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Research Article

Some New Recurrence Relations Concerning Jacobi Functions

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The aim of this paper is to obtain some new recurrence relations for the "modified" Jacobi functions $\Phi_\lambda^{\alpha,\beta}$. Based on an asymptotic relationship between the Jacobi function and the Bessel function, the expression of Bessel function in terms of elementary functions follows as particular cases.

1. Introduction

It is well known that the Jacobi function $t \to \varphi_\lambda^{\alpha,\beta}(t), \alpha, \beta, \lambda \in \mathbb{C}$, $\alpha \neq -1, -2, -3, \ldots$, is defined (cf. [1]) as the even C^∞-function on $\mathbb{R}$ which satisfies the differential equation

$$L_{\alpha,\beta} u(t) = -\left(\lambda^2 + (\alpha+\beta+1)^2\right) u(t), \qquad u(0) = 1, \quad u'(0) = 0, \tag{1}$$

where $L_{\alpha,\beta}$ is the Jacobi operator given by

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + \left[(2\alpha+1)\coth t + (2\beta+1)\tanh t\right]\frac{d}{dt}. \tag{2}$$

In this paper some new summation formulas for the "modified" Jacobi functions

$$t \longmapsto \Phi_\lambda^{\alpha,\beta}(t) = \frac{\Gamma\left((\alpha+\beta+1+i\lambda)/2\right)\Gamma\left((\alpha+\beta+1-i\lambda)/2\right)}{2^{-\beta+1}\Gamma(1+\alpha)} \times (\sinh t)^\alpha \varphi_\lambda^{\alpha,\beta}(t) \tag{3}$$

are proved. We establish, under some conditions, that

$$\begin{gathered}
\Phi_\lambda^{\alpha-1,\beta+1}(t) - \Phi_\lambda^{\alpha+1,\beta+1}(t) = \frac{2}{\cosh t}\frac{d}{dt}\left[\Phi_\lambda^{\alpha,\beta}\right](t), \\
\left[\frac{2}{\sinh(2t)}\frac{d}{dt}\right]^m \left[(\sinh t)^\alpha \Phi_\lambda^{\alpha,\beta}\right](t) = (\sinh t)^{\alpha-m}\Phi_\lambda^{\alpha-m,\beta+m}(t), \\
(\sinh t)^m \left[\Phi_\lambda^{\alpha-m,\beta+m}(t) - (-1)^m \Phi_\lambda^{\alpha+m,\beta+m}(t)\right] = \sum_{j=0}^{m-1}(-1)^j b_m^j(\alpha)(\sinh t)^j \Phi_\lambda^{\alpha+j,\beta+j}(t),
\end{gathered} \tag{4}$$

where $b_m^j(\alpha)$ stands for a constant that will be determined. A same of the last formula with respect to the dual variable is also shown, another formulas are proven and some examples are treated. We note that the subject for the generalization of recurrence relations concerning special functions was studied by many authors (cf. [2–5]).

The remaining part of the paper is organized as follows. In Section 2 we recall the main results about some necessary notions related to Jacobi functions, Bessel functions, and Macdonald's functions. Section 3 is devoted to establishing some new formulas concerning "modified" Jacobi functions and some examples are given.

2. Preliminaries

This section gives an introduction to the Jacobi function, Bessel function of the first kind, and Macdonald's function. Main references are [1, 6–11].

2.1. Jacobi Function. A Jacobi function $\varphi_\lambda^{\alpha,\beta}(\alpha,\beta,\lambda \in \mathbb{C}, \alpha \neq -1,-2,-3,\ldots)$ (cf. [1, 6]) can be expressed as a hypergeometric function under form

$$\varphi_\lambda^{\alpha,\beta}(t) = {}_2F_1\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 t\right): \quad \rho = \alpha+\beta+1. \tag{5}$$

This function satisfies the following properties.

(i) For every $t \in [0,+\infty[$, $(\Gamma(\alpha+1))^{-1}\varphi_\lambda^{\alpha,\beta}(t)$ is an entire function of α, β, and λ.

(ii) For each $\alpha,\beta \in \mathbb{C}$ and for each nonnegative integer n there exists a positive constant k such that

$$\forall t \geq 0, \quad \forall \lambda \in \mathbb{C},$$
$$\left|(\Gamma(\alpha+1))^{-1}\left(\frac{d}{dt}\right)^n \varphi_\lambda^{\alpha,\beta}(t)\right| \leq k(1+|\lambda|)^{n+k}(1+x)e^{(|\operatorname{Im}\lambda|-\operatorname{Re}(\rho))t}, \tag{6}$$

where $k=0$ if $\operatorname{Re}(\alpha) > -1/2$ and $k = [(1/2)-\operatorname{Re}(\alpha)]$ if $\operatorname{Re}(\alpha) \leq -1/2$.

(iii) The function $t \to \varphi_\lambda^{\alpha,\beta}(t)$ for $\operatorname{Re}(\alpha) > -1/2$ possesses the Laplace type integral representation

$$\forall t > 0, \ \forall \lambda \in \mathbb{C}, \quad \varphi_\lambda^{\alpha,\beta}(t) = \int_0^t \mathscr{K}(t,y)\cos(\lambda y)\,dy, \tag{7}$$

where $\mathscr{K}(t,\cdot)$ is explicitly given in [8] and it is a positive function on $(0,t)$ if $\alpha > -1/2$ and $|\beta| \leq \max\{1/2,\alpha\}$.

We note that the function $\varphi_\lambda^{\alpha,\beta}$ possesses also an integral representation with respect to the dual variable [12].

To finish this paragraph, we consider the "modified" Jacobi function

$$t \longmapsto \Phi_\lambda^{\alpha,\beta}(t) = \frac{\Gamma((\alpha+\beta+1+i\lambda)/2)\,\Gamma((\alpha+\beta+1-i\lambda/2))}{2^{-\beta+1}\Gamma(1+\alpha)} \times (\sinh t)^\alpha \varphi_\lambda^{\alpha,\beta}(t), \tag{8}$$

which can be written (cf. [7], page 693) as follows:

$$\forall t > 0, \quad \int_0^\infty x^\beta K_{i\lambda}(x) J_\alpha(x\sinh t)\,dx = \Phi_\lambda^{\alpha,\beta}(t), \quad |\operatorname{Im}\lambda| < \operatorname{Re}(\alpha+\beta+1), \tag{9}$$

where J_α and K_α denote, respectively, the Bessel function of the first kind and Macdonald's function.

Example 1. The functions $\varphi_\lambda^{1/2,1/2}$ and $\Phi_\lambda^{1/2,1/2}$ can be expressed in terms of elementary functions as

$$\forall t > 0, \quad \varphi_\lambda^{1/2,1/2}(t) = \frac{2\sin(\lambda t)}{\lambda\sinh(2t)}, \tag{10}$$

$$\forall t > 0, \quad \Phi_\lambda^{1/2,1/2}(t) = \sqrt{\frac{2}{\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right) \times \frac{1}{\lambda}\frac{\sin(\lambda t)}{\sqrt{\sinh t}\cosh t}, \quad |\operatorname{Im}\lambda| < 2, \tag{11}$$

2.2. Bessel Function of the First Kind. We recall that the Bessel function of the first kind and order α denoted by J_α is defined as an analytic function on $\{z \in \mathbb{C} : |\arg z| < \pi\}$ by

$$J_\alpha(z) = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)}\left(\frac{z}{2}\right)^{\alpha+2k} \tag{12}$$

and possesses the following integral representation:

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^1 \left(1-t^2\right)^{\alpha-1/2}\cos(zt)\,dt, \quad \operatorname{Re}(\alpha) > -\frac{1}{2}. \tag{13}$$

It satisfies the following functional relations:

$$J_{\alpha-1}(z) - J_{\alpha+1}(z) = 2J_\alpha'(z), \tag{14}$$

$$\frac{1}{z}\frac{d}{dz}\left(z^\alpha J_\alpha(z)\right) = z^{\alpha-1}J_{\alpha-1}(z). \tag{15}$$

It also verifies [10]

$$z^m\left(J_{\alpha-m}(z) - (-1)^m J_{\alpha+m}(z)\right) = \sum_{j=0}^{m-1}(-1)^j b_m^j(\alpha)\,z^j J_{\alpha+j}(z), \tag{16}$$

with

$$b_m^j(\alpha) = \Gamma(\alpha+1)\frac{2^{m-j}C_m^j}{\Gamma(\alpha+j-m+1)}, \tag{17}$$

and satisfies

$$J_0(z) + 2\sum_{j=1}^\infty J_{2j}(z) = 1, \tag{18}$$

$$\sum_{j=0}^\infty (-1)^j J_{2j+1}(z) = \frac{\sin z}{2}, \tag{19}$$

$$J_0(z) + 2\sum_{j=1}^\infty (-1)^j J_{2j}(z) = \cos z, \tag{20}$$

$$\sum_{j=1}^{\infty}(-1)^{j+1}(2j)^2 J_{2j}(z) = \frac{z\sin z}{2}, \tag{21}$$

$$\sum_{j=0}^{\infty}(-1)^{j}(2j+1)^2 J_{2j+1}(z) = \frac{z\cos z}{2}. \tag{22}$$

It has, for $z \geq 0$ and $\alpha \geq 0$, the following asymptotic representation:

$$\begin{aligned} J_\alpha(z) &\asymp \frac{z^\alpha}{2^\alpha \Gamma(\alpha+1)}, \quad z \longrightarrow 0, \\ J_\alpha(z) &\asymp \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right), \quad z \longrightarrow \infty. \end{aligned} \tag{23}$$

2.3. Modified Bessel Function of the First Kind and Macdonald's Function. We recall that the modified Bessel function of the first kind I_α of order $\alpha \in \mathbb{C}$ and Macdonald's function K_α of order $\alpha \in \mathbb{C}$ are defined as analytic functions on $\{z \in \mathbb{C} : |\arg z| < \pi\}$ by the formulas

$$\begin{aligned} I_\alpha(z) &= \sum_{k=0}^{\infty}\frac{(z/2)^{\alpha+2k}}{\Gamma(k+1)\Gamma(k+\alpha+1)}, \\ K_\alpha(z) &= \begin{cases} \dfrac{\pi}{2}\dfrac{[I_{-\alpha}(z) - I_\alpha(z)]}{\sin(\pi\alpha)}, & \text{if } \alpha \notin \mathbb{Z}, \\ \lim\limits_{\nu\to\alpha} K_\nu(z), & \text{if } \alpha \in \mathbb{Z}. \end{cases} \end{aligned} \tag{24}$$

The Macdonald's function satisfies

$$K_\alpha(z) = K_{-\alpha}(z), \tag{25}$$

$$K_{\alpha-1}(z) + K_{\alpha+1}(z) = -2K'_\alpha(z). \tag{26}$$

It also verifies (cf. [10])

$$\begin{aligned} &z^m\left(K_{\alpha+m}(z) - K_{\alpha-m}(z)\right) \\ &\quad = \sum_{j=0}^{m-1}(-1)^{m-j-1} b_m^j(\alpha) z^j K_{\alpha+j}(z), \end{aligned} \tag{27}$$

where $b_m^j(\alpha)$ is the constant defined by (17).

For $\operatorname{Re} z > 0$ and $\alpha \in \mathbb{C}$, it can be written under the representation integral ([9], page 119)

$$K_\alpha(z) = \int_0^\infty e^{-z\cosh u}\cosh(\alpha u)\,du. \tag{28}$$

We recall also that the modified Bessel function of the first kind and Macdonald's function have, for $z > 0$ and $\alpha \geq 0$, the following asymptotic representations ([9], page 136):

$$I_\alpha(z) \asymp \frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}, \quad z \longrightarrow 0, \tag{29}$$

$$K_\alpha(z) \asymp \begin{cases} \log\dfrac{2}{z}, & \text{if } \alpha = 0 \\ 2^{\alpha-1}\dfrac{\Gamma(\alpha)}{z^\alpha}, & \text{if } \alpha > 0, \end{cases} \quad z \longrightarrow 0 \tag{30}$$

$$I_\alpha(z) \asymp \frac{e^z}{\sqrt{2\pi z}}, \quad z \longrightarrow \infty, \tag{31}$$

$$K_\alpha(z) \asymp \sqrt{\frac{\pi}{2z}}e^{-z}, \quad z \longrightarrow \infty. \tag{32}$$

Note that in the special case $\alpha = \pm 1/2$, Macdonald's function is given by

$$K_{\pm 1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}. \tag{33}$$

To complete this paragraph, we recall the following (cf. [7], page 684 and page 747).

(i) If $|\operatorname{Im}\lambda| < 1 + \operatorname{Re}(\beta)$, then we have

$$\int_0^\infty x^\beta K_{i\lambda}(x)\,dx = 2^{\beta-1}\Gamma\left(\frac{1+\beta+i\lambda}{2}\right)\Gamma\left(\frac{1+\beta-i\lambda}{2}\right). \tag{34}$$

(ii) If $|\operatorname{Im}\lambda| < 1 + \operatorname{Re}(\beta)$ and $t > 0$, then we have

$$\int_0^\infty x^\beta K_{i\lambda}(x)\cos(x\sinh t)\,dx = \sqrt{\frac{\pi}{2}\sinh t}\,\Phi_\lambda^{-1/2,\beta+1/2}(t). \tag{35}$$

(iii) If $|\operatorname{Im}\lambda| < 2 + \operatorname{Re}(\beta)$ and $t > 0$, then we have

$$\int_0^\infty x^\beta K_{i\lambda}(x)\sin(x\sinh t)\,dx = \sqrt{\frac{\pi}{2}\sinh t}\,\Phi_\lambda^{1/2,\beta+1/2}(t). \tag{36}$$

3. Recurrence Relations for $\Phi_\lambda^{\alpha,\beta}$

3.1. Some Reccurrence Relations with Respect to Parameters α and β

Proposition 2. *Let $\alpha, \beta, \lambda \in \mathbb{C}$. If $|\operatorname{Im}\lambda| < \operatorname{Re}(\rho)$, with $\rho = \alpha + \beta + 1$ and $m \in \mathbb{N}$, then one has*

$$\begin{aligned} &(\sinh t)^m\left[\Phi_\lambda^{\alpha-m,\beta+m}(t) - (-1)^m\Phi_\lambda^{\alpha+m,\beta+m}(t)\right] \\ &\quad = \sum_{j=0}^{m-1}(-1)^j b_m^j(\alpha)(\sinh t)^j\Phi_\lambda^{\alpha+j,\beta+j}(t), \end{aligned} \tag{37}$$

where $b_m^j(\alpha)$ stands for the constant given by (17).

Proof. Formula (16) asserts that

$$x^{\beta}(x\sinh t)^{m}\left(J_{\alpha-m}(x\sinh t)-(-1)^{m}J_{\alpha+m}(x\sinh t)\right) = \sum_{j=0}^{m-1}(-1)^{j}b_{m}(j)\,x^{\beta}(x\sinh t)^{j}J_{\alpha+j}(x\sinh t). \tag{38}$$

We multiply both sides of the last identity by $K_{i\lambda}(x)$ and integrate with respect to x from 0 to ∞. Then use formula (9) to obtain the required formula. □

Proposition 3. *For $\alpha,\beta,\lambda\in\mathbb{C}$ such that $|\operatorname{Im}\lambda|<\operatorname{Re}(\alpha+\beta+1)$, one has*

$$\Phi_{\lambda}^{\alpha-1,\beta+1}(t)-\Phi_{\lambda}^{\alpha+1,\beta+1}(t)=\frac{2}{\cosh t}\frac{d}{dt}\left[\Phi_{\lambda}^{\alpha,\beta}\right](t). \tag{39}$$

Proof. At first, we suppose that $\operatorname{Re}(\alpha)>-1/2$ and $|\operatorname{Im}\lambda|<\operatorname{Re}(\alpha+\beta+1)$. With the help of formulas (14) and (9), we obtain

$$\Phi_{\lambda}^{\alpha-1,\beta+1}(t)-\Phi_{\lambda}^{\alpha+1,\beta+1}(t) = \frac{2}{\cosh t}\int_{0}^{\infty}x^{\beta}K_{i\lambda}(x)\frac{d}{dt}\left[J_{\alpha}(x\sinh t)\right]dx. \tag{40}$$

On the other hand, by using formulas (14), (28), and (13), we can deduce that

$$\left|x^{\beta}K_{i\lambda}(x)\frac{d}{dt}\left[J_{\alpha}(x\sinh t)\right]\right| \le C(\alpha)\cosh t\cdot K_{|\operatorname{Im}(\lambda)|}(x)\,x^{\operatorname{Re}(\beta)+1} \times\left((x\sinh t)^{\operatorname{Re}(\alpha)-1}+(x\sinh t)^{\operatorname{Re}(\alpha)+1}\right), \tag{41}$$

where $C(\alpha)$ is a positive constant.

Taking account of the asymptotic formulas of Macdonald's function (30) and (32) (or formula (34)) and taking t in any compact of $(0,\infty)$, we can deduce the result in the case $\operatorname{Re}(\alpha)>-1/2$. According to principle of analytic continuation, the restriction $\operatorname{Re}(\alpha)>-1/2$ used can be dropped. □

Using Propositions 2 and 3, we obtain the following corollary.

Corollary 4. *If $|\operatorname{Im}\lambda|<\operatorname{Re}(\alpha+\beta+1)$, then one has*

$$\Phi_{\lambda}^{\alpha+1,\beta+1}(t)=\frac{\alpha}{\sinh t}\Phi_{\lambda}^{\alpha,\beta}(t)-\frac{1}{\cosh t}\frac{d}{dt}\left[\Phi_{\lambda}^{\alpha,\beta}\right](t), \tag{42}$$

$$\Phi_{\lambda}^{\alpha-1,\beta+1}(t)=\frac{\alpha}{\sinh t}\Phi_{\lambda}^{\alpha,\beta}(t)+\frac{1}{\cosh t}\frac{d}{dt}\left[\Phi_{\lambda}^{\alpha,\beta}\right](t). \tag{43}$$

Example 5. Using Example 1 and relation (42) for $\alpha=1/2$ and $\beta=1/2$, we obtain

$$\begin{aligned}\Phi_{\lambda}^{3/2,3/2}(t)&=\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{\lambda}\\&\quad\times\frac{2\sin(\lambda t)\cosh(2t)-\lambda\cos(\lambda t)\sinh(2t)}{(\sinh t)^{3/2}(\cosh t)^{3}},\\\varphi_{\lambda}^{3/2,3/2}(t)&=\frac{12}{\lambda(4+\lambda^{2})}\\&\quad\times\frac{2\sin(\lambda t)\cosh(2t)-\lambda\cos(\lambda t)\sinh(2t)}{(\sinh(2t))^{3}}.\end{aligned} \tag{44}$$

Example 6. Using Example 1 and relation (43) for $\alpha=1/2$ and $\beta=1/2$, we obtain

$$\begin{aligned}\Phi_{\lambda}^{-1/2,3/2}(t)&=\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{\lambda}\\&\quad\times\frac{\lambda\cos(\lambda t)\cosh t-\sin(\lambda t)\sinh t}{\sqrt{\sinh t}(\cosh t)^{3}},\\\varphi_{\lambda}^{-1/2,3/2}(t)&=\frac{\lambda\cos(\lambda t)\cosh t-\sin(\lambda t)\sinh t}{\sinh t(\cosh t)^{3}}.\end{aligned} \tag{45}$$

Example 7. Using Example 5 and relation (42) for $\alpha=1$ and $\beta=1$, we obtain

$$\begin{aligned}\Phi_{\lambda}^{5/2,5/2}(t)&=\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{4\lambda}\\&\quad\times\left(\left[16+\lambda^{2}-\left(\lambda^{2}-8\right)\cosh(4t)\right]\sin(\lambda t)\right.\\&\quad\left.-6\lambda\cos(\lambda t)\sinh(4t)\right)\\&\quad\times\left((\sinh t)^{5/2}(\cosh t)^{5}\right)^{-1},\\\varphi_{\lambda}^{5/2,5/2}(t)&=\frac{15}{8\lambda(16+\lambda^{2})(4+\lambda^{2})}\\&\quad\times\left(\left[16+\lambda^{2}-\left(\lambda^{2}-8\right)\cosh(4t)\right]\sin(\lambda t)\right.\\&\quad\left.-6\lambda\cos(\lambda t)\sinh(4t)\right)\\&\quad\times\left((\sinh t)^{5}(\cosh t)^{5}\right)^{-1}.\end{aligned} \tag{46}$$

Example 8. Using Example 7 and relation (42) for $\alpha = 1$ and $\beta = 1$, we obtain

$$\begin{aligned}
&\Phi_{\lambda}^{7/2,7/2}(t) \\
&\quad = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{8\lambda} \\
&\qquad \times\left(\frac{12\left[16+\lambda^2-\left(\lambda^2-4\right)\cosh(4t)\right]\cosh(2t)\sin(\lambda t)}{(\sinh t)^{7/2}(\cosh t)^7}\right. \\
&\qquad\quad \left. + \frac{\lambda\left[-76-\lambda^2+\left(\lambda^2-44\right)\cosh(4t)\right]\sinh(2t)\cos(\lambda t)}{(\sinh t)^{7/2}(\cosh t)^7}\right), \\
&\varphi_{\lambda}^{7/2,7/2}(t) \\
&\quad = \frac{105}{16\lambda\left(4+\lambda^2\right)\left(16+\lambda^2\right)\left(36+\lambda^2\right)} \\
&\qquad \times\left(\frac{12\left[16+\lambda^2-\left(\lambda^2-4\right)\cosh(4t)\right]\cosh(2t)\sin(\lambda t)}{(\sinh t)^{7}(\cosh t)^7}\right. \\
&\qquad\quad \left. + \frac{\lambda\left[-76-\lambda^2+\left(\lambda^2-44\right)\cosh(4t)\right]\sinh(2t)\cos(\lambda t)}{(\sinh t)^{7}(\cosh t)^7}\right).
\end{aligned} \tag{47}$$

Example 9. Using Example 5 and relation (43) for $\alpha = 3/2$ and $\beta = 3/2$, we obtain

$$\begin{aligned}
\Phi_{\lambda}^{1/2,5/2}(t) = {}& \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{\lambda} \\
&\times\left(\left[4+\lambda^2+\left(\lambda^2-2\right)\cosh(2t)\right]\sin(\lambda t)\right. \\
&\left. +3\lambda\cos(\lambda t)\sinh(2t)\right) \\
&\times\left((\sinh t)^{1/2}(\cosh t)^5\right)^{-1}.
\end{aligned} \tag{48}$$

Using the previous relation and relation (42) for $\alpha = 1/2$ and $\beta = 5/2$, we obtain

$$\begin{aligned}
&\Phi_{\lambda}^{3/2,7/2}(t) \\
&\quad = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{1}{4\lambda} \\
&\qquad \times\left(\frac{\left[\left(16+\lambda^2\right)(-3+4\cosh(2t))+\left(7\lambda^2-8\right)\cosh(4t)\right]\sin(\lambda t)}{(\sinh t)^{3/2}(\cosh t)^7}\right. \\
&\qquad\quad \left. + \frac{2\lambda\left[16+\lambda^2+\left(\lambda^2-14\right)\cosh(2t)\right]\sinh(2t)\cos(\lambda t)}{(\sinh t)^{3/2}(\cosh t)^7}\right).
\end{aligned} \tag{49}$$

Proposition 10. *For $\alpha, \beta, \lambda \in \mathbb{C}$ such that $|\operatorname{Im}\lambda| < \operatorname{Re}(\alpha+\beta+1)$, one has*

$$\begin{aligned}
\forall t > 0, \quad & \frac{2}{\sinh(2t)}\frac{d}{dt}\left[(\sinh t)^{\alpha}\Phi_{\lambda}^{\alpha,\beta}\right](t) \\
&= (\sinh t)^{\alpha-1}\Phi_{\lambda}^{\alpha-1,\beta+1}(t).
\end{aligned} \tag{50}$$

Proof. According to formula (15), we obtain

$$\begin{aligned}
&\frac{1}{\sinh(t)\cosh(t)}\frac{d}{dt}\left[(\sinh t)^{\alpha}J_{\alpha}(x\sinh t)\right] \\
&\quad = x(\sinh t)^{\alpha-1}J_{\alpha-1}(x\sinh t)
\end{aligned} \tag{51}$$

and consequently

$$\begin{aligned}
&(\sinh t)^{\alpha-1}\Phi_{\lambda}^{\alpha-1,\beta+1}(t) \\
&\quad = \frac{2}{\sinh(2t)}\int_0^{\infty} x^{\beta}K_{i\lambda}(x)\frac{d}{dt}\left[(\sinh t)^{\alpha}J_{\alpha}(x\sinh t)\right]dx.
\end{aligned} \tag{52}$$

By using formulas (14), (28), (13), and (34), we deduce the result. □

Corollary 11. *For $\alpha, \beta, \lambda \in \mathbb{C}$ such that $|\operatorname{Im}\lambda| < \operatorname{Re}(\alpha+\beta+1)$ and $m \in \mathbb{N}$ one has*

$$\begin{aligned}
&\left[\frac{2}{\sinh(2t)}\frac{d}{dt}\right]^m\left[(\sinh t)^{\alpha}\Phi_{\lambda}^{\alpha,\beta}\right](t) \\
&\quad = (\sinh t)^{\alpha-m}\Phi_{\lambda}^{\alpha-m,\beta+m}(t), \\
&\Phi_{\lambda}^{\alpha+m,\beta+m} \\
&\quad = \frac{(-1)^m}{(\sinh t)^{\alpha-m}}\left[\frac{2}{\sinh(2t)}\frac{d}{dt}\right]^m\left[(\sinh t)^{\alpha}\Phi_{\lambda}^{\alpha,\beta}\right] \\
&\qquad + \frac{(-1)^{m+1}}{(\sinh t)^m}\sum_{j=0}^{m-1}(-1)^j b_m(j)(\sinh t)^j\Phi_{\lambda}^{\alpha+j,\beta+j}.
\end{aligned} \tag{53}$$

Proof. The result follows by induction argument. □

3.2. Some Reccurrence Relations with Respect to the Dual Variable. We give now some recurrence relations with respect to the dual variable λ in the next proposition.

Proposition 12. *If $|\operatorname{Im}\lambda| < \operatorname{Re}(\alpha+\beta+1)$ and $m \in \mathbb{N}$, then one has*

$$\begin{aligned}
&\forall t > 0, \\
&\Phi_{\lambda-im}^{\alpha,\beta+m}(t) - \Phi_{\lambda+im}^{\alpha,\beta+m}(t) \\
&\quad = \sum_{k=0}^{m-1}(-1)^{m-k-1}b_m^k(i\lambda)\Phi_{\lambda-ik}^{\alpha,\beta+k}(t),
\end{aligned} \tag{54}$$

where $b_m^k(\alpha)$ is the constant given by (17). *For all $t > 0$, we have*

$$\begin{aligned}
&\Phi_{\lambda+i}^{\alpha,\beta+1}(t) + \Phi_{\lambda-i}^{\alpha,\beta+1}(t) \\
&\quad = 2(\beta+1)\Phi_{\lambda}^{\alpha,\beta}(t) + 2\tanh t\frac{d}{dt}\Phi_{\lambda}^{\alpha,\beta}(t).
\end{aligned} \tag{55}$$

Proof. The first equality is an immediate consequence of formulas (27) and (9) while the second is proved by using (26) and the asymptotic representations of J_{α} and K_{α}. □

As a consequence of this proposition, we have the following corollary.

Corollary 13. *If* $|\operatorname{Im}\lambda| < \operatorname{Re}(\alpha+\beta+1)$ *and* $t>0$, *one has*

$$\begin{aligned}\Phi_{\lambda-i}^{\alpha,\beta+1}(t) &= (\beta+1+i\lambda)\,\Phi_{\lambda}^{\alpha,\beta}(t)+\tanh t\frac{d}{dt}\Phi_{\lambda}^{\alpha,\beta}(t),\\ \Phi_{\lambda+i}^{\alpha,\beta+1}(t) &= (\beta+1-i\lambda)\,\Phi_{\lambda}^{\alpha,\beta}(t)+\tanh t\frac{d}{dt}\Phi_{\lambda}^{\alpha,\beta}(t).\end{aligned} \tag{56}$$

Example 14. By using formulas (5.32) (in [1], page 44) and (10), we can see that

$$\varphi_{\lambda}^{1/2,-1/2}(t) = \frac{\sin(\lambda t)}{\lambda\sinh t}, \tag{57}$$

and therefore

$$\Phi_{\lambda}^{1/2,-1/2}(t) = \frac{\Gamma((1+i\lambda)/2)\,\Gamma((1-i\lambda)/2)}{\sqrt{2\pi}}\frac{\sin(\lambda t)}{\lambda\sqrt{\sinh t}}. \tag{58}$$

Using now the last corollary, we can get

$$\Phi_{\lambda+i}^{1/2,1/2}(t) = -i\frac{\Gamma((1+i\lambda)/2)\,\Gamma((1-i\lambda)/2)}{\sqrt{2\pi}}\frac{\sin((i+\lambda)t)}{\cosh t\sqrt{\sinh t}}, \tag{59}$$

which coincided with the value of $\Phi_{\Lambda}^{1/2,1/2}(t)$ already calculated at $\Lambda = i+\lambda$ and we have

$$\begin{aligned}\Phi_{\lambda+i}^{1/2,3/2}(t) = &-\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{2+i\lambda}{2}\right)\Gamma\left(\frac{2-i\lambda}{2}\right)\frac{i}{\lambda}\\ &\times\frac{(\lambda+i)\sin((\lambda+i)t)\cosh t+\cos((\lambda+i)t)\sinh t}{\sqrt{\sinh t}\cosh^3 t}.\end{aligned} \tag{60}$$

Remark 15. By using (33) and the fact that (cf., e.g., [7], page 707)

$$\int_0^{\infty} e^{-x}J_{\alpha}(x\sinh t)\,dx = \frac{1}{\cosh t}\left(\frac{\cosh t-1}{\sinh t}\right)^{\alpha}, \quad t>0,\ \operatorname{Re}(\alpha)>-1, \tag{61}$$

we can get

$$\begin{aligned}\Phi_{i/2}^{\alpha,1/2}(t) &= \sqrt{\frac{\pi}{2}}\frac{1}{\cosh t}\left(\frac{\cosh t-1}{\sinh t}\right)^{\alpha}\\ &= \sqrt{\frac{\pi}{2}}\frac{1}{\cosh t}\tanh^{\alpha}\left(\frac{t}{2}\right), \quad t>0,\ \operatorname{Re}(\alpha)>-1.\end{aligned} \tag{62}$$

As an application of the last corollary, we obtain

$$\begin{aligned}\Phi_{3(i/2)}^{\alpha,3/2}(t) &= \frac{\sqrt{\pi}}{4}\frac{1}{\cosh^3 t}\tanh^{\alpha}\left(\frac{t}{2}\right)(3+2\alpha\cosh t+\cosh(2t)),\\ \Phi_{-i/2}^{\alpha,3/2}(t) &= \frac{\sqrt{\pi}}{2}\frac{1}{\cosh^3 t}\tanh^{\alpha}\left(\frac{t}{2}\right)(1+\alpha\cosh t).\end{aligned} \tag{63}$$

3.3. Some Summations Intervening the "Modified" Jacobi Functions

Proposition 16. *If* $|\operatorname{Im}\lambda| < \operatorname{Re}(1+\beta)$ *and* $t\in(0,\operatorname{arg}\sinh 1)$, *then one has*

$$\Phi_{\lambda}^{0,\beta}(t)+2\sum_{k=1}^{\infty}\Phi_{\lambda}^{2k,\beta}(t) = 2^{\beta-1}\Gamma\left(\frac{1+\beta+i\lambda}{2}\right)\Gamma\left(\frac{1+\beta-i\lambda}{2}\right). \tag{64}$$

$$\Phi_{\lambda}^{0,\beta}(t)+2\sum_{k=1}^{\infty}(-1)^k\Phi_{\lambda}^{2k,\beta}(t) = \sqrt{\frac{\pi}{2}\sinh t}\,\Phi_{\lambda}^{-1/2,\beta+1/2}(t). \tag{65}$$

Proof. Using formulas (18), (34), and (9), we obtain

$$\begin{aligned}&\Phi_{\lambda}^{0,\beta}(t)+2\int_0^{\infty}\left(\sum_{k=1}^{\infty}J_{2k}(x\sinh t)\right)x^{\beta}K_{i\lambda}(x)\,dx\\ &\quad= 2^{\beta-1}\Gamma\left(\frac{1+\beta+i\lambda}{2}\right)\Gamma\left(\frac{1+\beta-i\lambda}{2}\right).\end{aligned} \tag{66}$$

With the help of formulas (13), (28), and (34), we can see that

$$\begin{aligned}&\int_0^{\infty}\left|J_{2k}(x\sinh t)\,x^{\beta}K_{i\lambda}(x)\right|dx\\ &\quad\le\int_0^{\infty}\frac{((x\sinh t)/2)^{2k}}{(2k)!}x^{\operatorname{Re}(\beta)}K_{|\operatorname{Im}(\lambda)|}(x)\,dx\\ &\quad=\frac{((\sinh t)/2)^{2k}}{(2k)!}2^{2k+\operatorname{Re}(\beta)-1}\\ &\qquad\times\Gamma\left(\frac{1+2k+\operatorname{Re}(\beta)-|\operatorname{Im}(\lambda)|}{2}\right)\\ &\qquad\times\Gamma\left(\frac{1+2k+\operatorname{Re}(\beta)+|\operatorname{Im}(\lambda)|}{2}\right),\end{aligned} \tag{67}$$

which permits to conclude the convergence of series

$$\sum_{k=1}^{\infty}\int_0^{\infty}\left|J_{2k}(x\sinh t)\,x^{\beta}K_{i\lambda}(x)\right|dx, \quad \forall t\in(0,\operatorname{arg}\sinh 1), \tag{68}$$

and consequently we obtain the first equality. The second identity is obtained by using formulas (20), (9), and (35). □

Remark 17. With the help of formulas (62) and (65), we deduce that

$$\Phi_{-i/2}^{-1/2,2}(t) = \frac{1-\sinh^2 t}{\sqrt{\sinh t}\cosh^4 t}. \tag{69}$$

Using now Corollary 13, we see that

$$\Phi_{(3/2)i}^{-1/2,2}(t) = \frac{2}{\sqrt{\sinh t}\cosh^4 t}. \tag{70}$$

Proposition 18. *If* $|\operatorname{Im}\lambda| < 2 + \operatorname{Re}(\beta)$ *then one has*

$$\sum_{k=1}^{\infty}(-1)^{k+1}(2k)^2\Phi_{\lambda}^{2k,\beta}(t) = \sqrt{\pi}(\sinh t)^{3/2}\Phi_{\lambda}^{1/2,\beta+3/2}(t), \quad \forall t \in (0, \arg\sinh 1),$$
$$\sum_{k=0}^{\infty}(-1)^{k}(2k+1)^2\Phi_{\lambda}^{2k+1,\beta}(t) = \sqrt{\frac{\pi}{2}}(\sinh t)^{3/2}\Phi_{\lambda}^{-1/2,\beta+3/2}(t), \quad \forall t \in (0, \arg\sinh 1). \tag{71}$$

Proof. The first identity deduced from (21), (9), and (36). The second follows from (22), (9), and (35). □

Remark 19. By using the fact

$$\sum_{k=1}^{\infty}(-1)^{k+1}(2k)^2\tanh^{2k}\left(\frac{t}{2}\right) = \frac{\sinh^2 t}{\cosh^3 t},$$
$$\sum_{k=0}^{\infty}(-1)^{k}(2k+1)^2\tanh^{2k+1}\left(\frac{t}{2}\right) = \frac{(3-\cosh(2t))}{4}\frac{\sinh t}{\cosh^3 t}, \tag{72}$$

formula (62), and the last proposition, we can see that

$$\Phi_{i/2}^{1/2,2}(t) = \frac{1}{\sqrt{2}}\frac{\sqrt{\sinh t}}{\cosh^4 t},$$
$$\Phi_{i/2}^{-1/2,2}(t) = \frac{1}{4}\frac{(3-\cosh(2t))}{\sqrt{\sinh t}\cosh^4 t}. \tag{73}$$

Remark 20. It is well known that the hypergeometric function ${}_2F_1(a,b;c;z)$ tends to the confluent hypergeometric function ${}_0F_1(c;Z)$ as $a,b \to \infty$ and $z \to \infty$ such a way that $abz \to Z$. Consequently, as $\varepsilon \to 0$,

$$\varphi_{\lambda/\varepsilon}^{\alpha,\alpha}(\varepsilon x) = {}_2F_1\left(\alpha+\frac{1}{2}+\frac{i\lambda}{2\varepsilon}, \alpha+\frac{1}{2}-\frac{i\lambda}{2\varepsilon}; \alpha+1; -\sinh^2(\varepsilon x)\right) \tag{74}$$

tends to the normalized Bessel function

$$j_{\alpha}(\lambda x) = {}_0F_1\left(\alpha+1; -\left(\frac{\lambda x}{2}\right)^2\right) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)}\left(\frac{\lambda x}{2}\right)^{2k}. \tag{75}$$

Using this remark, the expression of Bessel function in terms of elementary functions for some particular values of the parameters follows as particular cases of our findings; we get the results given in [7, 9, 11].

References

[1] T. H. Koornwinder, "Jacobi functions and analysis on noncompact semi simple Lie groups," in *Special Functions: Group Theoretical Aspects and ApplicaTions*, R. A. Askey, T. H. Koornwinder, and W. Schempp, Eds., D. Reidel, Dordrecht, The Netherlands, 1984.

[2] H. Exton, "Some new summation formulae for the generalised hypergeometric function of higher order," *Journal of Computational and Applied Mathematics*, vol. 79, no. 2, pp. 183–187, 1997.

[3] H. Exton, "Summation formulae involving the Laguerre polynomial," *Journal of Computational and Applied Mathematics*, vol. 100, no. 2, pp. 225–227, 1998.

[4] M. Neher, "A note on a sum associated with the generalized hypergeometric function," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1527–1534, 2007.

[5] R. Tremblay and B. J. Fugère, "Products of two restricted hypergeometric functions," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 3, pp. 844–852, 1996.

[6] M. Flensted-Jensen and T. H. Koornwinder, "Jacobi functions: the addition formula and the positivity of the dual convolution structure," *Arkiv för Matematik*, vol. 17, no. 1, pp. 139–151, 1979.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Fourth edition prepared by Ju. V. Geronimus and M. Ju. Ceĭtlin. Translated from the Russian by Scripta Technica, Inc. Translation edited by Alan Jeffrey, Academic Press, New York, NY, USA, 1965.

[8] T. Koornwinder, "A new proof of a Paley-Wiener type theorem for the Jacobi transform," *Arkiv för Matematik*, vol. 13, pp. 145–159, 1975.

[9] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, NY, USA, 1972.

[10] S. G. Samko, "On some index relations for Bessel functions," *Integral Transforms and Special Functions*, vol. 11, no. 4, pp. 397–402, 2001.

[11] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Merchant Books, 2008.

[12] N. Ben Salem and K. Trimèche, "Mehler integral transforms associated with Jacobi functions with respect to the dual variable," *Journal of Mathematical Analysis and Applications*, vol. 214, no. 2, pp. 691–720, 1997.