Research Article

Strong Convergence Theorems for a Pair of Strictly Pseudononspreading Mappings

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Let H be a real Hilbert space. Let $T_1, T_2 : H \rightarrow H$ be k_1 -, k_2 -strictly pseudononspreading mappings; let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. For given $x_0 \in H$, the sequence $\{x_n\}$ is generated iteratively by $x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{w_1}(\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) T_{w_2} x_n)$, $\forall n \in \mathbb{N}$, where $T_{w_i} = (1 - w_i)I + w_i T_i$ with $i = 1, 2$ and $B : H \rightarrow H$ is strongly monotone and Lipschitzian. Under some mild conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, we prove that the sequence $\{x_n\}$ converges strongly to the set $F_{ix}(T_1) \cap F_{ix}(T_2)$ of fixed points of a pair of strictly pseudononspreading mappings T_1 and T_2 .

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, and let C be a nonempty, closed, and convex subset of H , respectively. Recall the following definitions.

Definition 1. Let $A : C \rightarrow H$ be a nonlinear mapping.

- (1) A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \quad (1)$$

- (2) A is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C. \quad (2)$$

For such a case, A is said to be α -strongly-monotone.

- (3) A is said to be inverse strongly if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (3)$$

For such a case, A is said to be α -inverse-strongly monotone.

The classical variational inequality which is denoted by $VI(A, C)$ is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (4)$$

The variational inequality has been extensively studied in the literature; see, for example, [1, 2] and the reference therein. Recall that T is a nonexpansive mapping of C into itself; that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (5)$$

The set of fixed points of T is the set $F_{ix}(T) = \{x \in H, Tx = x\}$.

In 2011, Osilike and Isiogugu [3] introduced a new class of mappings, the so-called k -strictly pseudononspreading; that is, a mapping $T : D(T) \subseteq H \rightarrow H$ is k -strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 \\ &\quad + 2\langle x - Tx, y - Ty \rangle, \end{aligned} \quad (6)$$

for all $x, y \in D(T)$. They showed that the class of nonspreading mappings is properly contained in the class of strictly pseudononspreading mappings.

The iteration procedure of Mann's type for approximating fixed points of a nonexpansive mapping T is the following: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in N, \quad (7)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$; see [4]. For two nonexpansive mappings S and T , Takahashi and Tamura [5] considered the following iteration procedure: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S(\beta_n x_n + (1 - \beta_n) T x_n), \quad \forall n \in N, \quad (8)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2010, Tian [6] introduced the following general viscosity iterative scheme for finding an element of set of solutions to the fixed point of nonexpansive mapping in Hilbert space. Define sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) T x_n, \quad n \geq 0, \quad (9)$$

where B is k -Lipschitzian and η -strongly monotone operator and T is a nonexpansive mapping on H ; then he proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (9) converges strongly to the unique solution $x^* \in C$ of the variational inequality

$$\langle (\gamma f - \mu B) x^*, x - x^* \rangle \leq 0, \quad \forall x \in C, \quad (10)$$

where $C = \text{Fix}(T)$.

In this paper, motivated by Takahashi and Tamura [5] and Tian [6], we introduce the following iteration scheme for finding a common point of the set $F_{ix}(T_1) \cap F_{ix}(T_2)$ of fixed points of a pair of strictly pseudononspreading mappings T_1 and T_2 :

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1 - \beta_n) T_{\omega_1} \\ &\times (\alpha_n \gamma f(x_n) \\ &+ (I - \mu \alpha_n B) T_{\omega_2} x_n), \quad \forall n \in N, \end{aligned} \quad (11)$$

where $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$ with $i = 1, 2$ and $B : H \rightarrow H$ is η -strongly monotone and k -Lipschitzian on H with $k > 0, \eta > 0$. Under suitable conditions, we prove a strong convergence theorem, which is different from the results of general viscosity iterative scheme in [6].

2. Preliminaries

We need some facts and tools in Hilbert space H which are listed as in the following lemmas.

Definition 2. A mapping T is said to be demiclosed, if for any sequence $\{x_n\}$ which weakly converges to y , and if the sequence $\{T x_n\}$ strongly converges to z , then $T(y) = z$.

Definition 3. $T : H \rightarrow H$ is called demicontractive on H , if there exists a constant $\alpha < 1$ such that

$$\begin{aligned} \|Tx - q\|^2 &\leq \|x - q\|^2 + \alpha \|x - Tx\|^2, \\ \forall (x, q) &\in H \times F_{ix}(T). \end{aligned} \quad (12)$$

Remark 4. Every k -strictly pseudononspreading mapping with a nonempty fixed point set $F_{ix}(T)$ is demicontractive (see [7, 8]).

Remark 5 (see [9]). Let T be a α -demicontractive mapping on H with $F_{ix}(T) \neq \emptyset$ and $T_\omega = (1 - \omega)I + \omega T$ for $\omega \in (0, \infty)$.

(A1) $T\alpha$ -demicontractive is equivalent to

$$\begin{aligned} \langle x - Tx, x - q \rangle &\geq \frac{1}{2} (1 - \alpha) \|x - Tx\|^2, \\ \forall (x, q) &\in H \times F_{ix}(T). \end{aligned} \quad (13)$$

(A2) $F_{ix}(T) = F_{ix}(T_\omega)$ if $\omega \neq 0$.

Remark 6. According to $I - T_\omega = \omega(I - T)$ with T being a k -strictly pseudononspreading mapping, we obtain

$$\begin{aligned} \langle x - T_\omega x, x - q \rangle &\geq \frac{\omega(1 - k)}{2} \|x - Tx\|^2, \\ \forall (x, q) &\in H \times F_{ix}(T). \end{aligned} \quad (14)$$

Proposition 7 (see [3]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be a k -strictly pseudononspreading mapping. If $F_{ix}(T) \neq \emptyset$, then it is closed and convex.

Proposition 8 (see [3]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be a k -strictly pseudononspreading mapping. Then $(I - T)$ is demiclosed at 0.

Lemma 9. Let H be a real Hilbert space. The following expressions hold:

$$\begin{aligned} \text{(i)} \quad &\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \\ &\forall x, y \in H, \forall t \in [0, 1]. \\ \text{(ii)} \quad &\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \end{aligned}$$

Lemma 10 (see [10]). Let C be a closed convex subset of a Hilbert space H , and let $T : H \rightarrow H$ be a k -strictly pseudononspreading mapping with a nonempty fixed point set. Let $k \leq \omega < 1$ be fixed, and define $T_\omega : C \rightarrow C$ by

$$T_\omega x = (1 - \omega)x + \omega T x, \quad \forall x \in C. \quad (15)$$

Then $F_{ix}(T_\omega) = F_{ix}(T)$.

Lemma 11 (see [11]). Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 0, \quad (16)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$(i) \sum_{n=1}^{\infty} \gamma_n = \infty,$$

$$(ii) \limsup_{n \rightarrow \infty} (\delta_n / \gamma_n) = 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Lemma 12 (see [12]). Let $\{\mathcal{T}_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\mathcal{T}_{n_j}\}_{j \geq 0}$ of $\{\mathcal{T}_n\}$ which satisfies $\mathcal{T}_{n_j} < \mathcal{T}_{n_j+1}$ for all $j \geq 0$. Also consider that the sequence of integers $\{\delta(n)\}_{n \geq n_0}$ is defined by

$$\delta(n) = \max \{k \leq n \mid \mathcal{T}_k < \mathcal{T}_{k+1}\}. \quad (17)$$

Then $\{\delta(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} \delta(n) = \infty$, $\forall n \geq n_0$; it holds that $\mathcal{T}_{\delta(n)} < \mathcal{T}_{\delta(n)+1}$ and one has

$$\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}. \quad (18)$$

Lemma 13. Let K be a closed convex subset of a real Hilbert space H , given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in K. \quad (19)$$

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T_1, T_2 : C \rightarrow C$ be k_1 -, k_2 -strictly pseudononspreading mappings with nonempty fixed point set $\Omega = F_{ix}(T_1) \cap F_{ix}(T_2)$. Let f be a L -Lipschitz mapping on H with coefficient $L > 0$. Assume that the set Ω is nonempty. Since Ω is closed and convex, the nearest point projection from C onto Ω is well defined. Recall that $B : H \rightarrow H$ is η -strongly monotone and k -Lipschitzian on H with $k > 0$, $\eta > 0$.

Lemma 14 (see [13]). Let H be a real Hilbert space and let B be a k -Lipschitzian and η -strongly monotone operator with $k > 0$, $\eta > 0$. Let $0 < \mu < (2\eta/k^2)$ and $\tau = \mu(\eta - (\eta k^2/2))$. Then for $t \in \min\{1, (1/\tau)\}$, $I - t\mu B$ is in contradiction with constant $1 - t\tau$.

Lemma 15. Let T be a k -strictly pseudononspreading mapping on H , and $T_\omega x = (1 - \omega)x + \omega Tx$ with $0 < k < \omega < 1/2$. Then

$$\begin{aligned} & \|T_\omega x - T_\omega y\|^2 \\ & \leq \|x - y\|^2 + \frac{2}{\omega} \langle x - T_\omega x, y - T_\omega y \rangle, \quad \forall x, y \in C. \end{aligned} \quad (20)$$

Proof. For $x, y \in H$, we have

$$\begin{aligned} & \|T_\omega x - T_\omega y\|^2 \\ & = (1 - \omega) \|x - y\|^2 + \omega \|Tx - Ty\|^2 \\ & \quad - \omega(1 - \omega) \|x - Tx - (y - Ty)\|^2 \\ & \leq (1 - \omega) \|x - y\|^2 \\ & \quad + \omega [\|x - y\|^2 + k \|x - Tx - (y - Ty)\|^2 \\ & \quad + 2 \langle x - Tx, y - Ty \rangle] \\ & \quad - \omega(1 - \omega) \|x - Tx - (y - Ty)\|^2 \\ & = \|x - y\|^2 + 2\omega \langle x - Tx, y - Ty \rangle \\ & \quad - \omega(1 - \omega)(1 - k) \|x - Tx - (y - Ty)\|^2 \\ & \leq \|x - y\|^2 + 2\omega \langle x - Tx, y - Ty \rangle \\ & = \|x - y\|^2 + \frac{2}{\omega} \langle x - T_\omega x, y - T_\omega y \rangle. \end{aligned} \quad (21)$$

□

Remark 16. If $y = p \in F_{ix}(T)$ and from Lemma 15, we can easily claim that $\|T_\omega x - p\| \leq \|x - p\|$, $\forall x \in C$.

Theorem 17. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $T_1, T_2 : C \rightarrow C$ be k_1 -, k_2 -strictly pseudononspreading mappings and demiclosed. Let f be L -Lipschitz mapping on H with coefficient $L > 0$ and let $B : H \rightarrow H$ be η -strongly monotone and k -Lipschitzian on H with $k > 0$, $\eta > 0$, $0 < k_1$, and $k_2 < \omega < 1/2$. Let the sequence of $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (c1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (c2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_{n+1}/\alpha_n = 1$,
- (c3) $0 < c \leq \beta_n \leq d < 1$.

Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - (\mu k^2/2))/L = \tau/L$, and let the following sequence $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$$\begin{aligned} x_{n+1} & = \beta_n x_n + (1 - \beta_n) \\ & \quad \times T_{\omega_1} (\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) T_{\omega_2} x_n), \end{aligned} \quad (22)$$

where $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$ with $0 < k_i < \omega_i < 1$, $i = 1, 2$. Then $\{x_n\}$ converges strongly to the unique element x^* in $\Omega = F_{ix}(T_1) \cap F_{ix}(T_2)$ verifying

$$x^* = (P_\Omega \cdot f) x^*, \quad (23)$$

which equivalently solves the following variational inequality problem:

$$x^* \in \Omega, \quad \langle (\gamma f - \mu B) x^*, v - x^* \rangle \leq 0, \quad \forall v \in \Omega. \quad (24)$$

Proof. Let $x^* \in \Omega$. According to Remark 16, we have

$$\|T_{\omega}x_n - x^*\| \leq \|x_n - x^*\|. \quad (25)$$

Let $y_n = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B)T_{\omega_2}x_n$. From (22), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta_n x_n + (1 - \beta_n)T_{\omega_1}y_n - x^*\| \\ &\leq \beta_n \|x_n - x^*\| + (1 - \beta_n) \|T_{\omega_1}y_n - x^*\| \\ &\leq \beta_n \|x_n - x^*\| + (1 - \beta_n) \|y_n - x^*\|, \end{aligned} \quad (26)$$

And, from Lemma 14, we also obtain that

$$\begin{aligned} \|y_n - x^*\| &= \|\alpha_n \gamma (f(x_n) - f(x^*)) \\ &\quad + (I - \mu \alpha_n B)(T_{\omega_2}x_n - x^*) \\ &\quad + \alpha_n (\gamma f(x^*) - \mu Bx^*)\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(x^*)\| \\ &\quad + (1 - \alpha_n \tau) \|T_{\omega_2}x_n - x^*\| \\ &\quad + \alpha_n \|\gamma f(x^*) - \mu Bx^*\| \\ &\leq \alpha_n \gamma L \|x_n - x^*\| \\ &\quad + (1 - \alpha_n \tau) \|x_n - x^*\| \\ &\quad + \alpha_n \|\gamma f(x^*) - \mu Bx^*\|. \end{aligned} \quad (27)$$

Together with (26) and (27), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [1 - \alpha_n (1 - \beta_n) (\tau - \gamma L)] \|x_n - x^*\| \\ &\quad + \alpha_n (1 - \beta_n) \|\gamma f(x^*) - \mu Bx^*\|. \end{aligned} \quad (28)$$

Putting $M_1 = \max\{\|x_0 - x^*\|, \|\gamma f(x^*) - \mu Bx^*\|\}$, we clearly obtain $\|x_n - x^*\| \leq M_1$. By induction, we can deduce that $\{x_n\}$ is bounded and the sequences $\{f(x_n)\}$ and $\{T_{\omega_i}x_n\}$, $i = 1, 2$, are also bounded.

Next, we show the following estimation:

$$\begin{aligned} \mathcal{T}_{n+1} &\leq (1 - \alpha_n \delta) \mathcal{T}_n \\ &\quad - \delta (\|x_n - T_2x_n\|^2 + \|T_{\omega_1}y_n - x_n\|^2) \\ &\quad + 2\alpha_n (1 - \beta_n) \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle, \end{aligned} \quad (29)$$

where $\mathcal{T}_n = \|x_n - x^*\|^2$ and $\delta > 0$ is chosen so that

$$\begin{aligned} \delta &\leq \min \{ \omega_2 (1 - \alpha_n \tau) (1 - \beta_n) (1 - \omega_2) (1 - k_2), \\ &\quad \beta_n (1 - \beta_n) \}. \end{aligned} \quad (30)$$

From Lemma 15, we obtain

$$\begin{aligned} \|T_{\omega_2}x_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \omega_2 (1 - \omega_2) (1 - k_2) \|x_n - T_2x_n\|^2. \end{aligned} \quad (31)$$

By (25) and (31), we get

$$\begin{aligned} \|y_n - x^*\|^2 &= \|\alpha_n (\gamma f(x_n) - \mu Bx^*) + (I - \mu \alpha_n B)(T_{\omega_2}x_n - x^*)\|^2 \\ &\leq (1 - \alpha_n \tau) \|T_{\omega_2}x_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle \\ &\leq (1 - \alpha_n \tau) \|x_n - x^*\|^2 \\ &\quad - \omega_2 (1 - \alpha_n \tau) (1 - \omega_2) (1 - k_2) \|x_n - T_2x_n\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle. \end{aligned} \quad (32)$$

Consequently,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|T_{\omega_1}y_n - x^*\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_{\omega_1}y_n - x_n\|^2 \\ &\leq \beta_n \|x_n - x^*\|^2 - \beta_n (1 - \beta_n) \|T_{\omega_1}y_n - x_n\|^2 \\ &\quad + (1 - \beta_n) \|y_n - x^*\|^2 \\ &\leq \beta_n \|x_n - x^*\|^2 - \beta_n (1 - \beta_n) \|T_{\omega_1}y_n - x_n\|^2 \\ &\quad + (1 - \beta_n) (1 - \alpha_n \tau) \|x_n - x^*\|^2 \\ &\quad - \omega_2 (1 - \alpha_n \tau) (1 - \beta_n) (1 - \omega_2) (1 - k_2) \\ &\quad \times \|x_n - T_2x_n\|^2 \\ &\quad + 2\alpha_n (1 - \beta_n) \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle \\ &\leq (1 - \alpha_n \tau (1 - \beta_n)) \|x_n - x^*\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_{\omega_1}y_n - x_n\|^2 \\ &\quad - \omega_2 (1 - \alpha_n \tau) (1 - \beta_n) (1 - \omega_2) (1 - k_2) \\ &\quad \times \|x_n - T_2x_n\|^2 \\ &\quad + 2\alpha_n (1 - \beta_n) \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle, \end{aligned} \quad (33)$$

and the desired inequality (29) follows.

Finally, we show $\lim_{n \rightarrow \infty} \mathcal{T}_n = 0$ by considering two possible cases. $\square$

Case 1. $\{\mathcal{T}_n\}$ is nonincreasing sequence with some $n \geq n_0$. In this case, $\{\mathcal{T}_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below). From (29), we have

$$\delta (\|x_n - T_2x_n\|^2 + \|T_{\omega_1}y_n - x_n\|^2) \leq \alpha_n M + (\mathcal{T}_n - \mathcal{T}_{n+1}), \quad (34)$$

where $0 < M = 2\langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle$. Consequently, both $\|x_n - T_2 x_n\|$ and $\|T_{\omega_1} y_n - x_n\|$ converge to zero. From (22) and (25), $\lim_{n \rightarrow \infty} \alpha_n = 0$ and the sequence $\{x_n\}$ is bounded, we obtain

$$\begin{aligned} & \|y_n - x_n\| \\ &= \|\alpha_n (\gamma f(x_n) - \mu Bx_n) + (I - \mu \alpha_n B)(T_{\omega_2} x_n - x_n)\| \\ &\leq \alpha_n \|\gamma f(x_n) - \mu Bx_n\| + (1 - \alpha_n \tau) \|T_{\omega_2} x_n - x_n\| \\ &\leq \alpha_n \|\gamma f(x_n) - \mu Bx_n\| + (1 - \alpha_n \tau) \omega_2 \|T_2 x_n - x_n\| \rightarrow 0, \end{aligned} \quad (35)$$

which implies

$$\begin{aligned} \|x_n - T_{\omega_1} x_n\|^2 &\leq \|x_n - T_{\omega_1} y_n\|^2 + \|T_{\omega_1} y_n - T_{\omega_1} x_n\|^2 \\ &\leq \|x_n - T_{\omega_1} y_n\|^2 + \|x_n - y_n\|^2 \\ &\quad + \frac{2}{\omega_1} \langle x_n - T_{\omega_1} x_n, y_n - T_{\omega_1} y_n \rangle \\ &\leq \|x_n - T_{\omega_1} y_n\|^2 + \|x_n - y_n\|^2 \\ &\quad + \frac{2}{\omega_1} \langle x_n - T_{\omega_1} x_n, y_n - x_n \rangle \\ &\quad + \frac{2}{\omega_1} \langle x_n - T_{\omega_1} x_n, x_n - T_{\omega_1} y_n \rangle. \end{aligned} \quad (36)$$

From $\lim_{n \rightarrow \infty} \|T_{\omega_1} y_n - x_n\| = 0$, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$, and $x_n - T_{\omega_1} x_n = \omega_1(x_n - T_1 x_n)$, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_{\omega_1} x_n\| = 0. \quad (37)$$

Using the demiclosedness principle (Proposition 8) and (35), we know that $\omega_w(x_n) \in \Omega$; hence

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle \\ &= \limsup_{n \rightarrow \infty} \langle \gamma f(x_n) - \mu Bx^*, x_n - x^* \rangle \\ &= \max_{w \in \omega_w(x_n)} \langle \gamma f(w) - \mu Bx^*, w - x^* \rangle \leq 0. \end{aligned} \quad (38)$$

It then follows from (29) that

$$\begin{aligned} \mathcal{T}_{n+1} &\leq (1 - \alpha_n \delta) \mathcal{T}_n \\ &\quad + 2\alpha_n (1 - \beta_n) \langle \gamma f(x_n) - \mu Bx^*, y_n - x^* \rangle. \end{aligned} \quad (39)$$

According to Lemma 11, we obtain $\lim_{n \rightarrow \infty} \mathcal{T}_n = 0$.

Case 2. Suppose there exists a subsequence $\{\mathcal{T}_{n_k}\}_{k \geq 0}$ of $\{\mathcal{T}_n\}_{n \geq 0}$ such that $\mathcal{T}_{n_k} \leq \mathcal{T}_{n_{k+1}}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{\delta(n)\}$ as defined in Lemma 12. It follows that $\mathcal{T}_{\delta(n+1)} - \mathcal{T}_{\delta(n)} > 0$, which by (29) amounts to

$$\delta (\|x_{\delta(n)} - T_2 x_{\delta(n)}\|^2 + \|T_{\omega_1} y_{\delta(n)} - x_{\delta(n)}\|^2) \leq \alpha_{\delta(n)} M. \quad (40)$$

In a similar way to Case 1, we obtain

$$\limsup_{n \rightarrow \infty} \langle \gamma f(x_{\delta(n)}) - \mu Bx^*, y_{\delta(n)} - x^* \rangle \leq 0. \quad (41)$$

From Lemma 12 and (29)

$$\mathcal{T}_{\delta(n)} \leq 2(1 - \beta_{\delta(n)}) \langle \gamma f(x_{\delta(n)}) - \mu Bx^*, y_{\delta(n)} - x^* \rangle, \quad (42)$$

for all $n > n_0$. Taking $\limsup$ in this inequality, we obtain $\lim_{n \rightarrow \infty} \mathcal{T}_{\delta(n)} = 0$. Moreover, it follows from (22) that

$$\begin{aligned} \mathcal{T}_{\delta(n)+1}^2 &= \|(x_{\delta(n)} - x^*) - (x_{\delta(n)} - x_{\delta(n)+1})\|^2 \\ &\leq \mathcal{T}_{\delta(n)}^2 + \|x_{\delta(n)} - x_{\delta(n)+1}\|^2 \\ &\leq \mathcal{T}_{\delta(n)}^2 + \|x_{\delta(n)} - T_{\omega_1} y_{\delta(n)}\|^2, \end{aligned} \quad (43)$$

which together with (40) implies $\lim_{n \rightarrow \infty} \mathcal{T}_{\delta(n)+1} = 0$. Consequently, from Lemma 12, we obtain $\lim_{n \rightarrow \infty} \mathcal{T}_n = 0$.

In addition, from (38), we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle \gamma f(x_n) - \mu Bx^*, x_n - x^* \rangle \\ &= \limsup_{n \rightarrow \infty} \langle \gamma f(x_n) - \gamma f(x^*), x_n - x^* \rangle \\ &\quad + \limsup_{n \rightarrow \infty} \langle \gamma f(x^*) - \mu Bx^*, x_n - x^* \rangle \\ &= \gamma L \lim_{n \rightarrow \infty} \mathcal{T}_n + \limsup_{n \rightarrow \infty} \langle \gamma f(x^*) - \mu Bx^*, x_n - x^* \rangle \\ &= \langle \gamma f(x^*) - \mu Bx^*, \bar{y} - x^* \rangle, \quad \bar{y} \in \Omega; \end{aligned} \quad (44)$$

then variational inequality (44) can be written as

$$\langle (I - \mu B + \gamma f)x^* - x^*, \bar{y} - x^* \rangle \geq 0, \quad \bar{y} \in \Omega. \quad (45)$$

So, by Lemma 13, it is equivalent to the fixed point equation

$$P_{F_{ix}(T)}(I - \mu B + \gamma f)x^* = x^*. \quad (46)$$

Remark 18. For a nonspreading mapping T , we have $\varrho = 0$ in Theorem 17 to obtain the following corollary.

Corollary 19. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $T_1, T_2 : H \rightarrow H$ be nonspreading mappings and demiclosed. Let f be L -Lipschitz mapping on H with coefficient $L > 0$ and let $B : H \rightarrow H$ be η -strongly monotone and k -Lipschitzian on H with $k > 0$, $\eta > 0$. Let the sequence of $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (c1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (c2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_{n+1}/\alpha_n = 1$,
- (c3) $0 < c \leq \beta_n \leq d < 1$.

Let the following sequence $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{\omega_1} (\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) T_{\omega_2} x_n), \quad (47)$$

where $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$, $\omega_i \in (0, 1)$, $i = 1, 2$. Then $\{x_n\}$ converges strongly to the unique element x^* in $\Omega = F_{ix}(T_1) \cap F_{ix}(T_2)$ verifying

$$x^* = (P_\Omega \cdot f) x^*, \quad (48)$$

which equivalently solves the following variational inequality problem:

$$x^* \in \Omega \quad \langle (\gamma f - \mu B) x^*, v - x^* \rangle \leq 0, \quad \forall v \in \Omega. \quad (49)$$

Remark 20. If $B = I$ and $\mu = 1$ in Corollary 19, we obtain the following corollary.

Corollary 21. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $T_1, T_2 : H \rightarrow H$ be nonspreading mappings and demiclosed. Let f be L -Lipschitz mapping on H with coefficient $L > 0$. Let the sequence of $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (c1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (c2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_{n+1}/\alpha_n = 1$,
- (c3) $0 < c \leq \beta_n \leq d < 1$.

Let the following sequence $\{x_n\}_{n=0}^{\infty}$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{\omega_1} (\alpha_n \gamma f(x_n) + (1 - \alpha_n) T_{\omega_2} x_n), \quad (50)$$

where $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$, $\omega_i \in (0, 1)$, $i = 1, 2$. Then $\{x_n\}$ converges strongly to the unique element x^* in $\Omega = F_{ix}(T_1) \cap F_{ix}(T_2)$ verifying

$$x^* = (P_\Omega \cdot f) x^*, \quad (51)$$

which equivalently solves the following variational inequality problem:

$$x^* \in \Omega, \quad \langle (\gamma f - I) x^*, v - x^* \rangle \leq 0, \quad \forall v \in \Omega. \quad (52)$$

4. Application

In this section, we constructed a numerical example to illustrate that our main results are well defined. The following example is introduced by Deng et al. [14]. They prove that T is $(1/3)$ -strictly pseudononspreading.

Example 22. Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \forall x = (x_1, x_2, \dots, x_n, \dots) \in X, \quad (53)$$

$C = \{x = (x_1, x_2, \dots, x_n, \dots) \mid x_i \in \mathbb{R}^1, i = 1, 2, \dots\}$, and let C be an orthogonal subspace of X (i.e., $\forall x, y \in C$, we have $\langle x, y \rangle = 0$). Then it is obvious that C is a nonempty closed convex subset of X . Now, for any $x = (x_1, x_2, \dots, x_n, \dots) \in C$, define a mapping $T : C \rightarrow C$ as follows:

$$Tx = \begin{cases} (x_1, x_2, \dots, x_n, \dots), & \prod_{i=1}^{\infty} x_i < 0, \\ (-2x_1, -2x_2, \dots, -2x_n, \dots), & \prod_{i=1}^{\infty} x_i \geq 0. \end{cases} \quad (54)$$

From the definition of this example, let $x = (x_1, x_2) \in \mathbb{R}^2$ and $f(x) = 8(\sin(x_1), \cos(x_2))$ be a Lipschitz mapping on $\mathbb{R}^2$. Assume that $T_1(x) = (0, 4x_1)$ and $T_2(x) = (4x_2, 0)$ be $(1/3)$ -strictly pseudononspreading mappings. Let $\alpha_n = \beta_n = 1/5$, $\gamma = 1/8$, $\mu = \omega_1 = \omega_2 = 1$, and $B = I$. According to (22), we can obtain the following algorithm:

$$x_{n+1} = \frac{1}{5}x_n + \frac{4}{200}T_1(f(x_n)) + \frac{16}{25}T_2(x_n). \quad (55)$$

From Theorem 17, we can easily know that algorithm (55) converges to the unique point in $\mathbb{R}^2$. Let $x_0 = (5, 4)$ and $|x_{n+1} - x_n| \leq 10^{-5}$, and let x^* be the fixed point of the algorithm (55). Using the software of MATLAB, we obtain $x^* = (0, 0)$.

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