Research Article

General Integral Operator of Analytic Functions Involving Functions with Positive Real Part

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Let $B_\beta$ be the integral operator defined by

$$B_\beta (z) = \left[ \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left[ \frac{(f_i(t))}{t} \right] \frac{\zeta_i(t)}{t} dt \right]^{1/\beta},$$

where $f_i \in \mathcal{A}$, $p_i \in \mathcal{P}$, $\beta \in \mathbb{C}^+ = \mathbb{C} \setminus \{0\}$, and $\gamma_i, \zeta_i \in \mathbb{C}$ for all $i = 1, \ldots, n$. The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \}$. Further, by $\mathcal{S}$ we shall denote the class of all functions which are univalent in $\mathcal{U}$. Also, let $\mathcal{P}$ be the class of all functions which are analytic in $\mathcal{U}$ and satisfy $p(0) = 1$, $\text{Re}\{ p(z) \} > 0$.

Frasin and Darus [1] (see also [2]) defined the family $\mathcal{B}(\delta)$, $0 \leq \delta < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \delta (z \in \mathcal{U}).$$

Very recently many authors studied the problem of integral operators which preserve the class $\mathcal{S}$ (see, e.g., [3–15]). In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $B_\beta (z)$ defined by

$$B_\beta (z) = \left[ \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left[ \frac{(f_i(t))}{t} \right] \frac{\zeta_i(t)}{t} dt \right]^{1/\beta},$$

where $f_i \in \mathcal{A}$, $p_i \in \mathcal{P}$, $\beta \in \mathbb{C}^+ = \mathbb{C} \setminus \{0\}$, and $\gamma_i, \zeta_i \in \mathbb{C}$ for all $i = 1, \ldots, n$.

Here and throughout in the sequel, every many-valued function is taken with the principal branch.

Remark 1. Note that the integral operator $B_\beta$ generalizes the following operators introduced and studied by several authors:

(1) If we let $\gamma_i = 0$, for all $i = 1, \ldots, n$, in (3), we obtain the integral operator:

$$I_\gamma (f_1, \ldots, f_n) (z) = \left\{ \beta \int_0^z t^{\beta-1} \left( \frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\gamma_n} dt \right\}^{1/\beta},$$

introduced and studied by D. Breaz and N. Breaz [16].

(2) If we let $\gamma_i = 0$, for all $i = 1, \ldots, n$, in (3), we obtain the integral operator:

$$I_\beta (p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n) (z) = \left\{ \int_0^z \beta t^{\beta-1} \left( p_1(t) \right)^{\alpha_1} \cdots \left( p_n(t) \right)^{\alpha_n} dt \right\}^{1/\beta},$$

introduced and studied by Frasin [17].
If we let $\beta = 1$ and $\zeta_i = 0$, for all $i = 1, \ldots, n$, in (3), we obtain the integral operator:

$$F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\gamma_n} dt,$$

introduced and studied by D. Breaz and N. Breaz [16].

In order to derive our main results, we have to recall here the following lemmas.

**Lemma 2** (see [18]). Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. If $h \in \mathcal{A}$ satisfies

$$1 - \left| \frac{\text{Re}(\alpha)}{\alpha} \right| \left| z h''(z) \right| \leq 1,$$

for all $z \in \mathcal{U}$, then, for any complex number $\beta$ with $\text{Re}(\beta) \geq \text{Re}(\alpha)$, the integral operator

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} h'(t) dt \right\}^{1/\beta}$$

is in the class $\mathcal{D}$.

**Lemma 3** (see [13]). Let $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0, c \in \mathbb{C}$ with $|c| \leq 1, c \neq -1$. If $h \in \mathcal{A}$ satisfies

$$\left| c |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{z h''(z)}{\beta h'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then the integral operator $F_\beta(z)$ defined by (8) is in the class $\mathcal{D}$.

**Lemma 4** (see [19]). If $p(z) \in \mathcal{P}$, then we have

$$\left| \frac{z f'(z)}{f(z)} \right| < \frac{2|z|}{1 - |z|^2}, \quad (z \in \mathcal{U}).$$

**Lemma 5** (see [20]). If $f(z) \in \mathcal{B}(\delta)$, then

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < \frac{(1 - \delta) (1 + |z|)}{1 - |z|}, \quad (z \in \mathcal{U}).$$

Also, we need the following general Schwarz Lemma.

**Lemma 6** (see [21]). Let the function $f$ be regular in the disk $\mathcal{U}_R = \{ z : |z| < R \}$, with $|f(z)| < M$ for fixed $M$. If $f(z)$ has one zero with multiplicity order bigger than $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathcal{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \left( \frac{M}{R^m} \right) z^m,$$

where $\theta$ is constant.

### 2. Univalence Conditions for the Operator $B_\beta$

We first prove the following theorem.

**Theorem 7.** Let $f(z) \in \mathcal{B}(\delta_i), \delta_i \geq 1$ and $p_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > a > 0$. If

$$\sum_{i=1}^n \left[ |\gamma_i| (1 - \delta_i) + |\xi_i| \right] \leq \min \left\{ \frac{a-1}{2}, \frac{1}{4} \right\},$$

then the integral operator $B_\beta(z)$ defined by (3) is in the class $\mathcal{D}$.

**Proof.** Define the regular function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t^{\gamma_i}} \right)^{\gamma_i} \left( p_i(t) \right)^{\xi_i} dt.$$

Then it is easy to see that

$$h'(z) = \prod_{i=1}^n \left( \frac{f_i(z)}{z^{\gamma_i}} \right)^{\gamma_i} p_i^{\xi_i} (z),$$

and $h(0) = h'(0) - 1 = 0$. Differentiating both sides of (16) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \gamma_i \left( \frac{f'_i(z)}{f_i(z)} - 1 \right) + \sum_{i=1}^n \xi_i p_i'(z).$$

Thus, we have

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \gamma_i \left| \frac{f'_i(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^n \xi_i \left| p_i'(z) \right|.$$

Since $f_i(z) \in \mathcal{B}(\delta_i)$ and $p_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$, from (18), (11), and (10), we obtain

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \gamma_i (1 - \delta_i) \left( 1 + |z| \right) + \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{2|z|}{1 - |z|^2}.$$

Multiply both sides of (19) by $(1 - |z|^{2\text{Re}(\alpha)})/\text{Re}(\alpha)$, we get

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\text{Re}(\alpha)}}{1 - |z|} \frac{2}{\text{Re}(\alpha)} \times \sum_{i=1}^n \left[ |\gamma_i| (1 - \delta_i) + |\xi_i| \right],$$

for all $z \in \mathcal{U}$.

Let us denote $|z| = x, x \in [0, 1], \text{Re}(\alpha) = a > 0$, and $\Phi(x) = (1 - x^2a)/(1 - x)$. It is easy to prove that

$$\Phi(x) \leq \begin{cases} 
1, & \text{if } 0 < a < \frac{1}{2} \\
2a, & \text{if } \frac{1}{2} < a < \infty.
\end{cases}$$

(21)
From (20), (21), and the hypothesis (14), we have

$$\frac{1 - |z|^{2a}}{\alpha} \left| \frac{z h''(z)}{h(z)} \right| \leq \left\{ \begin{array}{ll} \frac{2^n}{\alpha} \sum_{i=1}^{n} \left| y_i \right| (1 - \delta_i) + |k_i|, & \text{if } 0 < \alpha < \frac{1}{2} \\ 4 \sum_{i=1}^{n} \left| y_i \right| (1 - \delta_i) + |k_i|, & \text{if } \frac{1}{2} < \alpha < \infty. \end{array} \right.$$  \hspace{1cm} (22)

for all $z \in \mathcal{U}$. Applying Lemma 2 for the function $h(z)$, we prove that $B_{\beta}(z) \in \mathcal{S}$.

Letting $n = 1, \delta_1 = \delta, \gamma_1 = \gamma, \zeta_1 = \zeta,$ and $f_1 = f$ in Theorem 7, we obtain the following corollary.

**Corollary 8.** Let $f(z) \in \mathcal{S}(\delta); 0 \leq \delta < 1$ and $p(z) \in \mathcal{P}$. Also, let $\alpha, \gamma, \zeta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. If

$$|y| (1 - \delta) + |k| \leq \min \left\{ \frac{a}{2}, \frac{1}{4} \right\},$$  \hspace{1cm} (23)

then the integral operator $B_{\beta, \gamma, \zeta}(z)$ defined by

$$B_{\beta, \gamma, \zeta}(z) = \left[ \beta \int_{0}^{1} t^{\beta - 1} \left( \frac{f(t)}{t} \right)^{\gamma} t^{\beta} (t) dt \right]^{1/\beta}$$  \hspace{1cm} (24)

is in the class $\mathcal{S}$.

If we set $\delta = 0$ in Corollary 8, we have the following.

**Corollary 9.** Let $f(z) \in \mathcal{S}$ and $p(z) \in \mathcal{P}$. Also, let $\alpha, \gamma, \zeta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. If

$$|y| + |k| \leq \min \left\{ \frac{a}{2}, \frac{1}{4} \right\},$$  \hspace{1cm} (25)

then the integral operator $B_{\beta, \gamma, \zeta}(z)$ defined by (24) is in the class $\mathcal{S}$.

Next, we prove the following theorem.

**Theorem 10.** Let $f_i(z) \in \mathcal{A}$ satisfies $\text{Re} \left( f_i(z)/z \right) > 0$, and

$$\left( \gamma_i + \zeta_i \right) \left( \frac{z f'_i(z)}{f_i(z)} - 1 \right) \leq \left( 2a + 1 \right) \left( \frac{2a + 1}{2a} \right)^{1/2},$$  \hspace{1cm} (26)

for all $i = 1, \ldots, n$, where $\gamma_i, \zeta_i, \alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, then the integral operator $B_{\beta}(z)$ defined by (3) is in the class $\mathcal{S}$.

**Proof.** Suppose that $\text{Re}(f_i(z)/z) > 0$ for all $i = 1, \ldots, n$. Thus, we have

$$\frac{f_i(z)}{z} = p_i(z),$$  \hspace{1cm} (27)

where $p_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$. Differentiating both sides of (27) logarithmically, we obtain

$$\frac{z f'_i(z)}{f_i(z)} - 1 = \frac{z p'_i(z)}{p_i(z)}, \quad (i = 1, \ldots, n).$$  \hspace{1cm} (28)

Define the regular function $h(z)$ as in (15). Thus from (28) and (17), we have

$$\frac{z h''(z)}{h'(z)} = \sum_{i=1}^{n} \left( \gamma_i + \zeta_i \right) \left( \frac{z f'_i(z)}{f_i(z)} - 1 \right), \quad z \in \mathcal{U}.$$  \hspace{1cm} (29)

Form the hypothesis (26) and (29), we immediately have

$$\left| \frac{z h''(z)}{h'(z)} \right| \leq \frac{(2a + 1)(2a + 1)^{1/2}}{2},$$  \hspace{1cm} (30)

for all $z \in \mathcal{U}$. Applying Lemma 6, we obtain

$$\left| \frac{z h''(z)}{h'(z)} \right| \leq \frac{(2a + 1)(2a + 1)^{1/2}}{2} |z|, \quad z \in \mathcal{U}.$$  \hspace{1cm} (31)

Thus from (29) and (31) we have,

$$1 - |z|^{2a} \left| \frac{z h''(z)}{h'(z)} \right| \leq \frac{|z| (1 - |z|^{2a})}{a} \frac{(2a + 1)(2a + 1)^{1/2}}{2},$$  \hspace{1cm} (32)

for all $z \in \mathcal{U}$. Let us denote $|z| = x, x \in [0, 1]$, $\text{Re}(\alpha) > 0$, and $\Psi(x) = x(1 - x^{2a})$. It is easy to prove that the maximum is attained at the point $x = 1/(2a + 1)^{1/2a}$, and thus we have

$$\Psi(x) \leq \frac{2a}{(2a + 1)(2a + 1)^{1/2a}}.$$  \hspace{1cm} (33)

In view of this inequality and (32), we obtain

$$1 - |z|^{2a} \left| \frac{z h''(z)}{h'(z)} \right| \leq 1 \quad (z \in \mathcal{U}).$$  \hspace{1cm} (34)

Applying Lemma 2 for the function $h(z)$, we prove that $B_{\beta}(z) \in \mathcal{S}$.

Letting $n = 1, \gamma_1 = \gamma, \zeta_1 = \zeta$, and $f_1 = f$ in Theorem 10, we have the following corollary.

**Corollary 11.** Let $f(z) \in \mathcal{S}$ satisfies $\text{Re} \left( f(z)/z \right) > 0$, and

$$\left( \gamma + \zeta \right) \left( \frac{z f'(z)}{f(z)} - 1 \right) \leq \frac{(2a + 1)(2a + 1)^{1/2}}{2},$$  \hspace{1cm} (35)

where $\gamma, \zeta, \alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, then the integral operator $B_{\beta, \gamma, \zeta}(z)$ defined by (24) is in the class $\mathcal{S}$.

Using Lemma 3, we derive the following theorem.

**Theorem 12.** Suppose that each of the functions $f_i(z) \in \mathcal{A}$ satisfies $\text{Re} \left( f_i(z)/z \right) > 0$, and

$$\left( \gamma_i + \zeta_i \right) \left( \frac{z f'_i(z)}{f_i(z)} - 1 \right) \leq \frac{|\beta|}{n} \left( 1 - |c| \right),$$  \hspace{1cm} (36)

for all $i = 1, \ldots, n$, where $\gamma_i, \zeta_i, \beta \in \mathbb{C}$, $\text{Re} (\beta) > 0$, and $c \in \mathbb{C}$, $|c| < 1$, then the integral operator $B_{\beta}(z)$ defined by (3) is in the class $\mathcal{S}$. 
Proof. From (29), we have
\[
|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)}
\]
\[
= \left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \sum_{i=1}^{n} (y_i + \zeta_i) \left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right|
\]
\[
\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} |y_i + \zeta_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right|.
\]
(37)

Now by using the hypothesis (36), we obtain
\[
|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \leq 1.
\]
(38)

Finally, by applying Lemma 3, we conclude that \(B_{\beta}(z) \in \mathcal{S}\).

Letting \(n = 1, y_1 = \gamma, \zeta_1 = \zeta,\) and \(f_1 = f\) in Theorem 12, we have the following corollary.

Corollary 13. Suppose that the functions \(f(z) \in \mathcal{A}\) satisfy \(\text{Re}(f(z)/z) > 0,\) and
\[
\left|\left(y + \zeta\right) \left(\frac{zf'(z)}{f(z)} - 1\right)\right| \leq |\beta| \left(1 - |z|\right),
\]
(39)

where \(\gamma, \zeta, \beta \in \mathbb{C}, (\text{Re}(\beta) > 0)\) and \(c \in \mathbb{C}, (|c| < 1),\) then the integral operator \(B_{\beta \gamma} \chi(z)\) defined by (24) is in the class \(\mathcal{S}\).

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References
