








Research Article

Some Studies in Hemirings by the Falling Fuzzy k -Ideals

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In this article, we establish the idea of falling fuzzy k -ideals in hemirings through the falling shadow theory and fuzzy sets. We shall express the relations between fuzzy k -ideals and falling fuzzy k -ideals in hemirings. In particular, we shall establish different characterizations of k -hemiregular hemirings in the perfect positive correlation and independent probability space by means of falling fuzzy k -ideals.

1. Introduction

Semiring theory had been developed by Vandiver in order to generalize rings [1]. Semirings are used in major applied mathematical fields, like automata and computer sciences [2, 3]. In the structural theory of hemirings and other algebraic structures, like BCI-algebras and KU-algebras, ideals play a significant role in their studies. Several types of ideals like p -ideals, a -ideals, and q -ideals in BCI-algebras and KU-algebras have been discussed by Senapati and others in [4, 5]. The type of ideals, viz., m -ideals, was studied by Munir and others in [6] for semigroups and in [7] for semirings. In [8], Henriksen defined finite category of k -ideals related to the ideals in semirings. In [9], La Torre examined the h -ideals and k -ideals in hemirings.

Several authors have studied fuzzy h -ideals and fuzzy k -ideals in semirings [10, 11]. In [12], Shabir and Anjum have

examined the fuzzy right k -ideals by using the characterizations of right k -ideals in hemirings. Prime h -ideals and semiprime k -bi-ideals are also defined in hemirings in [13, 14]. The fuzzy sets can be used to study patterns in classification and information processing. The idea of the fuzzy sets has broader applicability, e.g., Goodman worked on the equivalence between a class of random sets and a fuzzy set [15]. In [16–19], Verma used the fuzzy sets in the multiple attribute decision-making and multicriteria decision making environments to study different problems. Some more applications of the fuzzy sets are found in the context of controlled fuzzy metric spaces [20] (see also [21–23]) and pattern recognition [24].

The relationship between probability theory and fuzzy sets is one of most contentious matters in an information science and an uncertainty modeling. The literature on this debate is obscure. Dubois and Prade, in their article [25], have addressed some traditional incomprehension between

fuzzy sets and probabilities. They considered probabilistic analysis of membership functions that helped in assessments of these functions. In 1981, a statistical experiment was conducted to determine the membership function of the fuzzy concept “young” by three different groups of students under the supervision of Wang in China. From that experiment, they approached the existence of the resulting membership function in the theory of fuzzy sets. After this, Wang developed this approach further and furnished a precise mathematical structure of this theory in his book [26]. In [27], Tan and others established a conceptual perspective to define a fuzzy inference relation based on the theory of falling shadows. In [28, 29], Jun and Yu established some connections between the probability theory and the fuzzy mathematics in algebraic structures, like BCK-algebras and hemirings.

In a future work, one can consider k -hemiregular using falling Yuan and Lee, [30], considered a fuzzy subgroup (subring, ideal) as the falling shadow of a cloud of the subgroups (subrings, ideals). By taking inspiration from [30], the notion of falling fuzzy k -ideals of a hemiring is introduced on the basis of the theory of falling shadows and fuzzy sets. This theory is widely applicable in information science and intelligent and fuzzy systems.

This research article is arranged such that the first section briefly describes the history and reviews the literature of fuzzy set theory, ideals in hemirings, and the theory of falling shadows. The second section provides an explanation of the fundamental concepts such as hemirings, ideals, fuzzy sets, and probability spaces. The falling fuzzy k -ideals in a probability space and their results are presented in Section III with intriguing examples. Section IV establishes characterizations of the k -hemiregular hemirings by a perfect positive correlation and an independent probability space through falling fuzzy k -ideals. Finally, the research is concluded in the fifth section.

2. Preliminaries

Definition 1. A nonempty set T , together with two binary operations $(+)$ and (\cdot) , is called a semiring if T is a semigroup under both the operations satisfying the following two distributive laws [13]:

$$t^1 \cdot (t^2 + t^3) = t^1 \cdot t^2 + t^1 \cdot t^3 \text{ and } (t^1 + t^2) \cdot t^3 = t^1 \cdot t^3 + t^2 \cdot t^3, \quad (1)$$

for all $t^1, t^2, t^3 \in T$.

Definition 2. A semiring $(T, +, \cdot)$ is a hemiring if $(T, +)$ is a commutative semigroup with a zero, i.e., $0 \in T$ such that $t^1 + 0 = 0 + t^1 = t^1$ and $t^1 \cdot 0 = 0 \cdot t^1 = 0$ for all $t^1 \in T$ [13].

Definition 3. $\acute{e} \in T$ is the identity element of T if $\acute{e} \cdot t^1 = t^1 \cdot \acute{e} = t^1 \forall t^1 \in T$ [12].

A hemiring $(T, +, \cdot)$ is commutative if (T, \cdot) is a commutative semigroup [12].

Definition 4. Let $X \neq \phi$ be a subset of hemiring T . It is called an left (right) ideal of T if

- (i) $\acute{i} + \acute{j} \in X$ for all $\acute{i}, \acute{j} \in X$
- (ii) $r\acute{i} \in X$ ($\acute{i}r \in X$) for all $\acute{i} \in X, r \in T$

If X is both a left and a right ideal of T , then it is called two-sided ideal or simply an ideal of T [12].

Definition 5. Referring to [12, 13], the left (right) ideal X in a hemiring T is called a left (right) k -ideal of T if $\acute{x}_1, \acute{y}_1 \in X$ and $\acute{a} \in T, \acute{a} + \acute{x}_1 = \acute{y}_1 \Rightarrow \acute{a} \in X$.

Definition 6. As in [12, 13], the k -closure \bar{X} of X in the hemiring T is defined as

$$\bar{X} = \{a \in T \mid a + x_i = x_j \text{ for all } x_i, x_j \in X\}. \quad (2)$$

Definition 7. Let F be a nonempty set. A membership function $\vartheta : F \rightarrow [0, 1]$ is called a fuzzy subset ϑ of F . Also, $\vartheta \neq \phi$ if $\vartheta(\acute{x}) > 0$ for all $\acute{x}_1 \in F$ [12].

Definition 8 [13]. A fuzzy set $\acute{\alpha}$ in a hemiring T is the fuzzy left (right) k -ideal of T if for $e, \acute{f}, \acute{t}, \acute{t} \in T$, and we have the following:

- (i) $\acute{\alpha}(e + \acute{f}) \geq \{\acute{\alpha}(e) \wedge \acute{\alpha}(\acute{f})\}$
- (ii) $\acute{\alpha}(e\acute{f}) \geq \acute{\alpha}(\acute{f}), (\acute{\alpha}(e\acute{f}) \geq \acute{\alpha}(e))$
- (iii) $e + \acute{t} = \acute{t} \rightarrow \acute{\alpha}(e) \geq \{\acute{\alpha}(\acute{t}) \wedge \acute{\alpha}(\acute{t})\}$

Also, α is said to be a fuzzy k -ideal of T if it is both a left k -ideal and a right k -ideal of T .

Recall that $\acute{\alpha}(\acute{0}) \geq \acute{\alpha}(e)$ for all $e \in T$.

Now, we spread some light on theory of the falling shadows. For more details, the references [15, 27] can be followed.

Let V be a universal set and $R(v)$ be a power set of V . For $v \in V$, consider

$$\acute{v} = \{F \mid v \in F, F \subseteq V\}. \quad (3)$$

Let $F \in R(V)$, and then

$$\acute{F} = \{\acute{v} \mid v \in F\}. \quad (4)$$

If \acute{A} is a δ -field in R and $\subseteq A$, then the ordered pair $(R(V), \acute{A})$ is called a hypermeasurable structure on V [27].

Let $(\Lambda, \mathcal{B}, R)$ be a probability space and $(R(V), \acute{A})$ be a hypermeasurable structure, then a random set on V is considered as a function $\zeta : \Lambda \rightarrow R(V)$ [27], which is β - A measurable, i.e.,

$$\zeta^{-1}(D) = \{q \mid q \in \Lambda, \zeta(q) \in D\} \in \mathcal{B}, \forall D \in A. \quad (5)$$

Definition 9. Referring to [27], let ζ be a random set on V ,

and then \tilde{K} is called a falling shadow of ζ if

$$\tilde{K}(v) = R(q|v \in \zeta(q)), \text{ for all } v \in V, \quad (6)$$

and ζ is cloud of \tilde{K} , where R is a probability measure.

Example 1. Given $(\Lambda, \beta, R) = \{[0, 1], \beta, n\}$, where β is a Borel field on $[0, 1]$ and n is the Lebesgue measure, suppose \tilde{K} is a fuzzy set of V and $\tilde{K}_s = \{v \in V | \tilde{K}(v) \geq s\}$ is a s -cut of \tilde{K} , then

$$\zeta : [0, 1] \longrightarrow R(V), s \mapsto \tilde{K}_s \quad (7)$$

is a random set, and ζ is a cloud of \tilde{K} . We shall call ζ as a cut cloud of \tilde{K} . It was defined in [29].

3. Falling Fuzzy k -Ideals in a Probability Space

In this section, we define the falling fuzzy k -ideals in a hemiring through a probability space. We shall also study the relation between fuzzy k -ideals and falling fuzzy k -ideals in a probability space.

Definition 10 [29]. Suppose (Λ, β, R) is a probability space and $\zeta : \Lambda \longrightarrow R(T)$ is a random set, let $\zeta(q)$ be a left (right) ideal of T for all $q \in \Lambda$, and then the random set of the falling shadow

$$\tilde{K}(x) = R(q|x \in \zeta(q)) \quad (8)$$

is known as a falling fuzzy left (right) ideal of T . Also, $\tilde{K}(x)$ is said to be a falling fuzzy ideal of T if it is both a falling fuzzy left ideal and a falling fuzzy right ideal of T .

Let (Λ, β, R) be a probability space. Given $\check{G}(T) = \{g | g : \Lambda \longrightarrow T\}$, where T is a hemiring. Define the two operations $(+)$ and (\cdot) on $\check{G}(T)$ by

$$(g_1^b + g_2^b)(q) = g_1^b(q) + g_2^b(q), \quad (9)$$

$(g_1^b \cdot g_2^b)(q) = g_1^b(q) \cdot g_2^b(q)$, where $q \in \Lambda$ and $g_1^b, g_2^b \in \check{G}(T)$.

Let $\varphi \in \check{G}(T)$ be defined by $\varphi(q) = 0$ for each $q \in \Lambda$, and then $(\check{G}(T), +, \cdot, \varphi)$ is a hemiring.

For any subset B of T and $g \in \check{G}(T)$, consider

$$B_g = \{q \in \Lambda | g(q) \in B\}. \quad (10)$$

Let

$$\zeta : \Lambda \longrightarrow R(\check{G}(T)) \quad (11)$$

be given as

$$q \mapsto \{g \in \check{G}(T) | g(q) \in B\}. \quad (12)$$

Then, $B_g \in \beta$.

Definition 11. Suppose (Λ, β, R) is a probability space and $\zeta : \Lambda \longrightarrow R(T)$ is a random set, if $\zeta(q)$ is a k -ideal of T for all $q \in \Lambda$, then the falling shadow of the random set ζ given as

$$\tilde{K}(v) = R(q|v \in \zeta(q)) \quad (13)$$

is called the falling fuzzy k -ideal of T .

Proposition 1. We have that $\zeta(q) = \{g \in \check{G}(T) | g(q) \in B\}$ is a k -ideal of $\check{G}(T)$ for any k -ideal B of T .

Proof. For $q \in \Lambda$, let $e, f \in \check{G}(T)$ be such that $e, f \in \zeta(q)$, then $e(q), f(q) \in B$. As B is a k -ideal of T , then $e(q) + f(q) \in B$. Note that

$$(e + f)(q) = e(q) + f(q) \in B. \quad (14)$$

This implies that $e + f \in \zeta(q)$.

Let $e \in \zeta(q)$ and $s \in \check{G}(T)$, then $e(q) \in B$. Since B is a k -ideal of T , one has

$$(s \cdot e)(q) = s(q) \cdot e(q) \in B. \quad (15)$$

Therefore, $s \cdot e \in \zeta(q)$, which shows that $\check{G}(T) \cdot \zeta(q) \in \zeta(q)$. Similarly, we can also show that $\zeta(q) \cdot \check{G}(T) \in \zeta(q)$. Let $f \in \zeta(q)$, and $s \in \check{G}(T)$, then $f(q) \in B$. Hence,

$$(s + e)(q) = s(q) + e(q) = f(q), \quad (16)$$

where B is a k -ideal of T ; so, $s(q) \in B$ and also $s \in \zeta(q)$. Hence, $\zeta(q)$ is a k -ideal of $\check{G}(T)$. \square

By the above result, \tilde{K} is a falling fuzzy ideal of $\check{G}(T)$ in (Λ, β, R) , where

$$\tilde{K}(f) = R(q|f(q) \in B). \quad (17)$$

Since

$$\zeta^{-1}(f) = \{q \in \Lambda | f \in \zeta(q)\} = \{q \in \Lambda | f(q) \in B\} = B_f \in \beta, \quad (18)$$

one asserts that ζ is a random set on $\check{G}(T)$; so, \tilde{K} is a falling fuzzy k -ideal on T by Proposition 1.

We present the following examples.

Example 2. Consider the set $T = \{0, e^b, f^b, g^b\}$ equipped with the two operations $(+)$ and (\cdot) defined by (see Table 1)

Then, $(T, +, \cdot)$ is a hemiring [29]. Given $(\Lambda, \mathfrak{B}, R) = ([0, 1], \mathfrak{B}, n)$. Let $\zeta : [0, 1] \longrightarrow R(T)$ be defined by

$$\zeta(s) = \begin{cases} \{0\}, & \text{if } s \in [0, 0.3), \\ \{0, e^b\}, & \text{if } s \in [0.3, 0.5), \\ \{0, f^b\}, & \text{if } s \in [0.5, 0.7), \\ \{0, e^b, f^b\}, & \text{if } s \in [0.7, 0.9), \\ T, & \text{if } s \in [0.9, 1]. \end{cases} \quad (19)$$

Then, $\zeta(s)$ is a k -ideal of T for all $s \in [0, 1]$. Hence, $\tilde{K} = R(s | w \in \zeta(s))$ is a falling fuzzy k -ideal of T .

Example 3. Let $T = \{0, e^b, x^b, y^b, z^b\}$ be a hemiring with the two operations $(+)$ and (\cdot) defined by the following Cayley table (see Table 2):

Then, $(T, +, \cdot)$ is a hemiring. Given $(\Lambda, \mathfrak{B}, R) = ([0, 1], \mathfrak{B}, n)$. Define $\zeta : [0, 1] \longrightarrow R(T)$ by

$$\zeta(s) = \begin{cases} \{0\}, & \text{if } s \in [0, 0.1), \\ \{0, x^b\}, & \text{if } s \in [0.1, 0.3), \\ \{0, z^b\}, & \text{if } s \in [0.3, 0.5), \\ \{0, x^b, z^b\}, & \text{if } s \in [0.5, 0.7), \\ \{0, e^b, x^b, y^b\}, & \text{if } s \in [0.7, 0.9), \\ T, & \text{if } s \in [0.9, 1]. \end{cases} \quad (20)$$

Then, $\zeta(s)$ is a k -ideal of T for all $s \in [0, 1]$. Hence, $\tilde{K} = R(s | w \in \zeta(s))$ is a falling fuzzy k -ideal of T .

Theorem 1. Each fuzzy k -ideal of T is also a falling fuzzy k -ideal of T , but the converse is not true.

Proof. Let $(\Lambda, \mathfrak{B}, R) = ([0, 1], \mathfrak{B}, n)$ be a probability space, i.e., R is a probability measure, \mathfrak{B} is a Borel field in $[0, 1]$, and n is the Lebesgue measure. Let μ be a fuzzy k -ideal of T , and then μ_s is a k -ideal of T for all $s \in [0, 1]$. Let $\zeta : [0, 1] \longrightarrow R(T)$ be a random set given as $\zeta(s) = \mu_s$ for all $s \in [0, 1]$. Hence, μ is also a falling fuzzy k -ideal of T . \square

For the converse, consider the hemiring $T = \{0, x^b, y^b, z^b\}$ with the two following binary operations $(+)$ and (\cdot) (see Table 3):

Given $(\Lambda, \mathfrak{B}, R) = ([0, 1], \mathfrak{B}, n)$. Define $\zeta : [0, 1] \longrightarrow R(T)$ by

$$\zeta(s) = \begin{cases} \{0\}, & \text{if } s \in [0, 0.3), \\ \{0, x^b\}, & \text{if } s \in [0.3, 0.5), \\ \{0, z^b\}, & \text{if } s \in [0.5, 0.9), \\ T, & \text{if } s \in [0.9, 1]. \end{cases} \quad (21)$$

Then, $\zeta(s)$ is a k -ideal of T for all $s \in [0, 1]$. Hence, $\tilde{K} = R(s | e \in \zeta(s))$ is the falling fuzzy k -ideal of T , which is represented by

$$\tilde{K}(e) = \begin{cases} 1, & \text{if } e = 0, \\ 0.3 & \text{if } e = x^b, \\ 0.5 & \text{if } e = z^b, \\ 0.1 & \text{if } e = y^b, \end{cases} \quad (22)$$

$$\tilde{K}_s = \begin{cases} \{0\}, & \text{if } s \in (0.5, 1], \\ \{0, z^b\}, & \text{if } s \in (0.3, 0.5], \\ \{0, x^b, z^b\}, & \text{if } s \in (0.1, 0.3], \\ T, & \text{if } s \in (0, 0.1]. \end{cases} \quad (22)$$

If $s \in (0.1, 0.3]$, then $\tilde{K}_s = \{0, x^b, z^b\}$ is not a k -ideal of T since $x^b + z^b = y^b \cup \{0, x^b, z^b\}$. Thus, \tilde{K} is not a fuzzy k -ideal of T .

Consider the probability space $(\Lambda, \mathfrak{B}, R)$ and the falling shadow of a random set $\zeta : \Lambda \longrightarrow R(T)$. Let $\Lambda(e, \zeta) = \{q \in \Lambda | w \in \zeta(q)\}$, then $\Lambda(e, \zeta) \in R$, where $e \in T$.

Theorem 2. If the falling shadow \tilde{K} of the random set $\zeta : \Lambda \longrightarrow R(T)$ is the falling fuzzy left (right) k -ideal in T for all $e, f, x, y \in T$, then we have the following assertions:

- (i) $\Lambda(e, \zeta) \cap \Lambda(f, \zeta) \subseteq \Lambda(e + f, \zeta)$
- (ii) $\Lambda(f, \zeta) \subseteq \Lambda(ef, \zeta)$, $(\Lambda(e, \zeta) \subseteq \Lambda(ef, \zeta))$
- (iii) $e + x = y$ implies that $\Lambda(x, \zeta) \cap \Lambda(y, \zeta) \subseteq \Lambda(e, \zeta)$

Proof.

- (i) Let $q \in \Lambda(e, \zeta) \cap \Lambda(f, \zeta)$ where $e, f \in \zeta(q)$. Since $\zeta(q)$ is a left k -ideal in T , we have $e + f \in \zeta(q)$, so

$$q \in \Lambda(e + f, \zeta) \Rightarrow \Lambda(e, \zeta) \cap \Lambda(f, \zeta) \subseteq \Lambda(e + f, \zeta). \quad (23)$$

Similarly, the statement is true when $\zeta(q)$ is a right k -ideal.

- (ii) Suppose $q \in \Lambda(f, \zeta)$ and $f \in \zeta(q)$, then $\zeta(q)$ is a left k -ideal on T , so $ef \in \zeta(q)$ and also $q \in \Lambda(ef, \zeta)$. It shows that

$$\Lambda(f, \zeta) \subseteq \Lambda(ef, \zeta). \quad (24)$$

We can also prove

$$\Lambda(e, \zeta) \subseteq \Lambda(ef, \zeta) \quad (25)$$

is for right k -ideals.

(iii) Let $e + \acute{x} = \acute{y}$. Suppose $q \in \Lambda(\acute{x}, \zeta) \cap \Lambda(\acute{y}, \zeta)$ and $e \in T$, then $\acute{x}, \acute{y} \in \zeta(q)$. As $\zeta(q)$ is a left k -ideal on T , so $e \in \zeta(q)$ and $q \in \Lambda(e, \zeta)$. We have

$$\Lambda(\acute{x}, \zeta) \cap \Lambda(\acute{y}, \zeta) \subseteq \Lambda(e, \zeta). \quad (26)$$

□

Theorem 3. If \tilde{K} is a falling fuzzy k -ideal of T , then $\tilde{K}(ef) \geq \tilde{K}(e) \vee \tilde{K}(f)$.

Proof. Since $\Lambda(ef, \zeta) \supseteq \Lambda(e, \zeta) \cap \Lambda(f, \zeta)$, one writes

$$\begin{aligned} \tilde{K}(ef) &= R(q|ef \in \zeta(q)) \geq R(q|e \in \zeta(q)) \cap (q|f \in \zeta(q)) \\ &\geq R(q|e \in \zeta(q)) + R(q|f \in \zeta(q)) - R(q|e \in \zeta(q)) \cap (q|f \in \zeta(q)) \\ &\geq R(q|e \in \zeta(q)) \vee R(q|f \in \zeta(q)) = \tilde{K}(e) \vee \tilde{K}(f). \end{aligned} \quad (27)$$

□

4. Characterizations of k -Hemiregular Hemirings

In this section, we shall characterize the k -hemiregular hemirings on a perfect positive correlation probability space (PPS) and an independent probability space (IPS) through falling fuzzy k -ideals.

Definition 12 [13]. A hemiring T is called a k -hemiregular if for all $x \in T$, there are $a_1, a_2 \in T$ such that

$$x + xa_1x = xa_2x. \quad (28)$$

Lemma 1 [13]. If X and Y are both a right k -ideal and a left k -ideal in a hemiring T , then

$$\bar{X}Y \subseteq X \cap Y. \quad (29)$$

Lemma 2. [13]. The hemiring T is a hemiregular hemiring if for each right and left k -ideal X and Y , we have $\bar{X}Y = X \cap Y$.

Definition 13. The probability space is called a perfect positive correlation [29] if

$$\Lambda(e, \zeta) \subseteq \Lambda(f, \zeta) \text{ or } \Lambda(f, \zeta) \subseteq \Lambda(e, \zeta) \forall e, f \in T. \quad (30)$$

Definition 14. Suppose Λ is a PPS, let \tilde{K}_ϕ and \tilde{K}_ψ be the falling fuzzy k -ideals of T , and then the I product of \tilde{K}_ϕ and \tilde{K}_ψ is

defined by

$$(\tilde{K}_\phi e_I \tilde{K}_\psi)(e) = \sum_{r=1}^u s_r t_r = \sum_{m=1}^v s'_m t'_m (\tilde{K}_\phi(s_r) \wedge \tilde{K}_\phi(\acute{s}_m) \wedge \tilde{K}_\psi(t_r) \wedge \tilde{K}_\psi(\acute{t}_m)) \quad (31)$$

and $(\tilde{K}_\phi e_I \tilde{K}_\psi)(e) = 0$ if e cannot expressed as $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$, where $r = 1, \dots, u$ and $m = 1, \dots, v$.

Theorem 4. Suppose Λ is a PPS and \tilde{K} is the falling fuzzy left (right) k -ideal on T for all $e, f, x, y \in T$, then

- (1) $\tilde{K}(e + f) \geq \tilde{K}(e) \wedge \tilde{K}(f)$
- (2) $\tilde{K}(ef) \geq \tilde{K}(f)$, $(\tilde{K}(ef) \geq \tilde{K}(e))$
- (3) $e + x = y \Rightarrow \tilde{K}(e) \geq \tilde{K}(x) \wedge \tilde{K}(y)$

Proof.

- (1) Let $\zeta(q)$ be a k -ideal on T for any $q \in \Lambda$. Since

$$\Lambda(e, \zeta) \cap \Lambda(f, \zeta) \subseteq \Lambda(e + f, \zeta), \quad (32)$$

one writes

$$\tilde{K}(e + f) = R(q|e + f \in \zeta(q)) \geq R(\{q|e \in \zeta(q)\} \cap \{q|f \in \zeta(q)\}). \quad (33)$$

If $\{q|e \in \zeta(q)\} \supseteq \{q|f \in \zeta(q)\}$, then $\tilde{K}(e + f) \geq \tilde{K}(e)$. While if $\{q|e \in \zeta(q)\} \subseteq \{q|f \in \zeta(q)\}$, then $\tilde{K}(e + f) \geq \tilde{K}(f)$. Hence,

$$\tilde{K}(e + f) \geq \{\tilde{K}(e) \wedge \tilde{K}(f)\}. \quad (34)$$

- (2) As $\zeta(q)$ is a left k -ideal on T , for any $q \in \Lambda$, so $\Lambda(f, \zeta) \subseteq \Lambda(ef, \zeta)$. Hence, we have

$$\tilde{K}(ef) = R(q|ef \in \zeta(q)) \geq R(q|f \in \zeta(q)) = \tilde{K}(f). \quad (35)$$

Similarly, for a right k -ideal, the statement is true.

- (3) If $\zeta(q)$ is a left k -ideal on T for any $q \in \Lambda$, then

$$e + x = y \text{ implies that } \Lambda(x, \zeta) \cap \Lambda(y, \zeta) \subseteq \Lambda(e, \zeta). \quad (36)$$

So, we have

$$\tilde{K}(e) = R(q|e \in \zeta(q)) \geq R(\{q|x \in \zeta(q)\} \cap \{q|y \in \zeta(q)\}). \quad (37)$$

If $\{q|x \in \zeta(q)\} \supseteq \{q|y \in \zeta(q)\}$, then $\tilde{K}(e) \geq \tilde{K}(x)$. While if $\{q|x \in \zeta(q)\} \subseteq \{q|y \in \zeta(q)\}$, then $\tilde{K}(e) \geq \tilde{K}(y)$. Hence,

$$\tilde{K}(e) \geq \{\tilde{K}(x) \wedge \tilde{K}(y)\}. \quad (38)$$

□

Proposition 2 [27]. *If Λ is a perfect positive correlation probability space, then*

$$(\tilde{K}_\phi \cap \tilde{K}_\psi)(e) = \tilde{K}_\phi(e) \wedge \tilde{K}_\psi(e). \quad (39)$$

Proposition 3. *Let Λ be a (PPS) perfect positive correlation probability space. Suppose that \tilde{K}_ϕ and \tilde{K}_ψ are falling fuzzy left (right) k -ideals on T , then $\tilde{K}_\phi \cap \tilde{K}_\psi$ is the falling fuzzy left (right) k -ideal on T .*

Proof.

(1) We have

$$\begin{aligned} (\tilde{K}_\phi \cap \tilde{K}_\psi)(e+f) &= \tilde{K}_\phi(e+f) \wedge \tilde{K}_\psi(e+f) \\ &\geq \{\tilde{K}_\phi(e) \wedge \tilde{K}_\phi(f)\} \wedge \{\tilde{K}_\psi(e) \wedge \tilde{K}_\psi(f)\} \\ &= \{\tilde{K}_\phi(e) \wedge \tilde{K}_\psi(e)\} \wedge \{\tilde{K}_\phi(f) \wedge \tilde{K}_\psi(f)\} \\ &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(e) \wedge (\tilde{K}_\phi \cap \tilde{K}_\psi)(f). \end{aligned} \quad (40)$$

(2) Now,

$$\begin{aligned} (\tilde{K}_\phi \cap \tilde{K}_\psi)(ef) &= \tilde{K}_\phi(ef) \wedge \tilde{K}_\psi(ef) \geq \tilde{K}_\phi(f) \wedge \tilde{K}_\psi(f) \\ &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(f). \end{aligned} \quad (41)$$

(3) Let $e+x=y$, then

$$\begin{aligned} (\tilde{K}_\phi \cap \tilde{K}_\psi)(e) &\geq \{(\tilde{K}_\phi(x) \wedge \tilde{K}_\phi(y)) \wedge \{\tilde{K}_\psi(x) \wedge \tilde{K}_\psi(y)\}\} \\ &= \{\tilde{K}_\phi(x) \wedge \tilde{K}_\psi(x) \wedge \tilde{K}_\phi(y) \wedge \tilde{K}_\psi(y)\} \\ &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(x) \wedge (\tilde{K}_\phi \cap \tilde{K}_\psi)(y). \end{aligned} \quad (42)$$

For right k -ideals, the proof is similar. □

Theorem 5. *If the falling fuzzy set \tilde{K} is the falling fuzzy left (right) k -ideal on T , then*

$$\chi_T \odot_I \tilde{K} \subseteq \tilde{K} (\tilde{K} \odot_I \chi_T \subseteq \tilde{K}). \quad (43)$$

Proof. Suppose \tilde{K} is a falling fuzzy k -ideal on T , let $e \in T$. If

$(\chi_T \odot_I \tilde{K})(e) = 0$, then $(\chi_T \odot_I \tilde{K})(e) \tilde{K}(e)$. Otherwise, there are $s_r, t_r, \acute{s}_m, \acute{t}_m \in T$ such that $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$, so

$$\begin{aligned} (\chi_T \odot_I \tilde{K})(e) &= \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \{\tilde{K}(s_r) \wedge \tilde{K}(t_r)\} \\ &\leq \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \{\tilde{K}(s_r t_r) \wedge \tilde{K}(\acute{s}_m \acute{t}_m)\} \\ &\leq \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \left\{ \tilde{K} \left(\sum_{r=1}^u s_r t_r \right) \wedge \tilde{K} \left(\sum_{m=1}^v \acute{s}_m \acute{t}_m \right) \right\} \\ &\leq \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \tilde{K}(e) = \tilde{K}(e). \end{aligned} \quad (44)$$

It implies that $\chi_T \odot_I \tilde{K} \subseteq \tilde{K}$.

The proof is similar for falling fuzzy right k -ideals. □

Theorem 6. *Let T be a hemiring and $X, Y \subseteq T$, then $(\chi_X \odot_I \chi_Y) = \chi_{XY}$.*

Proof. If $e \in XY$, then $\chi_{XY}(e) = 1$ and $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$ for all $s_r, \acute{s}_m \in X, t_r, \acute{t}_m \in Y$. Let $e \in T$. We have

$$\begin{aligned} (\chi_X \odot_I \chi_Y)(e) &= \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \{\chi_X(s_r) \wedge \chi_X(\acute{s}_m) \wedge \chi_Y(t_r) \wedge \chi_Y(\acute{t}_m)\} \\ &\geq \chi_X(s_r) \wedge \chi_X(\acute{s}_m) \wedge \chi_Y(t_r) \wedge \chi_Y(\acute{t}_m) = 1. \end{aligned} \quad (45)$$

That is, $(\chi_X \odot_I \chi_Y)(e) = 1 = \chi_{XY}(e)$. Hence, $(\chi_X \odot_I \chi_Y) = \chi_{XY}$. If $e \notin XY$, then $\chi_{XY}(e) = 0$. Let $(\chi_X \odot_I \chi_Y) \neq 0$, and then

$$(\chi_X \odot_I \chi_Y)(e) = \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \{\chi_X(s_r) \wedge \chi_X(\acute{s}_m) \wedge \chi_Y(t_r) \wedge \chi_Y(\acute{t}_m)\} \neq 0. \quad (46)$$

Hence, there are $s_r, \acute{s}_m, t_r, \acute{t}_m \in T$ such that $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$ and

$$\chi_X(s_r) \wedge \chi_X(\acute{s}_m) \wedge \chi_Y(t_r) \wedge \chi_Y(\acute{t}_m) \neq 0 \quad (47)$$

imply that

$$\chi_X(s_r) = \chi_X(\acute{s}_m) = \chi_Y(t_r) = \chi_Y(\acute{t}_m) = 1 \quad (48)$$

for all $s_r, \acute{s}_m \in X, t_r, \acute{t}_m \in Y$ and $e \in XY$, which is a contradiction to $\chi_{XY}(e) = 0$. So, our supposition is wrong, and $(\chi_X \odot_I \chi_Y)(e) = 0 = \chi_{XY}(e)$.

In both cases,

$$(\chi_X \odot_I \chi_Y) = \chi_{XY} \quad (49)$$

which completes the proof. □

Theorem 7. A hemiring T is a k -hemiregular if for any falling fuzzy right k -ideal \tilde{K}_ϕ and any falling fuzzy left k -ideal \tilde{K}_ψ of T , we have

$$\tilde{K}_\phi \odot_I \tilde{K}_\psi = \tilde{K}_\phi \cap \tilde{K}_\psi. \quad (50)$$

Proof. Suppose T is a k -hemiregular hemiring, let \tilde{K}_ϕ (resp., \tilde{K}_ψ) be a falling fuzzy right (resp., left) k -ideal of T . Then,

$$\begin{aligned} \tilde{K}_\phi \odot_I \tilde{K}_\psi &\subseteq \tilde{K}_\phi \odot_I \chi_T \subseteq \tilde{K}_\phi, \\ \tilde{K}_\phi \odot_I \tilde{K}_\psi &\subseteq \chi_T \odot_I \tilde{K}_\psi \subseteq \tilde{K}_\psi. \end{aligned} \quad \text{Then } \tilde{K}_\phi \odot_I \tilde{K}_\psi \subseteq \tilde{K}_\phi \cap \tilde{K}_\psi. \quad (51)$$

Conversely, suppose $e \in T$, as T is a k -hemiregular, so there are $x, \acute{x} \in T$ such that

$$\begin{aligned} e + exe &= e\acute{x}e, (\tilde{K}_\phi \odot_I \tilde{K}_\psi)(e) \\ &= \bigvee_{e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m} \{ \tilde{K}_\phi(s_r) \wedge \tilde{K}_\phi(\acute{s}_m) \wedge \tilde{K}_\psi(t_r) \wedge \tilde{K}_\psi(\acute{t}_m) \} \\ &\geq \tilde{K}_\phi(ex) \wedge \tilde{K}_\phi(\acute{e}\acute{x}) \wedge \tilde{K}_\psi(e) \geq \tilde{K}_\phi(e) \wedge \tilde{K}_\psi(e) \\ &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(e). \end{aligned} \quad (52)$$

This implies that

$$\tilde{K}_\phi \odot_I \tilde{K}_\psi \supseteq \tilde{K}_\phi \cap \tilde{K}_\psi. \quad (53)$$

Therefore,

$$\tilde{K}_\phi \odot_I \tilde{K}_\psi = \tilde{K}_\phi \cap \tilde{K}_\psi. \quad (54)$$

Conversely, suppose $\zeta_\phi(q)$ (resp., $\zeta_\psi(q)$) is a left k -ideal (resp., right) k -ideal of T , then $\tilde{K}_\phi(v)$ (resp., $\tilde{K}_\psi(v)$) is a falling fuzzy left (resp., right) k -ideal of T . So, $\chi_{\zeta_\phi(q)}$ (resp., $\chi_{\zeta_\psi(q)}$) is the characteristic function of the fuzzy left (resp., right) k -ideal of T . We have

$$\begin{aligned} \chi_{\zeta_\phi(q)} \bar{\chi}_{\zeta_\psi(q)} &= \chi_{\zeta_\phi(q)} \odot_I \chi_{\zeta_\psi(q)} = \chi_{\zeta_\phi(q)} \cap \chi_{\zeta_\psi(q)} = \chi_{\zeta_\phi(q) \cap \zeta_\psi(q)}, \\ \zeta_\phi(q) \bar{\zeta}_\psi(q) &= \zeta_\phi(q) \cap \zeta_\psi(q). \end{aligned} \quad (55)$$

Thus, we have

$$(\tilde{K}_\phi \bar{\tilde{K}}_\psi)(v) = R(q | v \in \zeta_\phi(q) \bar{\zeta}_\psi(q)) = R(q | v \in \zeta_\phi(q) \cap \zeta_\psi(q)) = (\tilde{K}_\phi \cap \tilde{K}_\psi)(v). \quad (56)$$

Hence, T is a k -hemiregular hemiring. \square

Definition 15 [29]. Let Λ be a probability space, and then Λ is called an independent probability space if

$$\Lambda(e, \zeta) \cap \Lambda(f, \zeta) = \Lambda(e, \zeta) \Lambda(f, \zeta), \forall e, f \in T. \quad (57)$$

Definition 16. Let Λ be an IPS, and suppose that \tilde{K}_ϕ and \tilde{K}_ψ are falling fuzzy k -ideals on T , then the L -product of \tilde{K}_ϕ and \tilde{K}_ψ is defined by

$$(\tilde{K}_\phi \odot_I \tilde{K}_\psi)(e) = \begin{cases} \tilde{K}_\phi(s_r) \tilde{K}_\phi(\acute{s}_m) \tilde{K}_\psi(t_r) \tilde{K}_\psi(\acute{t}_m), & \text{if } e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m, \\ 0, & \text{otherwise.} \end{cases} \quad (58)$$

Theorem 8. Let Λ be an IPS and \tilde{K} be a falling fuzzy left (right) k -ideal of T for all $e, f, x, y \in T$. Then,

- (1) $\tilde{K}(e + f) \geq \tilde{K}(e) \tilde{K}(f)$
- (2) $\tilde{K}(ef) \geq \tilde{K}(f) (\tilde{K}(ef) \geq \tilde{K}(e))$
- (3) $e + x = y \Rightarrow \tilde{K}(e) \geq \tilde{K}(x) \tilde{K}(y)$

Proof.

- (1) Let $\zeta(q)$ be a left k -ideal on T for any $q \in \Lambda$. As we know that $\Lambda(e, \zeta) \cap \Lambda(f, \zeta) \subseteq \Lambda(e + f, \zeta)$, then

$$\begin{aligned} \tilde{K}(e + f) &= R(q | e + f \in \zeta(q)) \geq R(\{q | e \in \zeta(q)\} \cap \{q | f \in \zeta(q)\}) \\ &= R(\{q | e \in \zeta(q)\}) R(\{q | f \in \zeta(q)\}) = \tilde{K}(e) \tilde{K}(f) \end{aligned} \quad (59)$$

- (2) Let $\zeta(q)$ be a left k -ideal of T for any $q \in \Lambda$. As $\Lambda(f, \zeta) \subseteq \Lambda(ef, \zeta)$, so we have

$$\tilde{K}(ef) = R(q | ef \in \zeta(q)) \geq R(q | f \in \zeta(q)) = \tilde{K}(f) \quad (60)$$

- (3) Let $\zeta(q)$ be a left k -ideal of T for any $q \in \Lambda$. So,

$$e + x = y \Rightarrow \Lambda(x, \zeta) \cap \Lambda(y, \zeta) \subseteq \Lambda(e, \zeta) \quad (61)$$

Hence,

$$\begin{aligned} \tilde{K}(e) &= R(q | e \in \zeta(q)) \geq R(\{q | x \in \zeta(q)\} \cap \{q | y \in \zeta(q)\}) \\ &= R(\{q | x \in \zeta(q)\}) R(\{q | y \in \zeta(q)\}), \\ \tilde{K}(e) &= \tilde{K}(x) \tilde{K}(y). \end{aligned} \quad (62)$$

\square

TABLE 1

+	0	e^b	f^b	g^b
0	0	e^b	f^b	g^b
e^b	e^b	e^b	f^b	g^b
f^b	f^b	f^b	f^b	g^b
g^b	g^b	g^b	g^b	f^b
.	0	e^b	f^b	g^b
0	0	0	0	0
e^b	0	e^b	e^b	e^b
f^b	0	e^b	e^b	e^b
g^b	0	e^b	e^b	e^b

TABLE 2

+	0	e^b	x^b	y^b	z^b
0	0	e^b	x^b	y^b	z^b
e^b	e^b	y^b	e^b	x^b	e^b
x^b	f^b	f^b	f^b	g^b	x^b
y^b	g^b	g^b	g^b	f^b	y^b
z^b	z^b	e^b	x^b	y^b	z^b
.	0	e^b	x^b	y^b	z^b
0	0	0	0	0	0
e^b	0	e^b	e^b	z^b	z^b
x^b	0	e^b	e^b	0	z^b
y^b	0	y^b	x^b	e^b	z^b
z^b	0	z^b	z^b	z^b	0

TABLE 3

+	0	x^b	y^b	z^b
0	0	x^b	y^b	z^b
x^b	x^b	0	z^b	y^b
y^b	y^b	z^b	0	x^b
z^b	z^b	y^b	x^b	0
.	0	x^b	y^b	z^b
0	0	0	0	0
x^b	0	0	0	0
y^b	0	0	z^b	z^b
z^b	0	0	z^b	z^b

Proposition 6. Suppose Λ is an IPS, let \tilde{K}_ϕ and \tilde{K}_ψ be falling fuzzy left (right) k -ideals of T , and then $\tilde{K}_\phi \cap \tilde{K}_\psi$ is also a falling fuzzy left (right) k -ideal of T .

Proof. Let \tilde{K}_ϕ and \tilde{K}_ψ be falling fuzzy left k -ideals of T , and then

$$\begin{aligned}
 (\tilde{K}_\phi \cap \tilde{K}_\psi)(e + f) &= \tilde{K}_\phi(e + f)\tilde{K}_\psi(e + f) \geq \tilde{K}_\phi(e)\tilde{K}_\phi(f)\tilde{K}_\psi(e)\tilde{K}_\psi(f) \\
 &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(e)(\tilde{K}_\phi \cap \tilde{K}_\psi)(f), \\
 (\tilde{K}_\phi \cap \tilde{K}_\psi)(ef) &= \tilde{K}_\phi(ef)\tilde{K}_\psi(ef) \geq \tilde{K}_\phi(f)\tilde{K}_\psi(f) = (\tilde{K}_\phi \cap \tilde{K}_\psi)(f).
 \end{aligned}
 \tag{66}$$

(1) Let $e+x = y$, and then

$$(\tilde{K}_\phi \cap \tilde{K}_\psi)(e) \geq \tilde{K}_\phi(x)\tilde{K}_\phi(y)\tilde{K}_\psi(x)\tilde{K}_\psi(y) = (\tilde{K}_\phi \cap \tilde{K}_\psi)(x)(\tilde{K}_\phi \cap \tilde{K}_\psi)(y).
 \tag{67}$$

For the case of right k -ideals, the proof is found on similar lines. \square

Proposition 4 [27]. If Λ is an independent probability space, then

$$(\tilde{K}_\phi \cap \tilde{K}_\psi)(e) = \tilde{K}_\phi(e)\tilde{K}_\psi(e).
 \tag{63}$$

Proposition 5. Let Λ be an independent probability space and \tilde{K} be a falling fuzzy k -ideal on T , and then

$$\tilde{K}(ef) \geq \tilde{K}(e) + \tilde{K}(f) - \tilde{K}(e)\tilde{K}(f).
 \tag{64}$$

Proof. Since $\Lambda(ef, \zeta) \supseteq \Lambda(e, \zeta) \cap \Lambda(f, \zeta)$, one gets

$$\begin{aligned}
 \tilde{K}(ef) &= R\{q|ef \in \zeta(q)\} \geq R\{q|e \in \zeta(q)\} \cap \{q|f \in \zeta(q)\} \\
 &\geq R\{q|e \in \zeta(q)\} + R\{q|f \in \zeta(q)\} - R(\{q|e \in \zeta(q)\} \cap \{q|f \in \zeta(q)\}) \\
 &= R\{q|e \in \zeta(q)\} + R\{q|f \in \zeta(q)\} - R(\{q|e \in \zeta(q)\}R\{q|f \in \zeta(q)\}) \\
 &\geq \tilde{K}(e) + \tilde{K}(f) - \tilde{K}(e)\tilde{K}(f).
 \end{aligned}
 \tag{65}$$

\square

Theorem 9. If the falling fuzzy set \tilde{K} of T is a falling fuzzy left (right) k -ideal of T , then

$$\chi_T \odot_L \tilde{K} \subseteq \tilde{K}(\tilde{K} \odot_L \chi_T \subseteq \tilde{K}).
 \tag{68}$$

Proof. Suppose \tilde{K} is a falling fuzzy k -ideal of T and $e \in T$, if $(\chi_T \odot_L \tilde{K})(e) = 0$, then $(\chi_T \odot_L \tilde{K})(e)\tilde{K}(e)$. Otherwise, there are $s_r, t_r, \acute{s}_m, \acute{t}_m \in T$ such that $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$; so,

$$(\chi_T \odot_L \tilde{K})(e) = \tilde{K}(t_r)\tilde{K}(\acute{t}_m)\tilde{K}(s_r t_r)\tilde{K}(\acute{s}_m \acute{t}_m)\tilde{K}\left(\sum_{r=1}^u s_r t_r\right)\tilde{K}\left(\sum_{m=1}^v \acute{s}_m \acute{t}_m\right)\tilde{K}(e).
 \tag{69}$$

Thus, $\chi_T \odot_L \tilde{K} \subseteq \tilde{K}$.

The proof is similar for right k -ideals. □

Theorem 10. *If T is a hemiring and $X, Y \subseteq T$, then $\chi_X e_L \chi_Y = \chi_{XY}$.*

Proof. Let $e \in T$. If $e \in \bar{X}\bar{Y}$, then $\chi_{\bar{X}\bar{Y}}(e) = 1$ and $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$, for all $s_r, \acute{s}_m \in X, t_r, \acute{t}_m \in Y$. So, we have

$$\begin{aligned} (\chi_X \odot_L \chi_Y)(e) &= \chi_X(s_r) \chi_X(\acute{s}_m) \chi_Y(t_r) \chi_Y(\acute{t}_m) \\ &\geq \chi_X(s_r) \chi_X(\acute{s}_m) \chi_Y(t_r) \chi_Y(\acute{t}_m) = 1. \end{aligned} \tag{70}$$

Thus, $(\chi_X \odot_L \chi_Y)(e) = 1 = \chi_{\bar{X}\bar{Y}}(e)$. If $e \notin \bar{X}\bar{Y}$, then $\chi_{\bar{X}\bar{Y}}(e) = 0$. Let us suppose that $(\chi_X \odot_L \chi_Y) \neq 0$, and then

$$(\chi_X \odot_L \chi_Y)(e) = \chi_X(s_r) \chi_X(\acute{s}_m) \chi_Y(t_r) \chi_Y(\acute{t}_m) \neq 0. \tag{71}$$

Therefore, there are $s_r, \acute{s}_m, t_r, \acute{t}_m \in T$ such that $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$ and

$$\chi_X(s_r) \chi_X(\acute{s}_m) \chi_Y(t_r) \chi_Y(\acute{t}_m) \neq 0, \chi_X(s_r) = \chi_X(\acute{s}_m) = \chi_Y(t_r) = \chi_Y(\acute{t}_m) = 1. \tag{72}$$

Hence, for all $s_r, \acute{s}_m \in X, t_r, \acute{t}_m \in Y$ and $e \in \bar{X}\bar{Y}$, which is a contradiction to $\chi_{\bar{X}\bar{Y}}(e) = 0$, our supposition is wrong, and

$$(\chi_X \odot_L \chi_Y)(e) = 0 = \chi_{\bar{X}\bar{Y}}(e). \tag{73}$$

This implies that

$$(\chi_X \odot_L \chi_Y)(e) = \chi_{\bar{X}\bar{Y}}(e). \tag{74}$$

□

Theorem 11. *A hemiring T is a k -hemiregular if for any falling fuzzy right k -ideal \tilde{K}_ϕ of T and any falling fuzzy left k -ideal \tilde{K}_ψ on T , we have $\tilde{K}_\phi e_L \tilde{K}_\psi = \tilde{K}_\phi \cap \tilde{K}_\psi$.*

Proof. Suppose T is a k -hemiregular hemiring, let \tilde{K}_ϕ (resp., \tilde{K}_ψ) be the falling fuzzy right (resp., left) k -ideal of T . Then,

$$\begin{aligned} \tilde{K}_\phi \odot_L \tilde{K}_\psi &\subseteq \tilde{K}_\phi e_L \chi_T \subseteq \tilde{K}_\phi, \\ \tilde{K}_\phi \odot_L \tilde{K}_\psi &\subseteq \chi_T e_L \tilde{K}_\psi \subseteq \tilde{K}_\psi. \end{aligned} \tag{75}$$

It implies that

$$\tilde{K}_\phi \odot_L \tilde{K}_\psi \subseteq \tilde{K}_\phi \cap \tilde{K}_\psi. \tag{76}$$

Suppose $e \in T$, since T is a k -hemiregular, there are $x, \acute{x} \in T$ such that $e + exe = e\acute{x}e$ where $e + \sum_{r=1}^u s_r t_r = \sum_{m=1}^v \acute{s}_m \acute{t}_m$,

for all $s_r, \acute{s}_m, t_r, \acute{t}_m \in T$. We have

$$\begin{aligned} (\tilde{K}_\phi \odot_L \tilde{K}_\psi)(e) &= \tilde{K}_\phi(s_r) \tilde{K}_\phi(\acute{s}_m) \tilde{K}_\psi(t_r) \tilde{K}_\psi(\acute{t}_m) \\ &\geq \tilde{K}_\phi(exe) \tilde{K}_\psi(e\acute{x}e) \geq \tilde{K}_\phi(xe) \tilde{K}_\psi(e\acute{x}e) \\ &\geq \tilde{K}_\phi(e) \tilde{K}_\psi(e) = (\tilde{K}_\phi \cap \tilde{K}_\psi)(e). \end{aligned} \tag{77}$$

It yields that $\tilde{K}_\phi \odot_L \tilde{K}_\psi \supseteq \tilde{K}_\phi \cap \tilde{K}_\psi$.

Conversely, suppose that $\zeta_\phi(q)$ and $\zeta_\psi(q)$ are right k -ideal and left k -ideal of T , respectively, then $\tilde{K}_\phi(\bar{v})$ and $\tilde{K}_\psi(\bar{v})$ are both a falling fuzzy right k -ideal and a falling fuzzy left k -ideal of T . So, $\chi_{\zeta_\phi(q)}$ and $\chi_{\zeta_\psi(q)}$. Hence, the characteristic function of $\zeta_\phi(q)$ and $\zeta_\psi(q)$ is a fuzzy right k -ideal and a fuzzy left k -ideal of T , respectively. We have

$$\begin{aligned} \bar{v}_{\zeta_\phi(q)} \bar{\zeta}_\psi(q) &= \bar{v}_{\zeta_\phi(q)} \odot_L \bar{v}_{\zeta_\psi(q)} = \bar{v}_{\zeta_\phi(q)} \cap \bar{v}_{\zeta_\psi(q)} = \bar{v}_{\zeta_\phi(q) \cap \zeta_\psi(q)}, \\ \zeta_\phi(q) \bar{\zeta}_\psi(q) &= \zeta_\phi(q) \cap \zeta_\psi(q). \end{aligned} \tag{78}$$

Thus,

$$\begin{aligned} (\tilde{K}_\phi \tilde{K}_\psi)(\bar{v}) &= R(q | \bar{v} \in \zeta_\phi(q) \bar{\zeta}_\psi(q)) = R(q | \bar{v} \in \zeta_\phi(q) \cap \zeta_\psi(q)) \\ &= (\tilde{K}_\phi \cap \tilde{K}_\psi)(\bar{v}). \end{aligned} \tag{79}$$

Hence, T is a k -hemiregular hemiring. □

5. Conclusion

In this paper, we introduced the notion of falling fuzzy k -ideals of a hemiring and illustrated it with the help of examples. Then, we investigated some characteristics of k -hemiregulars by means of falling fuzzy k -ideals based on a perfect positive correlation and an independent probability space. In future works, one can consider k -hemiregulars using falling fuzzy k -bi-ideals and falling fuzzy k -quasi-ideals. One also can apply fuzzy inference relations to hemirings. Further, one can investigate this theory to information science, intelligent and fuzzy systems, etc. Moreover, the idea of falling fuzzy k -ideals can be extended to BCI-algebras [4] and KU-Algebras [5].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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