 0)| < \mathfrak{Y}^* < \infty$  and  $\sup_{z \in [s_0, T]} |\hat{f}_*(z, 0)| < \mathfrak{C}^* < \infty$ , we select  $\varepsilon > 0$  with

$$\varepsilon \geq \frac{\mathfrak{Y}^* \Lambda_2 + \mathfrak{C}^* \Lambda_1}{1 - (\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2)}. \quad (44)$$

We claim that the inclusion  $\mathfrak{Q}\mathbb{B}_\varepsilon^* \subseteq \mathbb{B}_\varepsilon^*$  is valid in which  $\mathbb{B}_\varepsilon^* = \{\omega^* \in \mathfrak{C} : \|\omega^*\| \leq \varepsilon\}$ . Simply, for any  $\omega^* \in \mathbb{B}_\varepsilon^*$ , from the previous arguments done in the proof of Theorem 9, we get

$$\|\mathfrak{Q}\omega^*\| \leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2) \varepsilon + \mathfrak{Y}^* \Lambda_2 + \mathfrak{C}^* \Lambda_1, \quad (45)$$

which illustrates that  $\mathfrak{Q}\mathbb{B}_\varepsilon^* \subseteq \mathbb{B}_\varepsilon^*$ . For any  $z \in [s_0, T]$  and any  $\omega_1^*, \omega_2^* \in \mathfrak{C}$ , we have

$$\begin{aligned} & |\mathfrak{Q}\omega_1^*(z) - \mathfrak{Q}\omega_2^*(z)| \\ & \leq \frac{|m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* - \rho^* - 1} \\ & \quad \times |\omega_1^*(q) - \omega_2^*(q)| dq + \frac{p_1}{m_1^* \Gamma(\sigma^* + \delta_1^*)} \int_{s_0}^z \varphi'(q) \\ & \quad \times (\varphi(z) - \varphi(q))^{\sigma^* + \delta_1^* - 1} |\hat{h}_*(q, \omega_1^*(q)) - \hat{h}_*(q, \omega_2^*(q))| dq \\ & \quad + \frac{p_2}{m_1^* \Gamma(\sigma^* + \delta_2^*)} \int_{s_0}^z \varphi'(q) (\varphi(z) - \varphi(q))^{\sigma^* + \delta_2^* - 1} \\ & \quad \times |\hat{f}_*(q, \omega_1^*(q)) - \hat{f}_*(q, \omega_2^*(q))| dq + \frac{\varphi(T) - \varphi(s_0)}{|Y|} \\ & \quad \times \left[ \frac{m_2^* |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^*)} \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_1^* - 1} \right. \\ & \quad \times |\omega_1^*(q) - \omega_2^*(q)| dq + \frac{p_1 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) \\ & \quad \times (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^* - 1} |\hat{h}_*(q, \omega_1^*(q)) - \hat{h}_*(q, \omega_2^*(q))| dq \\ & \quad + \frac{p_2 m_2^*}{m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^*)} \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_2^* - 1} \\ & \quad \times |\hat{f}_*(q, \omega_1^*(q)) - \hat{f}_*(q, \omega_2^*(q))| dq + \frac{(1 - m_2^*) |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^*)} \\ & \quad \times \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* - \rho^* + \theta_2^* - 1} |\omega_1^*(q) - \omega_2^*(q)| dq \\ & \quad + \frac{p_1 (1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^*)} \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_1^* + \delta_1^* - 1} \\ & \quad \times |\hat{h}_*(q, \omega_1^*(q)) - \hat{h}_*(q, \omega_2^*(q))| dq + \frac{p_2 (1 - m_2^*)}{m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^*)} \\ & \quad \times \int_{s_0}^T \varphi'(q) (\varphi(T) - \varphi(q))^{\sigma^* + \theta_2^* + \delta_2^* - 1} \\ & \quad \times |\hat{f}_*(q, \omega_1^*(q)) - \hat{f}_*(q, \omega_2^*(q))| dq \left. \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^*} |m_1^* - 1|}{m_1^* \Gamma(\sigma^* - \rho^* + 1)} \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_1^*} p_1}{m_1^* \Gamma(\sigma^* + \delta_1^* + 1)} \mathfrak{N}^* \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \delta_2^*} p_2}{m_1^* \Gamma(\sigma^* + \delta_2^* + 1)} \mathfrak{M}^* \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_1^* + 1} m_2^* |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_1^* + 1)} \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_2^* + 1} p_2 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_2^* + 1)} \mathfrak{N}^* \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_1^* + \delta_1^* + 1} p_1 m_2^*}{|Y| m_1^* \Gamma(\sigma^* + \theta_1^* + \delta_1^* + 1)} \mathfrak{N}^* \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* - \rho^* + \theta_2^* + 1} (1 - m_2^*) |m_1^* - 1|}{|Y| m_1^* \Gamma(\sigma^* - \rho^* + \theta_2^* + 1)} \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_1^* + 1} p_1 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_1^* + 1)} \mathfrak{N}^* \|\omega_1^* - \omega_2^*\| \\
 &\quad + \frac{(\varphi(T) - \varphi(s_0))^{\sigma^* + \theta_2^* + \delta_2^* + 1} p_2 (1 - m_2^*)}{|Y| m_1^* \Gamma(\sigma^* + \theta_2^* + \delta_2^* + 1)} \mathfrak{N}^* \|\omega_1^* - \omega_2^*\| \\
 &\leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2) \|\omega_1^* - \omega_2^*\|. \tag{46}
 \end{aligned}$$

Therefore,

$$\|\mathfrak{Q}\omega_1^* - \mathfrak{Q}\omega_2^*\| \leq (\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2) \|\omega_1^* - \omega_2^*\|. \tag{47}$$

As  $(\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2) < 1$ , thus  $\mathfrak{Q}$  is a contraction. Therefore, in virtue of the standard contraction result attributed to Banach, a fixed point exists uniquely for  $\mathfrak{Q}$ . In consequence, the nonlinear double- $\varphi$ -CapFr-integrodifferential BoVaPr (3) along with double-order  $\varphi$ -RLFr-integral conditions (4) possesses a unique solution on  $[s_0, T]$ .

### 4. Illustrative Examples

We here design three numerical examples which simulate the structure of the proposed nonlinear double- $\varphi$ -CapFr-integrodifferential BoVaPr (3) along with double-order  $\varphi$ -RLFr-integral conditions (4).

*Example 12.* As regards the nonlinear double- $\varphi$ -CapFr-integrodifferential BoVaPr

$$\begin{aligned}
 &(0.8 \mathfrak{D}_0^{1.5;(z+1)} + (1 - 0.8) \mathfrak{D}_0^{1.1;(z+1)}) \omega^*(z) \\
 &= 0.3 \mathcal{I}_0^{0.5;(z+1)} \left( \frac{\sin z}{1 + |z|^2} + \frac{7z^2 \omega^*(z)}{100} \right) \\
 &\quad + 0.4 \mathcal{I}_0^{0.6;(z+1)} \frac{1}{(z+3)^4 (1 + |\omega^*(z)|)}, \tag{48}
 \end{aligned}$$

furnished with double-order  $\varphi$ -RLFr-integral conditions

$$\begin{aligned}
 \omega^*(0) = 0, & 0.7 \mathcal{I}_0^{1;(z+1)} \omega^*(1) \\
 &+ (1 - 0.7) \mathcal{I}_0^{2;(z+1)} \omega^*(1) = 0. \tag{49}
 \end{aligned}$$

Here, we have taken values  $m_1^* = 0.8, m_2^* = 0.7, \sigma^* = 1.5, \varphi(z) = z + 1, \rho^* = 1.1, p_1 = 0.3, \delta_1^* = 0.5, p_2 = 0.4, \delta_2^* = 0.6, \theta_1^* = 1, \theta_2^* = 2, z \in [0, 1]$ , and  $\hat{h}_*, \hat{f}_* : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  by rules

$$\begin{aligned}
 \hat{h}_*(z, \omega^*) &= \frac{\sin z}{1 + |z|^2} + \frac{7z^2 \omega^*(z)}{100}, \hat{f}_*(z, \omega^*) \\
 &= \frac{1}{(z+3)^4 (1 + |\omega^*(z)|)}. \tag{50}
 \end{aligned}$$

Now for each  $\omega_1^*, \omega_2^* \in \mathbb{R}$ , clearly,  $|\hat{h}_*(z, \omega_1^*) - \hat{h}_*(z, \omega_2^*)| \leq \mathfrak{N}^* |\omega_1^* - \omega_2^*|$  with  $\mathfrak{N}^* = 0.07 > 0$ . On the other side, we obtain a continuous function  $\mu^*(z) = (1/((z+3)^4))$  such that  $|\hat{f}_*(z, \omega^*)| \leq \mu^*(z)$  for all  $\omega^* \in \mathbb{R}$ . Also, we have  $\|\mu^*\| = \sup_{z \in [0,1]} \mu^*(z) \approx 0.012$ . Moreover,  $Y \approx 0.40$  and  $\Lambda_0 + \mathfrak{N}^* \Lambda_1 \approx 0.61527 < 1$ . Obviously, all the presumptions of Theorem 9 are verified. Therefore, the nonlinear double- $\varphi$ -CapFr-integrodifferential BoVaPr (48) along with double-order  $\varphi$ -RLFr-integral conditions (49) possesses a solution in  $[0, 1]$ .

The following example illustrates Theorem 10.

*Example 13.* Regard the nonlinear double- $\varphi$ -CapFr-integrodifferential BoVaPr

$$\begin{aligned}
 &(0.8 \mathfrak{D}_0^{1.5;(z+1)} + (1 - 0.8) \mathfrak{D}_0^{1.1;(z+1)}) \omega^*(z) \\
 &= 0.3 \mathcal{I}_0^{0.5;(z+1)} \frac{1}{\sqrt[3]{4913 + z^3}} \left( \frac{1}{(z+1)^2} + \frac{|\omega^*(z)|}{2} \right) \\
 &\quad + 0.4 \mathcal{I}_0^{0.6;(z+1)} \frac{1}{(10 + z^3)^2} \left( \frac{0.9347}{(z^3 + 1)^2} + \frac{\omega^*(z)}{17} \right), \tag{51}
 \end{aligned}$$

furnished with double-order  $\varphi$ -RLFr-integral conditions

$$\begin{aligned}
 \omega^*(0) = 0, & 0.7 \mathcal{I}_0^{1;(z+1)} \omega^*(1) \\
 &+ (1 - 0.7) \mathcal{I}_0^{2;(z+1)} \omega^*(1) = 0. \tag{52}
 \end{aligned}$$

Here, we have taken values  $m_1^* = 0.8, m_2^* = 0.7, \sigma^* = 1.5, \varphi(z) = z + 1, \rho^* = 1.1, p_1 = 0.3, \delta_1^* = 0.5, p_2 = 0.4, \delta_2^* = 0.6, \theta_1^* = 1, \theta_2^* = 2, z \in [0, 1]$ , and  $\hat{h}_*, \hat{f}_* : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  by rules

$$\begin{aligned}
 \hat{h}_*(z, \omega^*) &= \frac{1}{\sqrt[3]{4913 + z^3}} \left( \frac{1}{(z+1)^2} + \frac{|\omega^*(z)|}{2} \right), \\
 \hat{f}_*(z, \omega^*) &= \frac{1}{(10 + z^3)^2} \left( \frac{0.9347}{(z^3 + 1)^2} + \frac{\omega^*(z)}{17} \right). \tag{53}
 \end{aligned}$$

Clearly,

$$\begin{aligned} \left| \widehat{h}_*(z, \omega^*) \right| &\leq \frac{1}{\sqrt[3]{4913 + z^3}} (1 + \|\omega^*\|), \left| \widehat{f}_*(z, \omega^*) \right| \\ &\leq \frac{1}{(10 + z^3)^2} (1 + \|\omega^*\|), \end{aligned} \tag{54}$$

with  $\gamma_1(z) = 1/\sqrt[3]{4913 + z^3}$ ,  $\gamma_2(z) = 1/(10 + z^3)^2$  and  $\eta_1(\|\omega^*\|) = \eta_2(\|\omega^*\|) = 1 + \|\omega^*\|$ . Now,  $\|\gamma_1\| \approx 0.0588$ ,  $\|\gamma_2\| \approx 0.01$ , and  $\eta_1(\Omega^*) = \eta_2(\Omega^*) = 1 + \Omega^*$ . From the above values, we have  $Y \approx 0.4$ ,  $\Lambda_0 \approx 0.59367$ ,  $\Lambda_1 \approx 0.30859$ , and  $\Lambda_2 \approx 0.36938$ . By condition  $(\mathfrak{H}_4)$ , we obtain  $\Omega^* > 0.02282$ . Clearly, all the pre-suppositions of Theorem 10 are verified. Therefore, the nonlinear double- $\varphi$ -CapFr-integro-differential BoVaPr (51) along with double-order  $\varphi$ -RLFr-integral conditions (52) possesses a solution in  $[0, 1]$ .

The following example illustrates Theorem 11.

*Example 14.* As regards the nonlinear double- $\varphi$ -CapFr-integro-differential BoVaPr

$$\begin{aligned} &\left( 0.8 \mathcal{D}_0^{1.5;(z+1)} + (1 - 0.8) \mathcal{D}_0^{1.1;(z+1)} \right) \omega^*(z) \\ &= 0.3 \mathcal{I}_0^{0.5;(z+1)} \frac{3z^2 |\tan(\pi/4)z| |\omega^*(z)|}{73(1 + |\omega^*(z)|)^2} + 1 \\ &+ 0.4 \mathcal{I}_0^{0.6;(z+1)} \frac{z(\cot(\pi/4)z) |\omega^*(z)|}{101(z + |\omega^*(z)|)^2} + 2, \end{aligned} \tag{55}$$

furnished with double-order  $\varphi$ -RLFr-integral conditions

$$\begin{aligned} \omega^*(0) &= 0, 0.7 \mathcal{I}_0^{1;(z+1)} \omega^*(1) \\ &+ (1 - 0.7) \mathcal{I}_0^{2;(z+1)} \omega^*(1) = 0. \end{aligned} \tag{56}$$

Here, we have taken values  $m_1^* = 0.8$ ,  $m_2^* = 0.7$ ,  $\sigma^* = 1.5$ ,  $\varphi(z) = z + 1$ ,  $\rho^* = 1.1$ ,  $p_1 = 0.3$ ,  $\delta_1^* = 0.5$ ,  $p_2 = 0.4$ ,  $\delta_2^* = 0.6$ ,  $\theta_1^* = 1$ ,  $\theta_2^* = 2$ ,  $z \in [0, 1]$ , and  $\widehat{h}_*, \widehat{f}_* : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  by rules

$$\begin{aligned} \widehat{h}_*(z, \omega^*) &= \frac{3z^2 |\tan(\pi/4)z| |\omega^*(z)|}{73(1 + |\omega^*(z)|)^2} + 1, \widehat{f}_*(z, \omega^*) \\ &= \frac{z(\cot(\pi/4)z) |\omega^*(z)|}{101(z + |\omega^*(z)|)^2} + 2. \end{aligned} \tag{57}$$

Then,  $\mathfrak{N}^* \approx 0.041$  and  $\mathfrak{M}^* \approx 0.0099$ , since

$$\begin{aligned} \left| \widehat{h}_*(z, \omega_1^*) - \widehat{h}_*(z, \omega_2^*) \right| &\leq (0.0411) |\omega_1^* - \omega_2^*|, \\ \left| \widehat{f}_*(z, \omega_1^*) - \widehat{f}_*(z, \omega_2^*) \right| &\leq (0.0099) |\omega_1^* - \omega_2^*|. \end{aligned} \tag{58}$$

From the above-given values, we have  $Y \approx 0.40$  and  $\Lambda_0 + \mathfrak{N}^* \Lambda_1 + \mathfrak{M}^* \Lambda_2 \approx 0.61892 < 1$ . Clearly, all the assumptions of Theorem 11 are verified. Thus, the nonlinear double- $\varphi$ -CapFr-integro-differential BoVaPr (55) along with double-order  $\varphi$ -RLFr-integral conditions (56) possesses a solution in  $[0, 1]$ .

## 5. Concluding Notes

Fractional differential (FD) equations are used to model a number of natural phenomena arising in science and technology. For this reason, most researchers have utilized numerous applied fractional operators in the past years to model various forms of natural processes that occurred in the world. Here, we concentrate on the existence specifications of solutions in relation to a newly configured model of a double-order integrodifferential equation in the framework of the  $\varphi$ -CapFr derivative subject to double-order  $\varphi$ -integral boundary conditions in the context of the  $\varphi$ -RLFr integral. Accordingly, to arrive at this issue, we first extract one of the existence features by making use of the fixed point method expressed by Krasnosel'skii, and then, by using a nonlinear alternative criterion attributed to Leray-Schauder, another existence result of the solution is derived. In addition to these, we invoke the Banach principle to confirm the uniqueness of the existing solutions. Three examples are simulated to guarantee the consistency of the analytical findings. From a mathematical view, it is important that we try to obtain generalized notions for the existence definitions. By using this view, the novelty of this work provides an instance of the generalized structure. Also, the proposed structure in this work can be extended by designing different and complicated boundary conditions in the future.

## Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

## Conflicts of Interest

All authors declare that they have no competing interests.

## Authors' Contributions

All authors declare that the research study was done in collaboration with equal responsibility. All authors read and approved the final version of the present manuscript.

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