Research Article

The Fractional S-Transform on Spaces of Type S

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Certain properties of the S-transform are given and composition of S-transform is defined. We study the fractional S-transform on spaces of type S. It is shown that the fractional S-transform is a continuous linear map of the space of type S defined on $\mathbb{R}$ into the space of type $\bar{S}$ defined on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.

1. Introduction

In this section we give the definition and some basic properties of the S-transform. The S-transform was first introduced by Stockwell et al. [1] in 1996 as invertible time-frequency spectral localization technique. The S-transform is an extension to the ideas of continuous wavelet transform and is based on a moving and scalable localizing Gaussian window and has characteristics superior to both of the Fourier transform and the wavelet transform [2–5].

The one-dimensional continuous S-transform of $u(t)$ is defined as [4]

$$\begin{aligned} (Su)(\tau, f) &= S(u(t))(\tau, f) \\ &= \int_{\mathbb{R}} u(t) \omega(\tau - t, f) e^{-i2\pi ft} dt, \end{aligned} \quad (1)$$

where the window ω is assumed to satisfy the following:

$$\int_{\mathbb{R}} \omega(t, f) dt = 1 \quad \forall f \in \mathbb{R} \setminus \{0\}. \quad (2)$$

The most usual window ω is the Gaussian one

$$\omega(t, f) = \frac{|f|}{k\sqrt{2\pi}} e^{-(f^2 t^2 / 2k^2)}, \quad k > 0, \quad (3)$$

where f is the frequency, t is the time variable, and k is a scaling factor that controls the number of oscillations in the window.

Then, (1) can be rewritten as a convolution

$$(Su)(\tau, f) = (u(\cdot) e^{-i2\pi f \cdot} * \omega(\cdot, f))(\tau). \quad (4)$$

Applying the convolution property for the Fourier transform, we obtain

$$(Su)(\tau, f) = \mathcal{F}^{-1} \{ \hat{u}(\cdot + f) \hat{\omega}(\cdot, f) \}(\tau), \quad (5)$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. For the Gaussian window case (3),

$$\hat{\omega}(\alpha, f) = e^{-2(\pi k \alpha / f)^2}. \quad (6)$$

Thus we can write the S-transform in the following form:

$$(Su)(\tau, f) = \int_{\mathbb{R}} \hat{u}(\alpha + f) e^{-2(\pi k \alpha / f)^2} e^{i2\pi \alpha \tau} d\alpha. \quad (7)$$

Also, if $\hat{u}(f)$ and $(Su)(\tau, f)$ are the Fourier transform and S-transform of u , respectively, then

$$\hat{u}(f) = \int_{\mathbb{R}} (Su)(\tau, f) d\tau, \quad (8)$$

so that

$$u(t) = \mathcal{F}^{-1} \left(\int_{\mathbb{R}} (Su)(\tau, \cdot) d\tau \right)(t). \quad (9)$$

2. Some Important Properties of S-Transform

Some basic properties of S-transform can be found in [6, 7] and certain properties of S-transform are obtained in this section.

Theorem 1. If $u(t) \in L^1(\mathbb{R})$, then $\lim_{|\tau| \rightarrow \infty} |(Su)(\tau, f)| = 0$.

Proof. We have by definition

$$\begin{aligned} (Su)(\tau, f) &= \int_{\mathbb{R}} \hat{u}(\alpha + f) e^{-2(\pi k \alpha / f)^2} e^{i2\pi \alpha \tau} d\alpha \\ &= - \int_{\mathbb{R}} \hat{u}(\alpha + f) e^{-2(\pi k \alpha / f)^2} e^{i2\pi(\alpha + 1/2\tau)\tau} d\alpha \\ &= - \int_{\mathbb{R}} \hat{u}\left(x + f - \frac{1}{2\tau}\right) e^{-2[\pi k(x - 1/2\tau)/f]^2} e^{i2\pi \tau x} dx. \end{aligned} \quad (10)$$

Thus

$$\begin{aligned} (Su)(\tau, f) &= \frac{1}{2} \left[\int_{\mathbb{R}} \hat{u}(\alpha + f) e^{-2(\pi k \alpha / f)^2} e^{i2\pi \alpha \tau} d\alpha \right. \\ &\quad \left. - \int_{\mathbb{R}} \hat{u}\left(\alpha + f - \frac{1}{2\tau}\right) e^{-2[\pi k(\alpha - 1/2\tau)/f]^2} e^{i2\pi \alpha \tau} d\alpha \right]. \end{aligned} \quad (11)$$

Clearly,

$$\begin{aligned} \lim_{|\tau| \rightarrow \infty} |(Su)(\tau, f)| &\leq \frac{1}{2} \lim_{|\tau| \rightarrow \infty} \int_{\mathbb{R}} \left| \hat{u}(\alpha + f) - \hat{u}\left(\alpha + f - \frac{1}{2\tau}\right) \right| d\alpha = 0. \end{aligned} \quad (12)$$

□

Theorem 2. If $u(t) \in L^1(\mathbb{R})$, then $\lim_{|f| \rightarrow 0} |(Su)(\tau, f)| = 0$.

Theorem 3. If $e^{-2(\pi k \alpha / f)^2} \hat{u}(\alpha + f) \in L^2(\mathbb{R})$, then $\|(Su)(\tau, f)\|_{L^2} \leq \|u\|_{L^2}$.

Theorem 4. If $u(t) \in L^1(\mathbb{R})$, then $(Su)(\tau, f)$ is continuous with respect to the variables τ and f .

Proof. By the definition

$$\begin{aligned} |(Su)(\tau + h, f) - (Su)(\tau, f)| &= \left| \int_{\mathbb{R}} \hat{u}(\alpha + f) \hat{w}(\alpha, f) (e^{i2\pi \alpha h} - 1) e^{i2\pi \alpha \tau} d\alpha \right| \\ &\leq \int_{\mathbb{R}} |\hat{u}(\alpha + f)| |e^{i2\pi \alpha h} - 1| d\alpha, \\ &\longrightarrow 0 \quad \text{as } h \longrightarrow 0. \end{aligned} \quad (13)$$

Hence the S-transform is continuous with respect to the first variables τ . Similarly we can show that it is also continuous with respect to the second variables f . □

2.1. Composition of S-Transforms. Let S_1 and S_2 be two S-transforms defined as

$$\begin{aligned} (S_1 u)(\tau_1, f_1) &= \int_{\mathbb{R}} \hat{u}(\alpha + f_1) \hat{w}_1(\alpha, f_1) e^{i2\pi \alpha \tau_1} d\alpha, \\ (S_2 u)(\tau_2, f_2) &= \int_{\mathbb{R}} \hat{u}(\alpha + f_2) \hat{w}_2(\alpha, f_2) e^{i2\pi \alpha \tau_2} d\alpha, \end{aligned} \quad (14)$$

provided the integrals exist. Then their composition $S_1 \circ S_2$ is defined by

$$\begin{aligned} (S^* u)(\tau_1, f_1, f_2) &= ((S_1 \circ S_2) u)(\tau_1, f_1, f_2) \\ &= \int_{\mathbb{R}} \mathcal{F}\{(S_2 u)(\tau_2, f_2)\}(\alpha + f_1) \hat{w}_1(\alpha, f_1) e^{i2\pi \alpha \tau_1} d\alpha \\ &= \int_{\mathbb{R}} \hat{u}(\alpha + f_1 + f_2) \hat{w}_2(\alpha + f_1, f_2) \hat{w}_1(\alpha, f_1) e^{i2\pi \alpha \tau_1} d\alpha \\ &= \int_{\mathbb{R}} \hat{u}(\alpha + f_1 + f_2) \hat{w}(\alpha, f_1, f_2) e^{i2\pi \alpha \tau_1} d\alpha, \end{aligned} \quad (15)$$

where $\hat{w}(\alpha, f_1, f_2) = \hat{w}_2(\alpha + f_1, f_2) \hat{w}_1(\alpha, f_1)$, provided the integral exists. The composition $S_1 \circ S_2$ is an S-transform of three variables mapping from $\mathbb{R}$ into $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ and analogous to the composition of the wavelet transform [8, pages 52-53].

Theorem 5. If $u(t) \in L^1(\mathbb{R})$, then $(S^* u)(\tau, f_1, f_2)$ is continuous with respect to the variables τ, f_1 and f_2 .

3. Fractional Fourier Transform

The fractional Fourier transform (FRFT) has played an important role in signal processing. The a th order FRFT of a signal $u(t)$ is defined as [9]

$$F_u^a(f) = \int_{\mathbb{R}} u(t) K_a(t, f) dt, \quad (16)$$

where the transform kernel $K_a(t, f)$ is defined as

$$K_a(t, f) = \begin{cases} A_{\theta} e^{i\pi(f^2 \cot \theta - 2ft \csc \theta + t^2 \cot \theta)}, & \text{if } \theta \neq n\pi \\ \delta(t - f), & \text{if } \theta = 2n\pi \\ \delta(t + f), & \text{if } \theta + \pi = 2n\pi, \end{cases} \quad (17)$$

where $A_{\theta} = \sqrt{1 - i \cot \theta}$, $\theta = a\pi/2$, $a \in [0, 4]$, i is the complex unit, n is an integer, and f is the fractional Fourier frequency (FRFfr). The inverse FRFT of (16) is

$$u(t) = \int_{\mathbb{R}} F_u^a(f) \overline{K_a(t, f)} df. \quad (18)$$

We can write (16) as

$$\begin{aligned} F_u^a(f) &= \int_{\mathbb{R}} u(t) A_{\theta} e^{i\pi(f^2 \cot \theta - 2ft \csc \theta + t^2 \cot \theta)} dt \\ &= A_{\theta} e^{i\pi f^2 \cot \theta} \int_{\mathbb{R}} u(t) e^{-i2\pi f t \csc \theta} e^{i\pi t^2 \cot \theta} dt \\ &= A_{\theta} e^{i\pi f^2 \cot \theta} \left[\widehat{e^{i\pi t^2 \cot \theta} u(t)} \right] (f \csc \theta). \end{aligned} \quad (19)$$

Replacing $u(t)$ by $e^{-i\pi t^2 \cot \theta} \phi(t)$, we have

$$F_u^a(f) = A_{\theta} e^{i\pi f^2 \cot \theta} \left[\widehat{\phi(t)} \right] (f \csc \theta). \quad (20)$$

Putting $f = \xi \sin \theta$, we have

$$\begin{aligned} F_u^a(\xi \sin \theta) &= A_{\theta} e^{i\pi \xi^2 \sin 2\theta/2} \widehat{\phi}(\xi), \\ \widehat{\phi}(\xi) &= \frac{1}{A_{\theta}} e^{-i\pi \xi^2 \sin 2\theta/2} F_u^a(\xi \sin \theta), \end{aligned} \quad (21)$$

where $u(t) = e^{-i\pi t^2 \cot \theta} \phi(t)$.

4. The Fractional S-Transform

The fractional S-transform (FRST) is a generalization of the S-transform. The a th order continuous fractional S-transform of $u(t)$ is defined as [10]

$$\text{FRST}_u^a(\tau, f) = \int_{\mathbb{R}} u(t) g(\tau - t, f) K_a(t, f) dt, \quad (22)$$

where the window g is

$$g(t, f) = \frac{|f \csc \theta|^p}{k \sqrt{2\pi}} e^{-t^2 (f \csc \theta)^{2p} / 2k^2}, \quad k, p > 0 \quad (23)$$

and satisfies the condition

$$\int_{\mathbb{R}} g(t, f) dt = 1 \quad \forall f \in \mathbb{R} \setminus \{0\}. \quad (24)$$

Inverse fractional S-transform is defined by

$$u(t) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \text{FRST}_u^a(\tau, f) d\tau \right] \overline{K_a(t, f)} df. \quad (25)$$

Note that the fractional S-transform depends on a parameter θ and can be interpreted as a rotation by an angle θ in the time-frequency plane. An FRST with $\theta = \pi/2$ corresponds to the S-transform, and an FRST with $\theta = 0$ corresponds to the zero operator. The parameters p and k can be used to adjust the the window function space.

Let

$$h(t, \tau, f) = g(\tau - t, f) K_a(t, f), \quad (26)$$

$$\begin{aligned} H_a(\tau, f, f_1) &= \int_{\mathbb{R}} h(t, \tau, f) \overline{K_a(t, f_1)} \\ &= \int_{\mathbb{R}} g(\tau - t, f) K_a(t, f) \overline{K_a(t, f_1)} dt. \end{aligned} \quad (27)$$

Since

$$\begin{aligned} K_a(t, f) \overline{K_a(t, f_1)} &= A_{\theta} \overline{A_{\theta}} e^{i\pi[(f^2 - f_1^2) \cot \theta - 2(f - f_1)t \csc \theta]}. \end{aligned} \quad (28)$$

By using (23) and (28) in (27) we obtain

$$\begin{aligned} H_a(\tau, f, f_1) &= \int_{\mathbb{R}} \frac{|f \csc \theta|^p}{k \sqrt{2\pi}} e^{-(\tau - t)^2 (f \csc \theta)^{2p} / 2k^2} \\ &\quad \times A_{\theta} \overline{A_{\theta}} e^{i\pi[(f^2 - f_1^2) \cot \theta - 2(f - f_1)t \csc \theta]} dt \\ &= \frac{|f \csc \theta|^p}{k \sqrt{2\pi}} A_{\theta} \overline{A_{\theta}} e^{i\pi[(f^2 - f_1^2) \cot \theta]} \\ &\quad \times \int_{\mathbb{R}} e^{-(\tau - t)^2 (f \csc \theta)^{2p} / 2k^2} e^{-i2\pi(f - f_1)t \csc \theta} dt. \end{aligned} \quad (29)$$

By using the technique of [10], we obtain

$$\begin{aligned} H_a(\tau, f, f_1) &= A_{\theta} \overline{A_{\theta}} e^{i\pi[(f^2 - f_1^2) \cot \theta]} e^{-i2\pi(f - f_1)\tau \csc \theta} \\ &\quad \times e^{-2\pi^2 k^2 (f - f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}}, \end{aligned} \quad (30)$$

and by using (28) we can write

$$\begin{aligned} H_a(\tau, f, f_1) &= e^{-[2\pi^2 k^2 (f - f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}]} K_a(t, f) \overline{K_a(t, f_1)}. \end{aligned} \quad (31)$$

Also, the FRST can be also defined as operations on the fractional Fourier domain:

$$\begin{aligned} \text{FRST}_u^a(\tau, f) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} F_u^a(f) \overline{K_a(t, f)} df \right] \\ &\quad \times g(\tau - t, f) K_a(t, f) dt \\ &= \int_{\mathbb{R}} F_u^a(f_1) H_a(\tau, f, f_1) df_1. \end{aligned} \quad (32)$$

By using (19) and (30) we can write (32) as

$$\begin{aligned}
 \text{FRST}_u^a(\tau, f) &= \int_{\mathbb{R}} A_{\theta} e^{i\pi f_1^2 \cot \theta} \left[\widehat{e^{i\pi t^2 \cot \theta} u(t)} \right] \\
 &\quad \times (f_1 \csc \theta) A_{\theta} \overline{A_{\theta}} e^{i\pi[(f^2 - f_1^2) \cot \theta]} \\
 &\quad \times e^{-i2\pi(f - f_1)\tau \csc \theta} \\
 &\quad \times e^{-2\pi^2 k^2 (f - f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} df_1 \\
 &= A_{\theta} |A_{\theta}|^2 \int_{\mathbb{R}} e^{i\pi f_1^2 \cot \theta} \left[\widehat{e^{i\pi t^2 \cot \theta} u(t)} \right] \\
 &\quad \times (f_1 \csc \theta) e^{i\pi[(f^2 - f_1^2) \cot \theta]} \\
 &\quad \times e^{-i2\pi(f - f_1)\tau \csc \theta} \\
 &\quad \times e^{-2\pi^2 k^2 (f - f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} df_1 \\
 &= A_{\theta} |A_{\theta}|^2 e^{i\pi f^2 \cot \theta} \\
 &\quad \times \int_{\mathbb{R}} \left[\widehat{e^{i\pi t^2 \cot \theta} u(t)} \right] (f_1 \csc \theta) \\
 &\quad \times e^{-i2\pi(f - f_1)\tau \csc \theta} \\
 &\quad \times e^{-2\pi^2 k^2 (f - f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} df_1.
 \end{aligned} \tag{33}$$

5. The Fractional S-Transform on Spaces of Type S

The spaces of type S play an important role in the theory of linear partial differential equations as intermediate spaces between the spaces of the C^{∞} and the analytic functions. The Fourier transform has been studied on the spaces of type S by Friedman [11] and Gelfand and Shilov [12]. Pathak and Singh [13] studied the wavelet transform on the spaces of type S. Also, Singh [6] studied the S-transform on the spaces of type S. In this section we study the fractional S-transform (22) on these spaces.

Let us recall the definitions of these spaces.

Definition 6. The space $S_{\alpha}(\alpha \geq 0)$ consists of all infinitely differentiable functions $\phi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$\begin{aligned}
 \gamma_{k,q}(\phi) &:= \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \\
 &\leq C_q A^k k^{k\alpha} \quad (k, q = 0, 1, 2, \dots),
 \end{aligned} \tag{34}$$

where the constants A and C_q depend on the function ϕ . For $k = 0$, the expression $k^{k\alpha}$ is considered to be equal to 1.

Definition 7. The space $S^{\beta}(\beta \geq 0)$ consists of all infinitely differentiable functions $\phi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$\begin{aligned}
 \gamma_{k,q}(\phi) &:= \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \\
 &\leq C_k B^q q^{q\beta} \quad (k, q = 0, 1, 2, \dots),
 \end{aligned} \tag{35}$$

where the constants B and C_k depend on the function ϕ .

Definition 8. The space $S_{\alpha}^{\beta}(\alpha \geq 0, \beta \geq 0)$ consists of all infinitely differentiable functions $\phi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$\begin{aligned}
 \gamma_{k,q}(\phi) &:= \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \\
 &\leq C A^k B^q k^{k\alpha} q^{q\beta} \quad (k, q = 0, 1, 2, \dots),
 \end{aligned} \tag{36}$$

where the constants A, B , and C depend on the function ϕ .

The spaces of type S are closely interrelated by means of the Fourier transformation; namely, the formulas

$$\widehat{S}_{\alpha} = S^{\alpha}, \quad \widehat{S}^{\beta} = S_{\beta}, \quad \widehat{S}_{\alpha}^{\beta} = S_{\beta}^{\alpha} \tag{37}$$

hold.

We will make use of the following inequalities in our investigation:

$$\frac{q!}{(q-k)!} = k! \binom{q}{k} \leq k! \sum_{k=0}^q \binom{q}{k} = k! 2^q, \tag{38}$$

$$(m+q)^{(m+q)\beta} \leq m^{m\beta} q^{q\beta} e^{m\beta} e^{q\beta} \quad [11, \text{page 265}]. \tag{39}$$

In what follows we will also need the following similar test function space defined on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$, the space of type $\widetilde{S}$.

Definition 9. The space $\widetilde{S}_{\alpha}^{\beta}(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$, $\alpha, \beta \geq 0$, is defined to be the space of all functions $\phi \in C^{\infty}(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ such that for all $s, t \in \mathbb{N}_0$,

$$\begin{aligned}
 &\left| \frac{(f \csc \theta)^{2ps} \tau^s}{(1 + |f \csc \theta|)^t (2(f \csc \theta)^{2p} + |f \csc \theta| + 1)^s} \left(\frac{\partial}{\partial \tau} \right)^t \phi(\tau, f) \right| \\
 &\leq C_{\theta,s,t} A^s B^t s^{\alpha} t^{\beta},
 \end{aligned} \tag{40}$$

where the constants A, B , and $C_{\theta,s,t}$ depend on ϕ .

We show that the fractional S-transform is a continuous linear map of the space of type S into the space of type $\widetilde{S}$ defined mentioned previously.

Theorem 10. For $0 < p \leq 1$, the fractional S-transform is a continuous linear map of $S_{\alpha}(\mathbb{R})$ into $\widetilde{S}_{\alpha}^{\beta}(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ where $\tilde{\alpha} = (\alpha + 3)$ and $\tilde{\beta} = 0$.

Proof. Let $s, t \in \mathbb{N}$ and $I = \tau^s (\partial/\partial \tau)^t \text{FRST}_u^a(\tau, f)$. Then, after differentiation and integration by parts, we obtain

$$\begin{aligned}
 I &= \tau^s \left(\frac{\partial}{\partial \tau} \right)^t A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-f_1)\tau \csc \theta} \\
 &\quad \times e^{-2\pi^2 k^2 (f-f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} \\
 &\quad \times \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) df_1 \\
 &= \tau^s A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} \\
 &\quad \times \int_{\mathbb{R}} (-i2\pi(f-f_1) \csc \theta)^t e^{-i2\pi(f-f_1)\tau \csc \theta} \\
 &\quad \times e^{-2\pi^2 k^2 (f-f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} \\
 &\quad \times \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) df_1 \\
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^t \\
 &\quad \times \int_{\mathbb{R}} \frac{1}{(-i2\pi \csc \theta)^s} e^{-i2\pi(f-f_1)\tau \csc \theta} \\
 &\quad \times D_{f_1}^s (f-f_1)^t e^{-2\pi^2 k^2 (f-f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} \\
 &\quad \times \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) df_1 \\
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-f_1)\tau \csc \theta} \sum_{l=0}^s \binom{s}{l} D_{f_1}^{s-l} (f-f_1)^t \\
 &\quad \times D_{f_1}^l \left[e^{-2\pi^2 k^2 (f-f_1)^2 (\csc \theta)^2 / (f \csc \theta)^{2p}} \right. \\
 &\quad \left. \times \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) \right] df_1 \\
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
 &\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!(-1)^{s-l}}{(t-s+l)!} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-f_1)\tau \csc \theta} (f-f_1)^{t-s+l} \\
 &\quad \times \sum_{m=0}^l \binom{l}{m} D_{f_1}^m \left[e^{-2[\pi k \csc \theta (f-f_1)/(f \csc \theta)^p]^2} \right. \\
 &\quad \left. \times D_{f_1}^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) \right] df_1
 \end{aligned}$$

$$\begin{aligned}
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
 &\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!(-1)^{s-l}}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-f_1)\tau \csc \theta} (f-f_1)^{t-s+l} \\
 &\quad \times \left(-\frac{2\pi k \csc \theta}{(f \csc \theta)^p} \right)^m e^{-2[\pi k \csc \theta (f-f_1)/(f \csc \theta)^p]^2} \\
 &\quad \times \sum_{n=0}^{[m/2]} \frac{(-1)^n m! (4\pi k \csc \theta)^{m-2n}}{n! (m-2n)!} \\
 &\quad \times \left(\frac{f-f_1}{(f \csc \theta)^p} \right)^{(m-2n)} \\
 &\quad \times D_{f_1}^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) df_1 \\
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
 &\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!(-1)^{s-l}}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
 &\quad \times \left(-\frac{2\pi k \csc \theta}{(f \csc \theta)^p} \right)^m \\
 &\quad \times \sum_{n=0}^{[m/2]} \frac{(-1)^n m! (4\pi k \csc \theta)^{m-2n}}{n! (m-2n)!} \frac{1}{(f \csc \theta)^{p(m-2n)}} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-f_1)\tau \csc \theta} (f-f_1)^{t-s+l+m-2n} \\
 &\quad \times e^{-2[\pi k \csc \theta (f-f_1)/(f \csc \theta)^p]^2} \\
 &\quad \times D_{f_1}^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (f_1 \csc \theta) df_1 \\
 &= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
 &\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!(-1)^{s-l}}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
 &\quad \times \left(-\frac{2\pi k \csc \theta}{(f \csc \theta)^p} \right)^m \\
 &\quad \times \sum_{n=0}^{[m/2]} \frac{(-1)^n m! (4\pi k \csc \theta)^{m-2n}}{n! (m-2n)!} \frac{1}{(f \csc \theta)^{p(m-2n)}} \\
 &\quad \times \int_{\mathbb{R}} e^{-i2\pi(f-y \sin \theta)\tau \csc \theta} (f-y \sin \theta)^{t-s+l+m-2n} \\
 &\quad \times e^{-2[\pi k \csc \theta (f-y \sin \theta)/(f \csc \theta)^p]^2} \\
 &\quad \times (\sin \theta)^{-l+m+1} D_y^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) dy
 \end{aligned}$$

$$\begin{aligned}
&= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
&\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!(-1)^{s-l}}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \left(-\frac{2\pi k \csc \theta}{(f \csc \theta)^p} \right)^m \\
&\quad \times \sum_{n=0}^{[m/2]} \frac{(-1)^n m! (4\pi k \csc \theta)^{m-2n}}{n! (m-2n)!} \frac{1}{(f \csc \theta)^{p(m-2n)}} \\
&\quad \times (\sin \theta)^{t-s+2m-2n+1} \\
&\quad \times \int_{\mathbb{R}} e^{-i2\pi(f \csc \theta - y)\tau} (f \csc \theta - y)^{t-s+l+m-2n} \\
&\quad \times e^{-2[\pi k(f \csc \theta - y)/(f \csc \theta)^p]^2} \\
&\quad \times D_y^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) dy \\
&= A_\theta |A_\theta|^2 e^{i\pi f^2 \cot \theta} (-i2\pi \csc \theta)^{t-s} \\
&\quad \times \sum_{l=0}^s \binom{s}{l} \frac{t!}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \left(-\frac{2\pi k \csc \theta}{(f \csc \theta)^p} \right)^m \\
&\quad \times \sum_{n=0}^{[m/2]} \frac{(-1)^n m! (4\pi k \csc \theta)^{m-2n}}{n! (m-2n)!} \frac{1}{(f \csc \theta)^{p(m-2n)}} \\
&\quad \times (\sin \theta)^{t-s+2m-2n+1} \\
&\quad \times \int_{\mathbb{R}} e^{-i2\pi(f \csc \theta - y)\tau} \\
&\quad \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
&\quad \times (f \csc \theta)^{t-s+l+m-2n-r} (-y)^r \\
&\quad \times e^{-2[\pi k(f \csc \theta - y)/(f \csc \theta)^p]^2} \\
&\quad \times D_y^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) dy.
\end{aligned}$$

(41)

By using the convergence factor $1 + y^2$, we obtain

$$\begin{aligned}
|I| &\leq |A_\theta|^3 (2\pi)^{t-s} \sum_{l=0}^s \binom{s}{l} \frac{t!}{(t-s+l)!} \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \left| \frac{2\pi k}{(f \csc \theta)^p} \right|^m \\
&\quad \times \sum_{n=0}^{[m/2]} \frac{m! (4\pi k)^{m-2n}}{n! (m-2n)!} \frac{1}{|f \csc \theta|^{p(m-2n)}}
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
&\times |f \csc \theta|^{t-s+l+m-2n-r} \\
&\times \left| \int_{\mathbb{R}} \left| (1+y^2) y^r D_y^{l-m} \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) \right| \right. \\
&\quad \times \frac{1}{1+y^2} dy.
\end{aligned}$$

(42)

Using the property (38) we obtain

$$\begin{aligned}
|I| &\leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} (s-l)^{s-l} 2^t \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \frac{1}{|f \csc \theta|^{pm}} \\
&\quad \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
&\quad \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
&\quad \times |f \csc \theta|^{t-s+l+m-2n-r} \\
&\quad \times \left| \int_{\mathbb{R}} \left| (1+y^2) y^r D_y^{l-m} \right. \right. \\
&\quad \times \left. \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) \right| \frac{1}{1+y^2} dy.
\end{aligned}$$

(43)

Use the properties (35) and (39) to obtain

$$\begin{aligned}
|I| &\leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} s^s 2^t \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \frac{1}{|f \csc \theta|^{pm}} \\
&\quad \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
&\quad \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
&\quad \times |f \csc \theta|^{t-s+l+m-2n-r} \\
&\quad \times C_r B^{l-m} (l-m)^{(l-m)\alpha} \\
&\leq C''_{\theta,s,t} s^s 2^t \sum_{l=0}^s \binom{s}{l} \sum_{m=0}^l \binom{l}{m} \\
&\quad \times \frac{1}{|f \csc \theta|^{pm}}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r} B_1^l l^\alpha \\
& \leq C''_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} \sum_{l=0}^s \binom{s}{l} \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{1}{|f \csc \theta|^{p(m-2n)}} (1 + |f \csc \theta|)^{t-s+l+m-2n} \\
& = C''_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} \sum_{l=0}^s \binom{s}{l} \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \frac{(1 + |f \csc \theta|)^{t-s+l+m}}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \left(\frac{|f \csc \theta|^{2p}}{(1 + |f \csc \theta|)^2} \right)^n \\
& \leq C''_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} \sum_{l=0}^s \binom{s}{l} \\
& \times \sum_{m=0}^l \binom{l}{m} \frac{(1 + |f \csc \theta|)^{t-s+l+m}}{|f \csc \theta|^{2pm}} \left[\frac{m}{2} \right] \\
& \leq C_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} (1 + |f \csc \theta|)^t \\
& \times \sum_{l=0}^s \binom{s}{l} \sum_{m=0}^l \binom{l}{m} \frac{(1 + |f \csc \theta|)^m}{|f \csc \theta|^{2pm}} \\
& = C_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} (1 + |f \csc \theta|)^t \\
& \times \sum_{l=0}^s \binom{s}{l} \left(1 + \frac{1 + |f \csc \theta|}{|f \csc \theta|^{2p}} \right)^l \\
& = C_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} (1 + |f \csc \theta|)^t \\
& \times \left(2 + \frac{1 + |f \csc \theta|}{|f \csc \theta|^{2p}} \right)^s \\
& = C_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s} (1 + |f \csc \theta|)^t \\
& \times \left(\frac{2|f \csc \theta|^{2p} + |f \csc \theta| + 1}{|f \csc \theta|^{2p}} \right)^s.
\end{aligned} \tag{44}$$

Hence, we obtain the estimate

$$\begin{aligned}
& \left| \frac{(f \csc \theta)^{2ps} \tau^s}{(1 + |f \csc \theta|)^t (2(f \csc \theta)^{2p} + |f \csc \theta| + 1)^s} \right. \\
& \left. \times \left(\frac{\partial}{\partial \tau} \right)^t \text{FRST}_u^a(\tau, f) \right| \leq C_{\theta,s,t} A_1^s B_1^t s^{(\alpha+3)s},
\end{aligned} \tag{45}$$

where A_1 and B_1 are positive constants. $\square$

Theorem 11. For $0 < p \leq 1$, the fractional S -transform is a continuous linear map of $S^\alpha(\mathbb{R})$ into $\tilde{S}_{\tilde{\alpha}}^{\tilde{\beta}}(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ where $\tilde{\alpha} = (\alpha + 3)$ and $\tilde{\beta} = \alpha$.

Theorem 12. For $0 < p \leq 1$, the fractional S -transform is a continuous linear map of $S^\beta_\alpha(\mathbb{R})$ into $\tilde{S}_{\tilde{\alpha}}^{\tilde{\beta}}(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ where $\tilde{\alpha} = (\alpha + 3)$ and $\tilde{\beta} = \beta$.

Proof. By inequality (43), we have

$$\begin{aligned}
|I| & \leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} (s-l)^{s-l} 2^t \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r} \\
& \times \int_{\mathbb{R}} \left| (1 + y^2)^r y^r D_y^{l-m} \right. \\
& \quad \left. \times \left[e^{i\pi t^2 \cot \theta} u(t) \right] (y) \right| \frac{1}{1 + y^2} dy.
\end{aligned} \tag{46}$$

Next, using the properties (36) and (39) we have

$$\begin{aligned}
|I| & \leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} s^s 2^t \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r}
\end{aligned}$$

$$\begin{aligned}
& \times C \left(A_1^r r^{r\beta} + A_1^{(r+2)} (r+2)^{(r+2)\beta} \right) \\
& \times B_1^{(l-m)} (l-m)^{(l-m)\alpha} \\
& \leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} s^s 2^t \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r} \\
& \times C A_2^r (r+2)^{(r+2)\beta} B_1^l l^{\alpha} \\
& \leq C'_{\theta,s,t} \sum_{l=0}^s \binom{s}{l} s^s 2^t \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r} \\
& \times C A_2^r r^{\beta} 2^{2\beta} e^{r\beta} e^{2\beta} B_1^l l^{\alpha} \\
& \leq C_{\theta,s,t} A^s B^t s^{s(\alpha+3)} t^{t\beta} \\
& \times \sum_{l=0}^s \binom{s}{l} s^s 2^t \sum_{m=0}^l \binom{l}{m} \\
& \times \frac{1}{|f \csc \theta|^{pm}} \\
& \times \sum_{n=0}^{[m/2]} \frac{(4\pi k)^{m-2n} (2n)^{2n} 2^m}{n!} \frac{1}{|f \csc \theta|^{p(m-2n)}} \\
& \times \sum_{r=0}^{t-s+l+m-2n} \binom{t-s+l+m-2n}{r} \\
& \times |f \csc \theta|^{t-s+l+m-2n-r}.
\end{aligned} \tag{47}$$

Next, by using the technique of Theorem 10, we obtain

$$\begin{aligned}
|I| & \leq C_{\theta,s,t} A^s B^t s^{s(\alpha+3)} t^{t\beta} (1 + |f \csc \theta|)^t \\
& \times \left(\frac{2|f \csc \theta|^{2p} + |f \csc \theta| + 1}{|f \csc \theta|^{2p}} \right)^s.
\end{aligned} \tag{48}$$

Hence, we obtain the estimate

$$\begin{aligned}
& \left| \frac{(f \csc \theta)^{2ps} \tau^s}{(1 + |f \csc \theta|)^t (2(f \csc \theta)^{2p} + |f \csc \theta| + 1)^s} \right. \\
& \left. \times \left(\frac{\partial}{\partial \tau} \right)^t \text{FRST}_u^a(\tau, f) \right| \leq C_{\theta,s,t} A^s B^t s^{s(\alpha+3)} t^{t\beta},
\end{aligned} \tag{49}$$

where A and B are positive constants. $\square$

6. Future Directions

The results of this paper will be useful in the investigation of time-frequency behaviour of test functions and distributions. We may use these results to obtain new results in modulation spaces, which may be defined by means of the fractional S-transform, by using the techniques developed in [14, Chapter 11–13].

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