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Research Article

General Integral Operator of Analytic Functions Involving Functions with Positive Real Part

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Let B_β be the integral operator defined by $B_\beta(z) = [\beta \int_0^z t^{\beta-1} \prod_{i=1}^n [((f_i(t))/t)^{\gamma_i} p_i^{\zeta_i}(t)] dt]^{1/\beta}$ where each of the functions f_i and p_i are, respectively, analytic functions and functions with positive real part defined in the open unit disk for all $i = 1, \ldots, n$. The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

1. Introduction and Definitions

Let $\mathscr{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathscr{S}$ we shall denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathscr{U}$. Also, let $\mathscr{P}$ be the class of all functions which are analytic in $\mathscr{U}$ and satisfy $p(0) = 1$, $\operatorname{Re}\{p(z)\} > 0$.

Frasin and Darus [1] (see also [2]) defined the family $\mathscr{B}(\delta)$, $0 \le \delta < 1$ so that it consists of functions $f \in \mathscr{A}$ satisfying the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \delta \, (z \in \mathscr{U}). \tag{2}$$

Very recently many authors studied the problem of integral operators which preserve the class $\mathscr{S}$ (see, e.g., [3–15]). In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $B_\beta(z)$ defined by

$$B_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\gamma_i} p_i^{\zeta_i}(t) \right] dt \right]^{1/\beta}, \tag{3}$$

where $f_i \in \mathscr{A}$, $p_i \in \mathscr{P}$, $\beta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\gamma_i, \zeta_i \in \mathbb{C}$ for all $i = 1, \ldots, n$.

Here and throughout in the sequel, every many-valued function is taken with the principal branch.

Remark 1. Note that the integral operator B_β generalizes the following operators introduced and studied by several authors:

(1) If we let $\zeta_i = 0$, for all $i = 1, \ldots, n$, in (3), we obtain the integral operator:

$$I_\gamma(f_1, \ldots, f_n)(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left(\frac{f_n(t)}{t} \right)^{\gamma_n} dt \right\}^{1/\beta}, \tag{4}$$

introduced and studied by D. Breaz and N. Breaz [16].

(2) If we let $\gamma_i = 0$, for all $i = 1, \ldots, n$, in (3), we obtain the integral operator:

$$I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} (p_1(t))^{\zeta_1} \cdots (p_n(t))^{\zeta_n} dt \right\}^{1/\beta} \tag{5}$$

introduced and studied by Frasin [17].

(3) If we let $\beta = 1$ and $\zeta_i = 0$, for all $i = 1, \dots, n$, in (3), we obtain the integral operator:

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt, \tag{6}$$

introduced and studied by D. Breaz and N. Breaz [16].

In order to derive our main results, we have to recall here the following lemmas.

Lemma 2 (see [18]). *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. If $h \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left|\frac{zh''(z)}{h'(z)}\right| \le 1, \tag{7}$$

for all $z \in \mathcal{U}$, then, for any complex number β with $\operatorname{Re}(\beta) \ge \operatorname{Re}(\alpha)$, the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} h'(t)\, dt\right\}^{1/\beta} \tag{8}$$

is in the class $\mathcal{S}$.

Lemma 3 (see [13]). *Let $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) > 0, c \in \mathbb{C}$ with $|c| \le 1$, $c \ne -1$. If $h \in \mathcal{A}$ satisfies*

$$\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le 1, \tag{9}$$

for all $z \in \mathcal{U}$, then the integral operator $F_\beta(z)$ defined by (8) is in the class $\mathcal{S}$.

Lemma 4 (see [19]). *If $p(z) \in \mathcal{P}$, then we have*

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{2|z|}{1-|z|^2}, \quad (z \in \mathcal{U}). \tag{10}$$

Lemma 5 (see [20]). *If $f(z) \in \mathcal{B}(\delta)$, then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{(1-\delta)(1+|z|)}{1-|z|} \quad (z \in \mathcal{U}). \tag{11}$$

Also, we need the following general Schwarz Lemma.

Lemma 6 (see [21]). *Let the function f be regular in the disk $\mathcal{U}_R = \{z : |z| < R\}$, with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m}|z|^m \quad (z \in \mathcal{U}_R). \tag{12}$$

The equality can hold only if

$$f(z) = e^{i\theta}\left(\frac{M}{R^m}\right) z^m, \tag{13}$$

where θ is constant.

2. Univalence Conditions for the Operator B_β

We first prove the following theorem.

Theorem 7. *Let $f(z) \in \mathcal{B}(\delta_i)$; $0 \le \delta_i < 1$ and $p_i(z) \in \mathcal{P}$ for all $i = 1, \dots, n$. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) = a > 0$. If*

$$\sum_{i=1}^n \left[|\gamma_i|(1-\delta_i) + |\zeta_i|\right] \le \min\left\{\frac{a}{2}; \frac{1}{4}\right\}, \tag{14}$$

then the integral operator $B_\beta(z)$ defined by (3) is in the class $\mathcal{S}$.

Proof. Define the regular function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\gamma_i} p_i^{\zeta_i}(t)\right] dt. \tag{15}$$

Then it is easy to see that

$$h'(z) = \prod_{i=1}^n \left[\left(\frac{f_i(z)}{z}\right)^{\gamma_i} p_i^{\zeta_i}(z)\right], \tag{16}$$

and $h(0) = h'(0) - 1 = 0$. Differentiating both sides of (16) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \sum_{i=1}^n \zeta_i \frac{zp_i'(z)}{p_i(z)}. \tag{17}$$

Thus, we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^n |\gamma_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \sum_{i=1}^n |\zeta_i| \left|\frac{zp_i'(z)}{p_i(z)}\right|. \tag{18}$$

Since $f_i(z) \in \mathcal{B}(\delta_i)$ and $p_i(z) \in \mathcal{P}$ for all $i = 1, \dots, n$, from (18), (11), and (10), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\le \sum_{i=1}^n |\gamma_i|(1-\delta_i)\left(\frac{1+|z|}{1-|z|}\right) + \sum_{i=1}^n |\zeta_i|\left(\frac{2|z|}{1-|z|^2}\right) \\ &\le \sum_{i=1}^n |\gamma_i|(1-\delta_i)\left(\frac{2}{1-|z|}\right) + \sum_{i=1}^n |\zeta_i|\left(\frac{2}{1-|z|}\right).\end{aligned} \tag{19}$$

Multiply both sides of (19) by $(1-|z|^{2\operatorname{Re}(\alpha)})/\operatorname{Re}(\alpha)$, we get

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{1-|z|^{2\operatorname{Re}(\alpha)}}{1-|z|}\frac{2}{\operatorname{Re}(\alpha)} \\ &\quad \times \sum_{i=1}^n \left[|\gamma_i|(1-\delta_i) + |\zeta_i|\right],\end{aligned} \tag{20}$$

for all $z \in \mathcal{U}$.

Let us denote $|z| = x$, $x \in [0,1)$, $\operatorname{Re}(\alpha) = a > 0$, and $\Phi(x) = (1-x^{2a})/(1-x)$. It is easy to prove that

$$\Phi(x) \le \begin{cases} 1, & \text{if } 0 < a < \frac{1}{2} \\ 2a, & \text{if } \frac{1}{2} < a < \infty. \end{cases} \tag{21}$$

From (20), (21), and the hypothesis (14), we have

$$\begin{aligned}\frac{1-|z|^{2a}}{a}&\left|\frac{zh''(z)}{h'(z)}\right| \\ &\leq \begin{cases}\frac{2}{a}\sum_{i=1}^{n}\left[|\gamma_i|(1-\delta_i)+|\zeta_i|\right], & \text{if } 0<a<\frac{1}{2} \\ 4\sum_{i=1}^{n}\left[|\gamma_i|(1-\delta_i)+|\zeta_i|\right], & \text{if } \frac{1}{2}<a<\infty.\end{cases} \\ &\leq 1,\end{aligned} \tag{22}$$

for all $z \in \mathcal{U}$. Applying Lemma 2 for the function $h(z)$, we prove that $B_\beta(z) \in \mathcal{S}$. □

Letting $n = 1$, $\delta_1 = \delta$, $\gamma_1 = \gamma$, $\zeta_1 = \zeta$, and $f_1 = f$ in Theorem 7, we obtain the following corollary.

Corollary 8. *Let* $f(z) \in \mathcal{B}(\delta)$; $0 \leq \delta < 1$ *and* $p(z) \in \mathcal{P}$. *Also, let* $\alpha, \gamma, \zeta \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) = a > 0$. *If*

$$|\gamma|(1-\delta)+|\zeta| \leq \min\left\{\frac{a}{2};\frac{1}{4}\right\}, \tag{23}$$

then the integral operator $B_{\beta,\gamma,\zeta}(z)$ *defined by*

$$B_{\beta,\gamma,\zeta}(z) = \left[\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma} p^{\zeta}(t)\,dt\right]^{1/\beta} \tag{24}$$

is in the class $\mathcal{S}$.

If we set $\delta = 0$ in Corollary 8, we have the following.

Corollary 9. *Let* $f(z) \in \mathcal{S}$ *and* $p(z) \in \mathcal{P}$. *Also, let* $\alpha, \gamma, \zeta \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) = a > 0$. *If*

$$|\gamma|+|\zeta| \leq \min\left\{\frac{a}{2};\frac{1}{4}\right\}, \tag{25}$$

then the integral operator $B_{\beta,\gamma,\zeta}(z)$ *defined by* (24) *is in the class* $\mathcal{S}$.

Next, we prove the following theorem.

Theorem 10. *Let* $f_i(z) \in \mathcal{A}$ *satisfies* $\operatorname{Re}(f_i(z)/z) > 0$, *and*

$$\left|(\gamma_i+\zeta_i)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right| \leq \frac{(2a+1)^{(2a+1)/2a}}{2n}, \tag{26}$$

for all $i = 1, \ldots, n$, *where* $\gamma_i, \zeta_i, \alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) = a > 0$, *then the integral operator* $B_\beta(z)$ *defined by* (3) *is in the class* $\mathcal{S}$.

Proof. Suppose that $\operatorname{Re}(f_i(z)/z) > 0$ *for all* $i = 1, \ldots, n$. Thus, we have

$$\frac{f_i(z)}{z} = p_i(z), \tag{27}$$

where $p_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$. Differentiating both sides of (27) logarithmically, we obtain

$$\frac{zf_i'(z)}{f_i(z)}-1 = \frac{zp_i'(z)}{p_i(z)}, \quad (i = 1, \ldots, n). \tag{28}$$

Define the regular function $h(z)$ as in (15). Thus from (28) and (17), we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\left[\gamma_i+\zeta_i\right]\left(\frac{zf'_i(z)}{f_i(z)}-1\right), \quad z \in \mathcal{U}. \tag{29}$$

Form the hypothesis (26) and (29), we immediately have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{(2a+1)^{(2a+1)/2a}}{2}, \tag{30}$$

for all $z \in \mathcal{U}$. Applying Lemma 6, we obtain

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{(2a+1)^{(2a+1)/2a}}{2}|z|, \quad z \in \mathcal{U}. \tag{31}$$

Thus from (29) and (31) we have,

$$\frac{1-|z|^{2a}}{a}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{|z|\left(1-|z|^{2a}\right)}{a}\cdot\frac{(2a+1)^{(2a+1)/2a}}{2}, \tag{32}$$

for all $z \in \mathcal{U}$. Let us denote $|z| = x$, $x \in [0,1]$, $\operatorname{Re}(\alpha) = a > 0$, and $\Psi(x) = x(1-x^{2a})$. It is easy to prove that the maximum is attained at the point $x = 1/(2a+1)^{1/2a}$, and thus we have

$$\Psi(x) \leq \frac{2a}{(2a+1)^{(2a+1)/2a}}. \tag{33}$$

In view of this inequality and (32), we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 \quad (z \in \mathcal{U}). \tag{34}$$

Applying Lemma 2 for the function $h(z)$, we prove that $B_\beta(z) \in \mathcal{S}$. □

Letting $n = 1$, $\gamma_1 = \gamma$, $\zeta_1 = \zeta$, and $f_1 = f$ in Theorem 10, we have the following corollary.

Corollary 11. *Let* $f(z) \in \mathcal{A}$ *satisfies* $\operatorname{Re}(f(z))/z > 0$, *and*

$$\left|(\gamma+\zeta)\left(\frac{zf'(z)}{f(z)}-1\right)\right| \leq \frac{(2a+1)^{(2a+1)/2a}}{2}, \tag{35}$$

where $\gamma, \zeta, \alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) = a > 0$, *then the integral operator* $B_{\beta,\gamma,\zeta}(z)$ *defined by* (24) *is in the class* $\mathcal{S}$.

Using Lemma 3, we derive the following theorem.

Theorem 12. *Suppose that each of the functions* $f_i(z) \in \mathcal{A}$ *satisfies* $\operatorname{Re}((f_i(z))/z) > 0$, *and*

$$\left|(\gamma_i+\zeta_i)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right| \leq \frac{|\beta|}{n}(1-|c|), \tag{36}$$

for all $i = 1, \ldots, n$, *where* $\gamma_i, \zeta_i, \beta \in \mathbb{C}(\operatorname{Re}(\beta) > 0)$ *and* $c \in \mathbb{C}(|c| < 1)$, *then the integral operator* $B_\beta(z)$ *defined by* (3) *is in the class* $\mathcal{S}$.

Proof. From (29), we have

$$\begin{aligned}
&\left|c|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \\
&\quad=\left|c|z|^{2\beta}+\frac{\left(1-|z|^{2\beta}\right)}{\beta}\sum_{i=1}^{n}\left(\gamma_i+\zeta_i\right)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right| \\
&\quad\le|c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left|\gamma_i+\zeta_i\right|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|.
\end{aligned} \tag{37}$$

Now by using the hypothesis (36), we obtain

$$\left|c|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\le 1. \tag{38}$$

Finally, by applying Lemma 3, we conclude that $B_\beta(z)\in\mathcal{S}$. □

Letting $n=1$, $\gamma_1=\gamma$, $\zeta_1=\zeta$, and $f_1=f$ in Theorem 12, we have the following corollary.

Corollary 13. *Suppose that the functions $f(z)\in\mathcal{A}$ satisfy* $\operatorname{Re}(f(z)/z)>0$, *and*

$$\left|(\gamma+\zeta)\left(\frac{zf'(z)}{f(z)}-1\right)\right|\le|\beta|\left(1-|c|\right), \tag{39}$$

where $\gamma,\zeta,\beta\in\mathbb{C}$ $(\operatorname{Re}(\beta)>0)$ and $c\in\mathbb{C}$ $(|c|<1)$, then the integral operator $B_{\beta,\gamma,\zeta}(z)$ defined by (24) *is in the class $\mathcal{S}$.*

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