Research Article

$f(R, \phi)$ and $f(R)$ Models: A Self-Similar Approach

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We study the $f(R, \phi)$ models and their particular case, the so-called $f(R)$ -models under the self-similarity hypothesis. In particular, we calculate the exact form that each quantity may take in order that field equations (FEs) admit self-similar solutions. The methods employed allow us to obtain general results that are valid not only for the FRW metric, but also for all the Bianchi types as well as for the Kantowski-Sachs model (under the self-similarity hypothesis and the power-law hypothesis for the scale factors).

1. Introduction

The physical and mathematical importance of the modified gravitational models has been recently pointed out for several authors (see, e.g., [1–4]), since this kind of theories explains in a better way the dynamics of the very early universe as well as its current acceleration. In a recent paper [5] (for a review see [6]), we have studied some particular cases of the so-called generalized scalar-tensor theories (STTs or $F(\phi)$ models). Obviously, although such class of theories is more general than the usual Jordan-Brans-Dicke models, they may be generalized in order to incorporate corrections to the Ricci scalar term as the $f(R)$ models (see, e.g., [7–9] and for a review [10]). Therefore, the purpose of this brief paper is to study, under the self-similarity hypothesis, the generalized $f(R, \phi)$ theories, where the $f(R)$ models (without ϕ) result as subclasses and the generalized SST with $f = F(\phi)R$ studied in [5]. In particular, we study the form that the different quantities may take in order that the field equations (FEs) admit self-similar solutions. To carry out our study, we employ two techniques: the Lie group method (LG) [11] and the matter collineation approach (MC) [12]. In this way, we shall be able to state and proof general theorems that are valid not only for the FRW symmetries, otherwise for all kind of Bianchi types as well as for the Kantowski-Sach models.

The self-similarity property of a space-time is characterized by the existence of a homothetic vector field $\mathcal{H}$ (HVF). For a metric, g , and for a vector field $\mathcal{H} \in \mathcal{X}(\mathcal{M})$,

$(\mathcal{H} = \mathcal{H}_i(t, x, y, z)\partial_{x_i})_{i=1}^4$ the homothetic equation reads: $\mathcal{L}_{\mathcal{H}}g = 2g$ [13]. We study self-similar solutions since, as it has been pointed out by Rosquist and Jantzen [14], they correspond to equilibrium points. In the same way, Coley [15] has stressed the fact that this kind of solutions describes asymptotic behaviour of the Bianchi models playing a dominant role in the dynamics of Bianchi cosmological models (see, e.g., [16] for a review of results in the context of the general relativity).

The paper is organized as follows. In Section 2 we study through the LG method the $f(R, \phi)$ models while Section 3 is devoted to study the $f(R)$ models. We summarize the main results in Section 4.

2. $f(R, \phi)$ Models

Our purpose is to describe a class of scalar-tensor theories of gravity represented by the action [17, 18]:

$$S = \int d^4x \frac{\sqrt{-g}}{\beta} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} Z(\phi) \phi^{\mu\nu} \phi_{;\mu\nu} - U(\phi) + \beta L_m \right], \quad (1)$$

where R is the Ricci scalar, and ϕ stands for the scalar field, where we assume that the scalar field is homogeneous, that is, $\phi = \phi(t)$. We introduce the constant β in order to fix units,

and L_m is a classical matter Lagrangian. The gravitational field equations derived from the action (1) are

$$FG_{\mu\nu} = \beta T_{\mu\nu}^m + Z \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\sigma} \phi^{,\sigma} \right) + (F_{,\mu\nu} - g_{\mu\nu} \square F) - U g_{\mu\nu} + \frac{1}{2} (f - FR) g_{\mu\nu}, \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}^m$ is the energy-momentum tensor, $Z = Z(\phi)$, $U = U(\phi)$ and

$$f = f(R, \phi), \quad F = \partial_R f(R, \phi). \quad (3)$$

The wave equation reads

$$2Z \square \phi = f_{,\phi} - Z \phi'^2 - 2U_{,\phi}, \quad (4)$$

where a prime denotes derivation with respect to time.

We will work under the assumptions

$$f = K(R) g(\phi), \quad K = (R)^r, \quad r \in \mathbb{R}, \quad (5)$$

that is, f may be split in separable variables and the function K takes this particular form, which is appropriate under the self-similarity hypothesis.

Theorem 1. *The scaling and in particular the self-similar solutions admitted for the FEs (2) and (4) under the assumptions (5) have the following form:*

$$\begin{aligned} \phi &= \phi_0 t^n, \quad Z(\phi) = Z_0 \phi^{-m}, \\ U(\phi) &= U_0 \phi^{-(mn-2n+2)}, \\ g(\phi) &= C_1 + C_2 \phi^A, \end{aligned} \quad (6)$$

with $A = (2 - m - (2/n)(1 - r))$, and therefore

$$\begin{aligned} f &= K(R) \cdot g(\phi) = f_0 t^{2n-mn-2}, \\ F &= K_R(R) \cdot g(\phi) = F_0 t^{n(2-m)}, \end{aligned} \quad (7)$$

with $K = R^r = (at^{-2})^r$, $a \in \mathbb{R}$.

Proof. We are going to study (4) through the LG method, that is, we study the kind of functions $f = f(R, \phi)$, $Z(\phi)$, $U(\phi)$, and ϕ such that the FEs (2) and (4) admit power-law and in particular self-similar solutions.

We start by rewriting it in an appropriate way:

$$2Z(\phi'' + ht^{-1}\phi') = -Kg_{\phi} + Z\phi'^2 + 2U_{\phi}, \quad (8)$$

since $H = ht^{-1}$, $h = \sum_{i=1}^3 a_i \in \mathbb{R}^+$, being a_i the exponents of the scale factors, that is, $a(t) = a_0 t^{a_1}$, and so forth, since we are considering not only the FRW case, but also the Bianchi types as well. We also assume that the scalar field is homogeneous, that is, $\phi = \phi(t)$; therefore, a prime denotes a derivative with respect to time. Note that in the self-similar framework, the scalar curvature always behaves as $R \sim t^{-2}$ [14]; therefore, we may assume that $K(R) \sim K(t)$.

Roughly speaking, a symmetry, $X = \xi(x, y)(\partial/\partial x) + \eta(x, y)(\partial/\partial y)$, of a differential equation is an invertible transformation that leaves it form-invariant. By applying the standard Lie procedure (see, e.g., [11]), we need to solve the following overdetermined system of linear partial differential equations for η and ξ (from the extended infinitesimal or prolonged transformations), which allows us to determine the set of the symmetries admitted by (8):

$$Z_{\phi} \xi_{\phi} - 2Z \xi_{\phi\phi} = 0, \quad (9)$$

$$\begin{aligned} (ZZ_{\phi\phi} - Z_{\phi}^2) \eta + ZZ_{\phi} \eta_{\phi} \\ + 4Z^2 ht^{-1} \xi_{\phi} + 2Z^2 \eta_{\phi\phi} - 4Z^2 \xi_{t\phi} = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} 3(2V_{\phi} - K(t)g_{\phi}) \xi_{\phi} + 2ht^{-2} Z(t\xi_t - \xi) \\ + 4Z \eta_{t\phi} - 2Z \xi_{tt} + 2Z_{\phi} \eta_t = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} [K(Z_{\phi} g_{\phi} - Z g_{\phi\phi}) + 2(ZU_{\phi\phi} - Z_{\phi} U_{\phi})] \eta \\ + Z(Kg_{\phi} - 2U_{\phi}) \eta_{\phi} + 2Z(2U_{\phi} - Kg_{\phi}) \xi_t \\ - K' Z g_{\phi} \xi + 2Z^2(ht^{-1} \eta_t + \eta_{tt}) = 0. \end{aligned} \quad (12)$$

The knowledge of one symmetry X might suggest the form of a particular solution as an invariant of the operator X , that is, a solution of $dx/(\xi(x, y)) = dy/(\eta(x, y))$. This particular solution is known as an invariant solution (generalization of similarity solution). The symmetry, $\xi = t$, $\eta = n\phi$, which induces the invariant solution, $\phi = t^n$, leads us to obtain the following restrictions. From (10),

$$Z_{\phi\phi} = \frac{Z_{\phi}^2}{Z} - \frac{Z_{\phi}}{\phi} \implies Z(\phi) = Z_0 \phi^{-m}, \quad (13)$$

with, $m \in \mathbb{R}$. From (12), we get

$$\begin{aligned} 2(ZU_{\phi\phi} - Z_{\phi} U_{\phi}) \eta - 2ZU_{\phi} \eta_{\phi} + 4ZU_{\phi} \xi_t = 0, \\ (KZ_{\phi} g_{\phi} - Z g_{\phi\phi}) \eta + KZ g_{\phi} \eta_{\phi} - 2KZ g_{\phi} \xi_t - K' Z g_{\phi} \xi = 0. \end{aligned} \quad (14)$$

Therefore,

$$U_{\phi\phi} = \left(\frac{n-2}{n} - m \right) \frac{U_{\phi}}{\phi}, \quad (15)$$

where we have taken into account that $Z(\phi) = Z_0 \phi^{-m}$. In this way,

$$U(\phi) = U_0 \phi^{-(1/n)(mn-2n+2)}, \quad (16)$$

while

$$g_{\phi\phi} = \left(1 - m - \frac{1}{n} \left(2 + t \frac{K'}{K} \right) \right) \frac{g_{\phi}}{\phi}, \quad (17)$$

obtaining the following solution

$$g(\phi) = C_1 + C_2 \phi^A, \quad A = \left(2 - m - \frac{1}{n} \left(2 + t \frac{K'}{K} \right) \right). \quad (18)$$

Compare this result with the obtained one in [5]. Note that if $K = t^{-2}$ (in the self-similar framework the Ricci scalar always varies as t^{-2} [14]), that is, $r = 1$, then

$$2 + t \frac{K'}{K} = 0, \quad (19)$$

and therefore

$$g(\phi) = C_1 + C_2 \phi^{2-m}. \quad (20)$$

In any case, when $K = R^r = t^{-2r}$, then

$$g(\phi) = C_1 + C_2 \phi^{2-m-(2/n)(1-r)}, \quad (21)$$

thus

$$\begin{aligned} f &= K(R) \cdot g(\phi) \approx R^r \left(t^{n(2-m-(2/n)(1-r))} \right) = f_0 t^{2n-mn-2}, \\ F &= K_R(R) \cdot g(\phi) \approx R^{r-1} \left(t^{n(2-m-(2/n)(1-r))} \right) = F_0 t^{n(2-m)}. \end{aligned} \quad (22)$$

Therefore, the so-called $f(R, \phi)$ models admit power-law and in particular self-similar solutions. $\square$

As we have shown, for different values of m and r we obtain different models. Note that the case $r = 1$ (the usual one) has been already studied in [5].

3. $f(R)$ Models

As a particular case of the above theories we may study the so-called $f(R)$ models. This kind of models is described by the action:

$$S = \int d^4x \sqrt{-g} [f(R) + L_m], \quad (23)$$

where R is the Ricci scalar, and L_m is a matter Lagrangian as in the above model. By defining $\partial f / \partial R (f = f(R))$, the gravitational field equations derived from the action (23) are [1]:

$$F G_{\mu\nu} = T_{\mu\nu}^m + F_{;\mu;\nu} - g_{\mu\nu} \square F + \frac{1}{2} (f - FR) g_{\mu\nu}, \quad (24)$$

where R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. The trace of the field equations read

$$3\square F + FR - 2f = T, \quad (25)$$

where T is the trace of the stress-energy tensor.

We deduce the form that can acquire the function $f(R)$ in order that the FE admit self-similar and power-law solutions (for the scale factors). For example, in the framework of FRW

symmetries it is already known that $f(R) = R^n$ if the scale factor follow a power-law (see, e.g., [19]). In this section, we extend this result to all the Bianchi types as well as for the Kantowski-Sachs model through two methods. The first one, the matter collineation approach allows us to determine that the unique admitted form for the function $f(R)$ is precisely $f(R) = R^n$ compatible with the self-similar hypothesis. With the second one, by studying the trace equation with the Lie group method we proof that $f(R) = R^n$ is the unique possible form for the function $f(R)$ under the power-law hypothesis for the scale factors.

3.1. Matter Collineation Approach

Theorem 2. *Under the hypothesis of self-similarity, the unique form admitted for the function $f(R)$ is $f(R) = R^n$.*

Proof. We may define an effective stress-energy tensor as follows:

$$T_{\mu\nu}^{\text{eff}} = F^{-1} T_{\mu\nu}^m + F^{-1} (F_{;\mu;\nu} - g_{\mu\nu} \square F) + \frac{1}{2} (f F^{-1} - R) g_{\mu\nu}, \quad (26)$$

which allows to use the standard Einstein equations by simply replacing the fluid quantities with the effective ones, so we have to calculate

$$\mathcal{L}_{\mathcal{H}} T_{\mu\nu}^{\text{eff}} = 0, \quad (27)$$

where $\mathcal{H}$ stands for a homothetic vector field. We calculate the Lie derivative of each component of the effective stress-energy tensor.

- (1) Thus, $\mathcal{L}_{\mathcal{H}} (F^{-1} T_{\mu\nu}^m) = 0$, in components, $\mathcal{L}_{\mathcal{H}} \cdot (F^{-1} T_{11}^m) = 0$,

$$\frac{\rho'}{\rho} - \frac{F'}{F} = -\frac{2}{t} \iff \rho F^{-1} = \rho_0 t^{-2}. \quad (28)$$

As it is noted, we are only able to obtain a relationship between the two quantities F and ρ , but we have not determine their particular form.

- (2) $\mathcal{L}_{\mathcal{H}} T_{\mu\nu}^f = \mathcal{L}_{\mathcal{H}} ((1/2)(f F^{(-1)} - R) g_{\mu\nu}) = 0$. The first term is $\mathcal{L}_{\mathcal{H}} (T_{11}^f) = 0$, that is,

$$t \left(f' - R' F - F' f F^{-1} \right) = -2(f - FR), \quad (29)$$

that we may rewrite in the following form:

$$t \left(FR' - FR' - f_{RR} R' f F^{-1} \right) = -2(f - FR), \quad (30)$$

that we may write as

$$t \left(-f_R^{-1} f_{RR} R' f \right) + 2(f - f_R R) = 0 \quad (31)$$

since $f' = (df/dR)R' = FR'$, and $F' = f_{RR} R'$. Now taking into account the fact that $tR' \approx -2R$, then the equation yields

$$f_{RR} = \frac{f_R^2}{f} - \frac{f}{R}, \quad (32)$$

whose solution is

$$f = C_1 R^n \quad n \in \mathbb{R}. \quad (33)$$

With the other components, $\mathcal{L}_{\mathcal{H}}(T_{ii}^f) = 0$, we only obtain restriction on the scale factors.

(3) $\mathcal{L}_{\mathcal{H}}T_{\mu\nu}^F = 0$, that is,

$$\mathcal{L}_{\mathcal{H}}(F^{-1}(F_{;\mu;\nu} - g_{\mu\nu}\square F)) = 0. \quad (34)$$

We would like to emphasize that this term depends on the metric, that is, on the scale factors so, we mainly obtain restriction on the scale factors. Nevertheless, the first component is $\mathcal{L}_{\mathcal{H}}(T_{11}^F) = 0$,

$$t \left(F'' H + F' H' - \frac{F'^2}{F} H \right) = -2HF', \quad (35)$$

now, taking into account that $H = ht^{-1}$, then we may rewrite it as follows:

$$F'' = \frac{F'^2}{F} - \frac{F'}{t}, \quad (36)$$

and whose solution is

$$F = F_0 t^{C_1}, \quad C_1 \in \mathbb{R}. \quad (37)$$

If, for example, we set $C_1 = 2 - 2n$, from the above result ($f(R) = R^n$), therefore, we arrive to the conclusion that

$$F = F_0 t^{2-2n} = nR^{n-1}. \quad (38)$$

Now if we take into account that $\rho = \rho_0 t^{-\alpha}$ (from the conservation equation for the matter field) being $\alpha = h(\gamma + 1)$ and the result from (28), we arrive to the conclusion that $n = \alpha/2$.

This concludes the proof of the theorem as it is required. $\square$

Since the hypothesis of self-similarity could be strong constraining the form of the function $f(R)$, in the next section we study if under the hypothesis of power-law solutions for the scale factors the function $f(R)$ could take a different form. For this purpose, we go next to study the trace equation through the Lie group method.

3.2. Lie Group Method

Theorem 3. *Under the hypothesis of power-law solutions for the scale factors, the unique form admitted for the function $f(R)$ is $f(R) = R^n$.*

Proof. By studying the trace equation (25), that we rewrite as follows:

$$F'' + ht^{-1}F' = \frac{r}{3}t^{-2}F - \left(\frac{3\gamma-1}{3}\right)\rho_0 t^{-\alpha} - \frac{2}{3}f, \quad (39)$$

where we have set

$$\begin{aligned} H &= ht^{-1}, & R &= rt^{-2}, \\ T &= (3\gamma-1)\rho_0 t^{-\alpha}, & f &= f(t), \end{aligned} \quad (40)$$

with $h, r \in \mathbb{R}$, that is, they are constants, $\alpha = h(\gamma + 1)$.

The general solution of (39) is

$$\begin{aligned} F &= C_2 t^{A_+} + C_1 t^{A_-} + D^{-1} t^{1/2} \\ &\times \left[t^{A_- - 1/2} \int \left(2t^{B_+} f + (3\gamma-1)\rho_0 t^{B_+ - \alpha} \right) dt \right. \\ &\quad \left. - t^{A_+ - 1/2} \int \left(2t^{B_-} f + (3\gamma-1)\rho_0 t^{B_- - \alpha} \right) dt \right], \end{aligned} \quad (41)$$

where

$$\begin{aligned} D &= \sqrt{9(h-1)^2 + 12r}, & C_1, C_2 &\in \mathbb{R}, \\ A_{\pm} &= \frac{1}{6}(3(1-h) \pm D), & B_{\pm} &= \frac{1}{6}(3(1+h) \pm D). \end{aligned} \quad (42)$$

Note that if in (39) we set $F = a = \text{const.}$, then

$$f = \frac{3}{2}rat^{-2} - (3\gamma-1)\rho_0 t^{-\alpha} \iff f(R) \approx R \approx t^{-2}, \quad (43)$$

which implies that $\alpha = 2$ (as in the standard case).

We will determine the form of the unknown function f in order that the FEs (24)-(25) admit power-law solutions. By studying (39) through the Lie group method, then the standard Lie procedure brings us to get following system of PDEs:

$$2\xi_{FF}t^2 = 0, \quad (44)$$

$$2\xi_F ht + \eta_{FF}t^2 - 2\xi_{tF}t^2 = 0, \quad (45)$$

$$\xi_t ht - \xi_F(rF - Ct^{-\alpha+2} - 2ft^2) + 2\eta_{tF}t^2 - \xi_{tt}t^2 - \xi_h = 0, \quad (46)$$

$$\begin{aligned} &\frac{1}{3}\xi \left(2f't^2 + 2rFt^{-1} - C\alpha t^{-\alpha+1} \right) + ht\eta_t \\ &+ \frac{2}{3}\xi_t \left(2t^2 f + Ct^{-\alpha+2} - rF \right) + t^2\eta_{tt} \\ &+ \frac{1}{3}\eta_F \left(rF - 2ft^2 - Ct^{-\alpha+2} \right) - \frac{1}{3}r\eta = 0. \end{aligned} \quad (47)$$

For example, the symmetry $= t/(2-2n)$, $\eta = F$, which induces the invariant solution $F = t^{2-2n}$, brings us to obtain the following restriction on the function f . From (47), we obtain

$$C(2n-\alpha) + 4nf t^\alpha + 2tt^\alpha f' = 0, \quad (48)$$

whose solution is

$$f = \frac{C_1}{t^{2n}} - \frac{1}{2} \frac{C}{t^\alpha} \quad f = t^{-\alpha} \iff n = \frac{\alpha}{2}, \quad (49)$$

and which is consistent with the previous result $F^{-1} \rho = t^{-2}$.

To finish this proof, we would like to emphasize that if one tries to find a symmetry (ξ, η) whose invariant solution coincides with the solution $F = nR^{n-1} = n(rt^{-2})^{n-1}$, from the assumption $f = R^n = (rt^{-2})^n$, as for example, $\xi = 1/(-2n(n-1)rt^{-3}(rt^{-2})^{n-2})$, $\eta = 1$, then it is easily checked that it does not verify (46). $\square$

We may come to the same conclusion, simply rewriting the trace equation in an appropriate way, by considering the relationship between the functions f and $F = f_R$ (i.e., $f_R = df/dR$). In this way, (39) yields

$$f_{RRR}R'^2 + f_{RR}R'' + f_{RR}R'H = \frac{1}{3}RF - \frac{1}{3}T - \frac{2}{3}f, \quad (50)$$

and taking into account that $R = rt^{-2}$, $r \in \mathbb{R}$, and $H = ht^{-1}$, then it may be rewritten in the following way:

$$\begin{aligned} \frac{4}{r}f_{RRR}R^3 + \left(\frac{6}{r} - \frac{2h}{r}\right)f_{RR}R^2 \\ = \frac{1}{3}RF - \frac{(1-3\gamma)\rho_0}{3r^{\alpha/2}}R^{\alpha/2} - \frac{2}{3}f, \end{aligned} \quad (51)$$

where we have identified $T = (1-3\gamma)\rho = (1-3\gamma)\rho_0 t^{-\alpha}$, and, therefore, we can write $T = (((1-3\gamma)\rho_0)/r^{(\alpha/2)})R^{\alpha/2} = CR^{\alpha/2}$. We have found that a particular solution of this third-order ODE is the next one:

$$f = f_0 R^{\alpha/2}, \quad (52)$$

induced by the symmetry $\xi = R$, $\eta = (\alpha/2)f$, which is a consistent solution with the previous one. Note that $f_0 = 6rC/(-4r + r\alpha - 3\alpha^2 + 9\alpha^2 - 6\alpha + 3\alpha h^2 - 6\alpha h)$.

4. Conclusions

We have studied under the self-similar hypothesis the modified gravitational $f(R, \phi)$ models as well as its particular case $f(R)$ models arriving at the conclusion that the $f(R, \phi)$ models admit power-law and in particular self-similar solutions if $f = K(R)g(\phi)$, and $K = (R)^r$ (see Theorem 1). In the same way, we have shown that the so-called $f(R)$ -models, admit power-law and self-similar solutions (see Theorems 2 and 3) if $f(R) = R^n$. This result coincides with the previous one only obtained for FRW symmetries (studying the Friedmann equation). In this work, we have shown that this result is also valid for any Bianchi type under the self-similarity hypothesis as well as under the power-law solution for the scale factors.

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