Research Article

Existence Results for Vector Mixed Quasi-Complementarity Problems

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We introduce strong vector mixed quasi-complementarity problems and the corresponding strong vector mixed quasi-variational inequality problems. We establish equivalence between strong mixed quasi-complementarity problems and strong mixed quasi-variational inequality problem in Banach spaces. Further, using KKM-Fan lemma, we prove the existence of solutions of these problems, under pseudomonotonicity assumption. The results presented in this paper are extensions and improvements of some earlier and recent results in the literature.

1. Introduction

In 1980, Giannessi [1] introduced vector variational inequalities in a finite-dimensional Euclidean space. Motivated by Giannessi [1], Chen and Cheng [2] studied vector variational inequalities in infinite-dimensional Euclidean space and applied them to the vector optimization problems. Since then, vector variational inequalities and their generalizations have been studied and applied to vector optimization problems, vector complementarity problems, game theory, and so forth; see, for example, [1–20] and references therein. It is well known that the complementarity problems are closely related to variational inequality problems. Complementarity theory is introduced by Lemke [21] and Cottle and Dantzig [5]. It has emerged as an active and interesting field for researcher with wide range of applications in pure and applied sciences. Complementarity problems have been extended and generalized in various directions to study a large class of problems arising in industry, finance, optimization, physical, mathematical and engineering sciences; see, for example [4–12, 14, 15, 20]. Recently, vector complementarity problems and their relations with vector variational inequality problems have been investigated under pseudomonotone-type conditions and positiveness-type conditions; see, for example [6, 8–10, 20]. However, to the best of our knowledge, only a few existence results on the strong version of the vector variational inequality and vector complementarity problems were established.

Recently, Huang et al. [12] discussed equivalence results among a vector complementarity problem, a vector variational inequality problem, a vector optimization problem, and weak minimal element problem, under some monotonicity conditions and some inclusive-type conditions in ordered Banach spaces. In 2005, Huang and Fang [9] introduced several classes of strong vector F-complementarity problems and give some existence results for these problems in Banach spaces and discussed the least element problems of feasible sets and presented their relations with the strong vector F-complementarity problems.

Very recently, Khan [22] introduced and studied a generalized vector implicit Quasi-Complementarity problem and generalized vector implicit quasi variational inequality problem. He investigated the nonemptiness and closedness of solution sets of these problems and proved that solution sets of both the problems are equivalent to each other under some suitable conditions.

Inspired and motivated by the work going in this direction, in this paper we introduce and analyze a new class of strong vector Quasi-Complementarity problem and the corresponding strong vector mixed quasi variational inequality problem in the setting of Banach space and establish
equivalence results between them. By using the KKM-Fan lemma, we derive the existence of solutions of strong vector mixed quasi variational inequalities under pseudomonotonicity assumption and show that the solution of the strong vector mixed quasi variational inequality is equivalent to the solution of strong vector mixed Quasi-Complementarity problems under suitable conditions. The results presented in this paper are the generalization and improvement of existing works of [6, 7, 9, 11, 15].

2. Preliminaries

Throughout this paper unless otherwise stated let $X$ and $Y$ be two real Banach spaces. Let $K$ be a nonempty, closed, convex subset of a real Banach space $X$. A nonempty subset $P \subseteq Y$ is called convex, pointed, connected, and reproduced cone, respectively, if it satisfies the following conditions: (i) $\lambda P \subseteq P$, for all $\lambda > 0$ and $P + P \subseteq P$; (ii) $P \cap -P = \{0\}$; (iii) $P \cup -P = X$; (iv) $P - P = X$.

Given $P$ in $Y$, we can define the relations “$\leq_p$” and “$\neq_p$” as follows:

$$x \leq_p y \Leftrightarrow y - x \in P, \quad x \neq_p y \Leftrightarrow y - x \notin P, \forall x, y \in Y.$$  

If “$\leq_p$” is a partial order, then $(Y, \leq_p)$ is called a Banach space ordered by $P$. Let $L(X, Y)$ denote the space of all continuous linear mappings from $X$ into $Y$.

Now, we recall the following concepts and results needed in this paper.

Definition 1. A mapping $f: K \times K \to Y$ is said to be $P$-convex in first argument, if

$$f(tx + (1-t)y, z) \leq_p tf(x, z) + (1-t)f(y, z), \quad \forall x, y, z \in K, \quad t \in [0, 1].$$  

Definition 2. Let $T: K \to L(X, Y)$ and $F: K \times K \to Y$ be the two nonlinear mappings. $T$ is said to be monotone with respect to $F$ if

$$\langle Tx - Ty, x - y \rangle + F(y, x) - F(x, x) \geq_p 0, \quad \forall x, y \in K.$$  

Definition 3. Let $T: K \to L(X, Y)$ and $F: K \times K \to Y$ be the two nonlinear mappings. $T$ is said to be pseudomonotone with respect to $F$ if, for any given $x, y \in K$,

$$\langle Tx, y - x \rangle + F(y, x) - F(x, x) \neq_p 0 \Rightarrow \langle Ty, y - x \rangle + F(y, x) - F(x, x) \neq_p 0.$$  

Remark 4. Every monotone with respect to $F$ is pseudomonotone with respect to $F$ but converse does not hold in general. Definition 3 is vector version of $\theta$-pseudomonotonicity studied by Kazmi et al. in [23, 24].

Example 5. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2$, and

$$T(x) = \begin{pmatrix} 0 \\ \sin x \cos x \end{pmatrix},$$  

$$F(y, x) = \begin{pmatrix} y + x \\ y + x \end{pmatrix}, \quad \forall x, y \in K.$$  

Now,

$$\langle Tx - Ty, x - y \rangle + F(y, x) - F(x, x)$$  

$$= \begin{pmatrix} y - x \\ (1 + \sin x \cos x)(y - x) \end{pmatrix} \neq_p 0.$$

We have $y \geq x$. It follows that

$$\langle Ty, y - x \rangle + F(y, x) - F(x, x)$$  

$$= \begin{pmatrix} y - x \\ (1 + \sin y \cos y)(y - x) \end{pmatrix} \geq_p 0.$$  

So, $T$ is pseudomonotone with respect to $F$. However, for $x = \pi$ and $y = \pi/2$, it follows that

$$\langle Tx - Ty, x - y \rangle + F(y, x) - F(x, x) = \begin{pmatrix} -\pi/2 \\ -\pi/2 \end{pmatrix} \neq_p 0.$$  

This shows that $T$ is not a monotone with respect to $F$.

Definition 6. A mapping $T: K \to L(X, Y)$ is said to be hemi-continuous if, for any $x, y \in K$, the mapping $t \mapsto \langle T(x + t(y - x)), y - x \rangle$ is continuous at $0^+$.

Definition 7. A mapping $F: K \times K \to Y$ is said to be positively homogeneous in first argument, if $F(tx, y) = tF(x, y)$ for all $x, y \in K$ and $t \geq 0$.

Definition 8. Let $K$ be a nonempty subset of a topological vector space $X$. A set-valued map $T: K \to 2^X$ is said to be a KKM mapping if, for each nonempty finite subset $\{x_1, \ldots, x_n\} \subseteq K$, $co\{x_1, \ldots, x_n\} \subseteq \bigcup_{x \in K} T(x)$, where $co$ denotes the convex hull.

Lemma 9 (KKM-Fan Lemma (see [25])). Let $K$ be a nonempty subset of Hausdorff topological vector space $X$. Let $T: K \to 2^X$ be a KKM-mapping such that for each $x \in K$, $T(x)$ is closed and for at least one $x \in K$, $T(x)$ is compact, then

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$  

3. Strong Vector Mixed Quasi-Complementarity Problems

Throughout this section, let $X$ be a real Banach space and let $K \subseteq X$ be a nonempty, closed, and convex subset of $X$. Let $(Y, \leq_p)$ be an ordered Banach space induced by a pointed, closed, convex cone $P$ with nonempty interior. Let
First, we will investigate the equivalences among (SVMQCP)\(_1\), (SVMQCP)\(_2\), and (SVMQVIP), under some suitable assumptions.

**Theorem 11.** (i) Suppose that \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\). If \(x\) solves (SVMQCP)\(_1\) then \(x\) solves (SVMQVIP).

(ii) Let \(F: K \times K \rightarrow Y\) satisfy \(F(2x, y) = 2F(x, y)\), for all \(x, y \in K\) and \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\). If \(x\) solves (SVMQVIP) then \(x\) also solves (SVMQCP)\(_1\).

**Proof.** (i) Let \(x \in K\) be the solution of (SVMQCP)\(_1\). Then \(x \in K\) such that

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0,
\]

\[
\langle Tx, y \rangle + F(y, x) \notin P_{\{0\}} 0, \quad \forall y \in K.
\]

Substituting \(y = x\) in Inclusion (11), we get

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0.
\]

Since \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\), we have

\[
\langle Tx, x \rangle + F(x, x) \geq 0 \quad \text{or} \quad \langle Tx, x \rangle + F(x, x) \leq 0.
\]

From Inclusions (10), (12), and (13), we have

\[
\langle Tx, x \rangle + F(x, x) = 0.
\]

From (11) and (14), we have

\[
\langle Tx, y \rangle + F(y, x) = \langle Tx, x \rangle - F(x, x)
\]

\[
= \langle Tx, y \rangle + F(y, x) - \langle Tx, x \rangle - F(x, x)
\]

\[
= \langle Tx, y \rangle + F(y, x) \notin P_{\{0\}} 0,
\]

for all \(y \in K\). Thus, \(x \in K\) is the solution of (SVMQVIP).

(ii) Now, let \(x \in K\) be the solution of (SVMQVIP), then

\[
\langle Tx, y \rangle + F(y, x) - F(x, x) \notin P_{\{0\}} 0, \quad \forall y \in K.
\]

Since \(F(2x, y) = 2F(x, y)\), for all \(x, y \in K\), therefore it follows that \(F(0, y) = 0\), for all \(y \in K\). By substituting \(y = 2x\) and \(y = 0\), respectively, in (16), we get

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0.
\]

Since \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\), we have

\[
\langle Tx, x \rangle + F(x, x) \geq 0 \quad \text{or} \quad \langle Tx, x \rangle + F(x, x) \leq 0.
\]

From (17) and (18), we have

\[
\langle Tx, x \rangle + F(x, x) = 0.
\]

By using Inclusions (16) and (19), we have

\[
\langle Tx, y \rangle + F(y, x)
\]

\[
= \langle Tx, y \rangle - F(x, x) + \langle Tx, x \rangle + F(x, x)
\]

\[
= \langle Tx, y \rangle + F(x, x) - F(x, x)
\]

\[
\notin P_{\{0\}} 0,
\]

First, we will investigate the equivalences among (SVMQCP)\(_1\), (SVMQCP)\(_2\), and (SVMQVIP), under some suitable assumptions.

**Theorem 11.** (i) Suppose that \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\). If \(x\) solves (SVMQCP)\(_1\) then \(x\) solves (SVMQVIP).

(ii) Let \(F: K \times K \rightarrow Y\) satisfy \(F(2x, y) = 2F(x, y)\), for all \(x, y \in K\) and \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\). If \(x\) solves (SVMQVIP) then \(x\) also solves (SVMQCP)\(_1\).

**Proof.** (i) Let \(x \in K\) be the solution of (SVMQCP)\(_1\). Then \(x \in K\) such that

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0,
\]

\[
\langle Tx, y \rangle + F(y, x) \notin P_{\{0\}} 0, \quad \forall y \in K.
\]

Substituting \(y = x\) in Inclusion (11), we get

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0.
\]

Since \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\), we have

\[
\langle Tx, x \rangle + F(x, x) \geq 0 \quad \text{or} \quad \langle Tx, x \rangle + F(x, x) \leq 0.
\]

From Inclusions (10), (12), and (13), we have

\[
\langle Tx, x \rangle + F(x, x) = 0.
\]

From (11) and (14), we have

\[
\langle Tx, y \rangle + F(y, x) - F(x, x)
\]

\[
= \langle Tx, y \rangle + F(y, x) - \langle Tx, x \rangle - F(x, x)
\]

\[
= \langle Tx, y \rangle + F(y, x) \notin P_{\{0\}} 0,
\]

for all \(y \in K\). Thus, \(x \in K\) is the solution of (SVMQVIP).

(ii) Now, let \(x \in K\) be the solution of (SVMQVIP), then

\[
\langle Tx, y \rangle + F(y, x) - F(x, x) \notin P_{\{0\}} 0, \quad \forall y \in K.
\]

Since \(F(2x, y) = 2F(x, y)\), for all \(x, y \in K\), therefore it follows that \(F(0, y) = 0\), for all \(y \in K\). By substituting \(y = 2x\) and \(y = 0\), respectively, in (16), we get

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0.
\]

\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} 0.
\]

Since \((Tz, z) + F(z, z) \in P \cup (-P)\), for all \(z \in K\), we have

\[
\langle Tx, x \rangle + F(x, x) \geq 0 \quad \text{or} \quad \langle Tx, x \rangle + F(x, x) \leq 0.
\]

From (17) and (18), we have

\[
\langle Tx, x \rangle + F(x, x) = 0.
\]

By using Inclusions (16) and (19), we have

\[
\langle Tx, y \rangle + F(y, x)
\]

\[
= \langle Tx, y \rangle - F(x, x) + \langle Tx, x \rangle + F(x, x)
\]

\[
= \langle Tx, y \rangle + F(x, x) - F(x, x)
\]

\[
\notin P_{\{0\}} 0,
\]
for all \( y \in K \), which implies that \( x \) solves \((\text{SVMQCP})_1\).

**Remark 12.** The condition \( F(2x, y) = 2F(x, y) \), for all \( x, y \in K \) holds if \( F \) is positively homogeneous; that is, \( F(tx, y) = tF(x, y) \) for all \( t \geq 0 \). Hence, Theorem 11 generalizes and improves the theorems in [6, 9, 11, 14, 15].

Here we give an example of a function \( F \), which satisfies the condition \( F(2x, y) = 2F(x, y) \), for all \( x, y \in K \) but not a positively homogeneous, which implies that previously known results in [6, 9, 11, 14, 15] cannot be applied.

**Example 13.** Let \( F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), defined by
\[
F(x, y) = \begin{cases} 
2x, & \text{if } x \text{ rational,} \\
0, & \text{if } x \text{ irrational.}
\end{cases}
\] (21)

Then \( F \) satisfies \( F(2x, y) = 2F(x, y) \) but it is not positively homogeneous.

**Theorem 14.** (a) If \( x \) solves Problem \((\text{SVMQCP})_2\) then \( x \) solves \((\text{SVMQVIP})\).

(b) Let \( F : K \times K \rightarrow Y \) satisfy \( F(2x, y) = 2F(x, y) \), for all \( x, y \in K \) and \( \langle Tz, z \rangle + F(z, z) \in P \cup (-P) \), for all \( z \in K \). If \( x \) solves \((\text{SVMQVIP})\) then \( x \) solves \((\text{SVMQCP})_2\).

**Proof.** (a) Let \( x \in K \) be the solution of \((\text{SVMQCP})_2\). Then \( x \in K \) such that
\[
\langle Tx, x \rangle + F(x, x) = 0,
\] (22)
\[
\langle Tx, y \rangle + F(y, x) \notin P_{\{0\}}, \quad \forall y \in K.
\] (23)

Now,
\[
\langle Tx, y - x \rangle + F(y, x) - F(x, x) = \langle Tx, y \rangle + F(y, x) - \langle Tx, x \rangle - F(x, x)
\] (24)
\[
= \langle Tx, y \rangle + F(y, x) \notin P_{\{0\}}, \quad \forall y \in K.
\]
(25)

Since \( F(2x, y) = 2F(x, y) \), for all \( x, y \in K \), therefore it follows that \( F(0, y) = 0 \), for all \( y \in K \). By substituting \( y = 2x \) and \( y = 0 \), respectively, in (24), we get
\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}},
\]
\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}}.
\]
(26)

Since \( \langle Tz, z \rangle + F(z, z) \in P \cup (-P) \), for all \( z \in K \), we have
\[
\langle Tx, x \rangle + F(x, x) \notin P_{\{0\}} \quad \text{or} \quad \langle Tx, x \rangle + F(x, x) \in P_{\{0\}}.
\]
(27)

From (25) and (26), we have
\[
\langle Tx, x \rangle + F(x, x) = 0.
\] (28)

By using (27), we have
\[
\langle Tx, y \rangle + F(y, x) = \langle Tx, y - x \rangle + F(y, x) - F(x, x)
\] (29)
\[
+ \langle Tx, x \rangle + F(x, x)
\]
\[
= \langle Tx, y - x \rangle + F(y, x) - F(x, x)
\] (30)
\[
\notin P_{\{0\}},
\]
for all \( y \in K \). Then (27) and (28) imply that \( x \) solves \((\text{SVMQCP})_2\).

**4. Existence Results**

First, we prove following Minty-type lemma with the help of pseudomonotone mapping with respect to \( F \).

**Lemma 15.** Let \( F : K \times K \rightarrow Y \) be \( P \)-convex in first argument and let \( T : K \rightarrow L(X, Y) \) be a hemicontinuous mapping and pseudomonotone with respect to \( F \). Then the following two problems are equivalent:

(A) \( x \in K, \quad \langle Tx, y - x \rangle + F(y, x) - F(x, x) \notin P_{\{0\}}, \quad \forall y \in K \).

(B) \( x \in K, \quad \langle Ty, y - x \rangle + F(y, x) - F(x, x) \geq P_{\{0\}}, \quad \forall y \in K \).

**Proof.** (29) \( \Rightarrow \) (30). The result directly follows from pseudomonotonicity with respect to \( F \).

Now, (30) \( \Rightarrow \) (29). For any given \( y \in K \), we know that \( y = ty + (1 - t)x \in K \), for all \( t \in (0, 1) \), as \( K \) is convex. Since \( x \in K \) is a solution of problem (30), so for each \( x \in K \), it follows that
\[
\langle Ty, y - x \rangle + F(y, x) - F(x, x) \geq P_{\{0\}}.
\]
(31)

Now, we have
\[
t \langle Ty, y - x \rangle + t \langle F(y, x) - F(x, x) \rangle
\]
\[
\geq \langle Ty, y - x \rangle + F(y, x) - F(x, x) \geq P_{\{0\}}.
\]
(32)

For \( t > 0 \), we get
\[
\langle Ty, y - x \rangle + F(y, x) - F(x, x) \geq P_{\{0\}}.
\]
(33)

Since \( T \) is hemicontinuous and \( P \) is closed, letting \( t \rightarrow 0^+ \) in inclusion (33), we get
\[
\langle Tx, y - x \rangle + F(y, x) - F(x, x) \geq P_{\{0\}}, \quad \forall y \in K.
\]
(34)

Hence,
\[
\langle Tx, y - x \rangle + F(y, x) - F(x, x) \notin P_{\{0\}}, \quad \forall y \in K.
\]
(35)

Therefore, \( x \in K \) is solution of problem (29). This completes the proof.
Now, with the help of Lemma 15, we have following existence theorem for (SVMQVIP).

**Theorem 16.** Let $X$ be real reflexive Banach space and let $Y$ be a Banach space. Let $K \subset X$ be a nonempty, bounded, closed, and convex subset of $X$. Let $F : K \times K \rightarrow Y$ be P-convex and upper semicontinuous in first and second arguments, respectively. Let $T : K \rightarrow L(X, Y)$ be hemicontinuous and pseudomonotone with respect to $F$. Then (SVMQVIP) has solution.

**Proof.** Define two set-valued mappings $F, G : K \rightarrow 2^K$ as follows:

$$G(y) = \{x \in K : \langle Tx, y - x \rangle + F(x, x) \in P \cup \{0\}, \forall y \in K, \}$$

$$H(y) = \{x \in K : \langle Ty, y - x \rangle + F(x, x) \geq 0, \forall y \in K, \}$$

$G(y)$ and $H(y)$ are nonempty, since $y \in G(y) \cap H(y)$. We claim that $G$ is a KKM mapping. If this is not true, then there exists a finite set $\{y_1, \ldots, y_n\} \subset K$ and $t_i \geq 0, i = 1, \ldots, n$ with $\sum_{i=1}^{n} t_i = 1$ such that $y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} G(y_i)$. Now, by the definition of $G$, we have

$$\langle Ty, y_i - y \rangle + F(y_i, y) - F(y, y) \leq P, i = 1, \ldots, n.$$ (37)

Now, we have

$$0 = \langle Ty, y - y \rangle + F(y, y) - F(y, y)$$

$$= \sum_{i=1}^{n} \langle Ty, t_i y_i - y \rangle + F(\sum_{i=1}^{n} t_i y_i, y) - F(y, y)$$

$$= \sum_{i=1}^{n} t_i \langle Ty, y_i - y \rangle + F(y_i, y) - F(y, y) \leq P, \forall y \in K,$$ (38)

which is not possible. Thus, our claim is verified. So $G$ is a KKM mapping.

Now, since $T$ is pseudomonotone with respect to $F$, therefore $G(y) \subset H(y)$ for every $y \in K$ and so $H$ is also a KKM mapping. Now we claim that for each $y \in K$, $H(y) \subset K$ is closed in the weak topology of $X$.

Indeed, suppose $\bar{y} \in \overline{H(y)}^\ast$, the weak closure of $H(y)$. Since $X$ is reflexive, there is a sequence $\{x_n\}$ in $H(y)$ such that $\{x_n\}$ converges weakly to $\bar{y}$ in $K$. Then

$$\langle Ty, y - x_n \rangle + F(x, x_n) \geq 0.$$ (39)

Since $F(y, \cdot)$ is upper semicontinuous and $P$ is closed, therefore,

$$\langle Ty, y - \bar{x} \rangle + F(y, \bar{x}) - F(x, \bar{x}) \geq 0 \quad (40)$$

and so $\bar{x} \in H(y)$. This shows that $H(y)$ is weakly closed, for each $y \in K$. Our claim is then verified. Since $X$ is reflexive and $K \subset X$ is nonempty, bounded, closed and convex, $K$ is a weakly compact subset of $X$ and so $H(y)$ is also weakly compact. According to Lemma 9 (KKM-Fan Lemma),

$$\bigcap_{y \in K} H(y) \neq \emptyset.$$ (41)

This implies that there exists $x \in K$ such that

$$\langle Ty, y - x \rangle + F(y, x) - F(x, x) \geq 0, \forall y \in K.$$ (42)

Therefore by Lemma 15, we conclude that there exists $x \in K$ such that

$$\langle Ty, y - x \rangle + F(y, x) - F(x, x) \in P \cup \{0\}, \forall y \in K.$$ (43)

This completes the proof. \qed

**Theorem 17.** Let $F : K \times K \rightarrow Y$ satisfy $F(2x, y) = 2F(x, y)$, for all $x, y \in K$. If all the assumptions of Theorem 16 hold, then (SVMQCP), is solvable. In addition, if $(Tz, z) + F(z, z) \in P \cup \{0\}$, for all $z \in K$, then (SVMQCP)$_2$ is solvable.

**Proof.** The conclusion follows directly from Theorems 11, 14, and 16. \qed

**References**


