Research Article

Eigenvalue for Densely Defined \((S_+)^L\) Perturbations of Multivalued Maximal Monotone Operators in Reflexive Banach Spaces

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Let \(X\) be a real reflexive Banach space and let \(X^*\) be its dual. Let \(\Omega \subset X\) be open and bounded such that \(0 \in \Omega\). Let \(T : X \supset \mathcal{D}(T) \to 2^X^*\) be maximal monotone with \(0 \in \mathcal{D}(T)\) and \(0 \in T(0)\). Using the topological degree theory developed by Kartsatos and Quarcoo, we study the eigenvalue problem \(Tx + \lambda Px \ni 0\), where the operator \(P : X \supset \mathcal{D}(P) \to X^*\) is a single-valued of class \((S_+)\). The existence of continuous branches of eigenvectors of infinite length then could be easily extended to the case where the operator \(P : X \to 2^X^*\) is multivalued and is investigated.

1. Preliminaries

In what follows we assume that \(X\) is a real or complex Banach space and has been renormed such that it and its dual \(X^*\) are locally uniformly convex. The normalized duality mapping is defined by

\[ J(x) = \{ x^* \in X : \|x^*\| = \|x\|, \langle x, x^* \rangle = \|x\|^2 \}. \]

The mapping \(T : X \supset \mathcal{D}(T) \to 2^X^*\) is said to be “monotone” if for every \(x, y \in \mathcal{D}(T), u \in Tx\), and \(v \in Ty\) we have

\[ \langle u - v, x - y \rangle \geq 0. \]

A monotone operator \(T\) is “maximal monotone” if \(G(T)\) is maximal in \(X \times X^*\), when \(X \times X^*\) is partially ordered by inclusion. In our setting, a monotone operator \(T\) is maximal if and only if \(R(T + \lambda I) = X^*\) for every \(\lambda > 0\). If \(T : X \supset \mathcal{D}(T) \to 2^X^*\) is maximal monotone operator, the operator \(T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*\) is called the Yosida approximant of \(T\) and the following, which can be found in [1, page 102] is true.

**Lemma 1.** Let \(T : X \supset \mathcal{D}(T) \to 2^X^*\) be a maximal monotone operator with \(0 \in \mathcal{D}(T)\) and \(0 \in T(0)\). Then

(i) \(T_t\) is a bounded maximal monotone mapping with \(0 = T_t(0)\) for each \(t > 0\);

(ii) \(T_t x \to T[0]_t x\) in \(X\) as \(t \to 0\) for all \(x \in \mathcal{D}(T)\), where \(T[0]_t x\) denotes the element \(y^* \in Tx\) of minimum norm;

(iii) \(\|T_t x\| \to \infty\) as \(t \to 0\) for all \(x \notin \mathcal{D}(T)\).

Also \(T_t x \in T J_t x\), where \(J_t = I - tJ^{-1}T_t : X \to X\) and satisfies: if \(x \in \conv \mathcal{D}(T)\), then \(J_t x \to x\) in \(X\) as \(t \to 0\), where \(\conv M\) denote the convex hull of the set \(M\).

The proof of the next lemma can be found in Kartsatos and Skripnik [2].

**Lemma 2.** Let \(T : X \supset \mathcal{D}(T) \to 2^X^*\) be maximal monotone such that \(0 \in \mathcal{D}(T)\) and \(0 \in T(0)\). Then the mapping \((t, x) \to T_t x\) is continuous on the set \((0, \infty) \times X\).

An operator \(T : X \supset \mathcal{D}(T) \to Y\), with \(Y\) another Banach space, is bounded if it maps bounded subsets of \(\mathcal{D}(T)\) onto bounded sets. It is compact if it is continuous and maps bounded subsets of \(\mathcal{D}(T)\) onto relatively compact subsets of \(Y\). It is demicontinuous if for every sequence \(\{x_n\}\) such that \(x_n \to x_0\) we have \(Tx_n \rightharpoonup Tx_0\).
We say that the operator \( T : X \supset D(T) \to 2^{X^*} \) satisfies condition \((S_q)\) on a set \( A \subset D(T) \) if for every sequence \( \{x_n\} \subset A \) such that \( x_n \to x_0 \in X \) and any \( y_n^* \inTx_n \), with \( y_n^* \to (some)y^* \in X^* \), we have \( x_n \to x_0 \). If \( A = D(T) \), then we say that \( T \) satisfies \((S_q)\).

**Definition 3.** Let \( X \) be a separable reflexive Banach space and let \( L \) be a dense subspace of \( X \). A mapping \( P : D(P) \subset X \to X^*\) is said to be of class \((S_q)\) if, for any sequence of finite dimensional subspaces \( F_n \) of \( L \) with \( \bigcup_{n=1}^{\infty} F_n = X, h \in X^* \), \( \{x_n\}_{n=1}^{\infty} \subset D(P) \) with \( x_n \to x_0 \) and

\[
\lim_{n \to \infty} \langle p_n - h, x_n \rangle \leq 0, \quad \lim_{n \to \infty} \langle p_n - h, v \rangle = 0
\]

for all \( v \in U_{n=1}^{\infty} F_n \), we have \( x_n \to x_0, x \in D(P) \) and \( Px_0 = h \). If \( h = 0 \), then we call \( P \) a mapping of class \((S_q)\).

**Definition 4.** Let \( X \) be a separable reflexive Banach space and let \( L \) be a dense subspace of \( X \). A multivalued mapping \( P : D(P) \subset X \to 2^{X^*} \) is said to be of class \((S_q)\) if it satisfies the following conditions:

(i) \( Px \) is bounded closed and convex for each \( x \in D(P) \),

(ii) \( P \) is weakly upper semicontinuous in each finite dimensional space, that is, for each finite dimensional space \( F \) of \( L \), \( F \cap D(P) \neq \emptyset, P : F \cap D(P) \to 2^{X^*} \) is upper semicontinuous in the weak topology,

(iii) if for any sequence of finite dimensional subspaces \( F_n \) of \( L \) with \( L \subset U_{n=1}^{\infty} F_n, h \in X^* \), \( \{x_n\}_{n=1}^{\infty} \subset D(P) \cap L \) and \( x_n \to x_0 \) such that

\[
\lim_{n \to \infty} \langle p_n - h, x_n \rangle \leq 0, \quad \lim_{n \to \infty} \langle p_n - h, v \rangle = 0
\]

for all \( v \in U_{n=1}^{\infty} F_n \) and some \( p_n \in Px_n \), then we have \( x_n \to x_0 \), \( x \in D(P) \) and \( h \in Px_0 \).

If \( h = 0 \), then we call \( P \) a mapping of class \((S_q)\).

We will need the following two conditions:

(P1) there exist a subspace \( L \) of \( X \) such that \( L \subset D(P), \bar{L} = X \) and the operator \( P \) satisfies condition \((S_q)\);

(P2) there exist a function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) which is nondecreasing such that

\[
\langle p, x \rangle \geq -\phi(||x||), \quad p \in Px, x \in D(P).
\]

The following lemma can be found in Zeidler [3, page 915].

**Lemma 5.** Let \( T : X \supset D(T) \to 2^{X^*} \) be maximal monotone. Thus the following are true:

(i) \( \{x_n\} \subset D(T)x_n \to x_0 \) and \( Tx_n \ni y_n \to y_0 \) imply \( x_0 \in D(T) \), and \( y_0 \in Tx_0 \);

(ii) \( \{x_n\} \subset D(T)x_n \to x_0 \) and \( Tx_n \ni y_n \to y_0 \) imply \( x_0 \in D(T) \), and \( y_0 \in Tx_0 \).

We will need the following lemma from Adhikari and Kartsatos [4].

**Lemma 6.** Assume that the operators \( T : X \supset D(T) \to 2^{X^*} \) and \( T_0 : X \supset D(T_0) \to X^* \) are maximal monotone, with \( 0 \in D(T) \cap D(T_0) \) and \( 0 \in T(0) \cap T_0(0) \). Assume further that \( T + T_0 \) is maximal monotone. Assume that there is a positive sequence \( \{t_n\} \) such that \( t_n \downarrow 0 \), a sequence \( \{x_n\} \subset D(T_0) \) and a sequence \( \{w_n\} \in T_0x_n \) such that \( x_n \to x_0 \) and \( T_n x_n + w_n \to y_n^* \in X^* \). Then the following are true:

(i) the inequality

\[
\lim_{n \to \infty} \langle T_n x_n + w_n, x_n - x_0 \rangle < 0
\]

is impossible;

(ii) \( \lim_{n \to \infty} \langle T_n x_n + w_n, x_n - x_0 \rangle = 0 \),

then \( x_0 \in D(T + T_0) \) and \( y^* \in (T + T_0)x_0 \).

The proof of the following lemma is given in [5] proof of Theorem 7 but we will repeat it here for completeness and future reference.

**Lemma 7.** Let \( T : X \supset D(T) \to 2^{X^*} \) be a quasibounded maximal monotone operator such that \( 0 \in T(0) \). Let \( \{t_n\} \subset (0, \infty) \) and \( \{x_n\} \subset X \) be such that

\[
||u_n|| \leq S, \quad \langle T_{t_n} u_n, u_n \rangle \leq S_1,
\]

where \( S, S_1 \) are positive constants. Then there exists a number \( K > 0 \) such that \( ||T_{t_n} u_n|| \leq K \) for all \( n = 1, 2, \ldots \).

**Proof.** Let

\[
w_n = T_{t_n} u_n = (T^{-1} + t_n J^{-1})^{-1} u_n.
\]

We have that

\[
w_n \in T_{t_n} u_n = T x_n, \quad t_n w_n = J (u_n - x_n),
\]

where \( x_n = J_{t_n} u_n \). Thus,

\[
\langle w_n, x_n \rangle = \langle w_n, u_n - t_n J^{-1} w_n \rangle = \langle w_n, u_n \rangle - t_n \langle w_n, J^{-1} w_n \rangle = \langle w_n, u_n \rangle - t_n ||w_n||^2 \leq \langle T_{t_n} u_n, u_n \rangle \leq S_1.
\]

From (11) we obtain

\[
t_n ||w_n||^2 = \langle w_n, u_n \rangle - \langle w_n, x_n \rangle.
\]

Now since \( 0 \in T(0) \) and \( w_n \in T x_n \), we have that \( \langle w_n, x_n \rangle \geq 0 \), which implies \( t_n ||w_n||^2 \leq S_1 \). We claim that \( \{w_n\} \) is bounded,
if not we may assume that \(|w_n| \to \infty\) and \(|w_n| \leq |w_n|^2\) for all \(n\). Thus \(t_n|w_n| \leq S_1\) and
\[
t_n \|w_n\| = \|f(u_n - x_n)\| = \|u_n - x_n\| \tag{13}
\]
which implies that \(\{x_n\}\) is bounded. Now since \(T\) is strongly quasibounded, the boundedness of \(\{x_n\}\) and \(\{\|w_n\|\}\) imply the boundedness of \(\{w_n\}\), that is, a contradiction. It follows that \(\{T_n u_n\}\) is bounded and the proof is complete. \(\Box\)

We denote by \(f_\psi\) the duality mapping with gauge function \(\psi\). The function \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) is continuous, strictly increasing such that \(\psi(0) = 0\) and \(\psi(r) \to \infty\) as \(r \to \infty\). This mapping \(f_\psi\) is continuous, bounded, surjective, strictly and maximal monotone, and satisfies condition \((S_1)\). Also,
\[
\langle f_\psi x, x \rangle = \psi(\|x\|)\|x\|, \quad \|f_\psi x\| = \psi(\|x\|), \quad x \in X \tag{14}
\]
for these facts we refer to Petryshyn [6, pages 32-33 and 132].

## 2. The Eigenvalue Result

In this section we are using the topological degree developed by Kartasov and Quarcou in [7]. The following result will also improve Theorem 4 of Kartasov and Skrypnik in [8] since we are no longer assuming that the perturbation is quasibounded.

**Theorem 8.** Let \(\Omega\) be open and bounded with \(0 \in \Omega\). Assume that the operator \(T : X \supset D(T) \to X^*\) is maximal monotone with \(0 \in D(T)\) and \(0 \in T(0)\). Assume that the operator \(P : X \supset D(P) \to X^*\), with \(L \subset D(P)\), satisfies condition \((S_n)\) and \((P_2)\). Let \(e, \epsilon_0\), and \(\lambda\) be positive numbers. Assume that

\[(\mathcal{P})\] there exists \(\lambda \in (0, \Lambda]\) such that
\[
Tx + \lambda Px + e f_\psi x \geq 0 \tag{15}
\]
has no solution in \(D(T + P) \cap \Omega\).

Then

(i) there exists \((\lambda_0, x_0) \in (0, \Lambda] \times (D(T + P) \cap \partial \Omega)\) such that
\[
Tx_0 + \lambda_0 Px_0 + e f_\psi x_0 \geq 0; \tag{16}
\]

(ii) if \(0 \notin T(D(T) \cap \partial \Omega)\), \(T\) satisfies \((S_0)\) on \(\partial \Omega\), and property \((\mathcal{P})\) is satisfied for every \(e \in (0, \epsilon_0]\), then there exists \((\lambda_0, x_0) \in (0, \Lambda] \times (D(T + P) \cap \partial \Omega)\) such that
\[
Tx_0 + \lambda_0 Px_0 \geq 0. \tag{17}
\]

**Proof.** (i) Assume that \((\mathcal{P})\) is true and that \((16)\) is false. We consider the homotopy inclusion
\[
H(t, x) = Tx + tAPx + e f_\psi x \geq 0, \quad t \in [0, 1]. \tag{17}
\]
It is clear that this inclusion has no solution \(x \in D(H(t, \cdot)) \cap \partial \Omega\) for \(t \in [0, 1]\), because \(tx \in (T + e f_\psi)(0)\) and the operator \(T + e f_\psi\) is strictly monotone, hence one-to-one.

We are going to show that \(H(t, x)\) is an admissible homotopy for the degree in [4]. To this end, we set \(T_0 = T,\ P_0 \equiv tAP + f_\psi\) and we recall the operators \(T_{s_0} = T_{s_0} (T^{-1} + sI)^{-1} : X \to X^*,\ s > 0\), and we set \(I_s = I - sI^{-1}T_{s_0} = T_{s_0} I = X \to X\). We have \(D(H(0, \cdot)) = D(T)\) and \(D(H(t, \cdot)) = D(T + P)\), \(t \in [0, 1]\). We also set \(D^0 = D(tAP) = D(P)\). We have \(D^0 = X\) and \(D^0 = D(P)\) for \(t \in (0, 1]\). Let \(\Omega\) be an open and bounded subset of \(X\). We know that the equation
\[
T^t x + P^t x \geq 0 \tag{18}
\]
has no solution \(x \in D(H(t, \cdot)) \cap \partial \Omega\) for any \(t \in [0, 1]\). Now we consider the equation
\[
T_n x + tAPx + e f_\psi x = 0, \tag{19}
\]
and we show that there exists \(s_1 > 0\) such that
\[
0 \notin \left( T_s + tAP + e f_\psi \right) \left( D^0 \cap \partial \Omega \right). \tag{20}
\]
Assume that this is not true, then there exist \(\{s_n\} \subset (0, \infty)\) with \(s_n \downarrow 0\), \(\{t_n\} \subset [0, 1]\), with \(t_n \to 0\), \(\{x_n\} \subset \partial \Omega\) with \(x_n \to x_0\), and
\[
T_s x_n + t_n APx_n + e f_\psi x_n = 0. \tag{21}
\]
Clearly we cannot have \(t_n = 0\) for any \(n\), since \(\langle T_s + f_\psi \rangle(0) = 0\) and the operator \(T_s + f_\psi\) is strictly monotone, hence one-to-one. Thus \(t_n > 0\), for all \(n\). From (21) we have that
\[
\langle T_s x_n, x_n \rangle = - t_n \langle P x_n, x_n \rangle - e \langle f_\psi x_n, x_n \rangle \leq t_n \langle \phi(\|x_n\|) - e \psi(\|x_n\|) \rangle \tag{22}
\]
where \(S_1\) is the bound of \(\{x_n\}\). Thus we have the boundedness of \(\{T_s x_n, x_n \}\). Using Lemma 7, we have the boundedness of \(T_s x_n\). We may thus assume that \(T_s x_n \to h_s^*\). From (21) we also have that the sequence \(\{P x_n\}\) is bounded and we may assume that \(P x_n \to h_0^*\).

If \(t_0 = 0\), then from
\[
e \psi(\|x_n\|)\|x_n\| = \langle e f_\psi x_n, x_n \rangle = - t_n \langle P x_n, x_n \rangle \to 0 \tag{23}
\]
we obtain \(x_n \to 0 \in \partial \Omega\), which is a contradiction to \(0 \in \Omega\). Hence it follows that \(t_0 > 0\). Since \(\{P x_n\}\) is bounded, we may assume that \(P x_n \to h_0^*\) and we have \(t_0 \phi h_0^* = - h_s^* - e h_0^*\).

From (21) it follows that
\[
\langle T_s x_n + t_n AP x_n + e f_\psi x_n, x \rangle = 0, \quad \forall x \in U_{\epsilon_0} \tag{24}
\]
Thus
\[
\lim_{n \to \infty} \langle T_s x_n + t_n AP x_n + e f_\psi x_n, v \rangle = 0, \quad \forall v \in U_{\epsilon_0}. \tag{25}
\]
Now
\[
\langle t_n APx_n + h_n^1 + e h_n^2, x_n \rangle = \langle T_s x_n + t_n APx_n + e f_p x_n, x_n \rangle \\
- \langle T_s x_n + e f_p x_n, x_n - x_0 \rangle \\
- \langle T_s x_n + e f_p x_n, x_0 \rangle \\
+ \langle h_n^1 + e h_n^2, x_n \rangle,
\]
where \( \lambda_n \) is the bound of \( \{x_n\} \). This show that \( \langle u_n^*, x_n \rangle \) is bounded and further we obtain the boundedness of \( \{u_n^*\} \) by Lemma 2. We may thus assume that \( \lambda_n \to \lambda_0 \), \( x_n \to x_0 \), \( u_n^* \to h_0^* \), then \( \lambda_n APx_n \to -h_0^* \).

If \( \lambda_0 = 0 \), then (30) implies that
\[
\lim_{n \to \infty} u_n^* = \lim_{n \to \infty} \left[ -\lambda_n P x_n - \left( \frac{1}{n} \right) I_p x_n \right] = 0.
\]
Since \( T \) satisfies \( (S_\lambda) \), it follows that \( x_n \to x_0 \in \partial \Omega \). Now, by the demiclosedness of \( T \) (see Lemma 6) we obtain that \( x_0 \in D(T), 0 \in TX_0 \), and this contradicts \( 0 \in T(D(T) \cap \partial \Omega) \). Hence, \( \lambda_0 > 0 \). Repeating the proof of (i) starting from (21), we get again that
\[
\lim_{n \to \infty} \langle \lambda_n P x_n + h_n^*, x_n \rangle \leq 0
\]
and since \( P \) is of class \( \langle S_\lambda \rangle \), we have \( x_n \to x_0 \), \( -h_n^* = \lambda_0 P x_0 \). Using the demiclosedness of \( T \), we obtain \( x_0 \in D(T) \cap \partial \Omega \), \( h_0 \in TX_0 \) and \( TX_0 + \lambda_0 P x_0 \geq 0 \). The proof is now complete.

\[ \Box \]

3. Continuous Branches of Eigenvectors

In this section we are interested in showing that the results obtained in previous sections could be used in order to obtain the existence of continuous branches of eigenvectors. We need the following definition.

**Definition 9.** Let \( T: X \supset D(T) \to 2^{X^*}, P: X \supset D(P) \to X^* \) be given and consider the problem
\[
Tx + \lambda Px \geq 0.
\]
An eigenvector \( x \) is solution of (34) for some eigenvalue \( \lambda \) with \( x \in \langle D(T) \cap D(P) \rangle \). We say that the nonzero eigenvectors of the problem (34) form a continuous branch of infinite length if there exists \( r_0 > 0 \) such that, for every \( r \geq r_0 \), the sphere \( B_r(0) \) contains at least one nonzero eigenvector of (34).

**Theorem 10.** Assume that the operator \( T : X \supset D(T) \to 2^{X^*} \) is maximal monotone with \( 0 \in D(T) \) and \( 0 \in T(0) \). Assume that the operator \( P : X \supset D(P) \to X^* \) with \( L \subset D(P) \) is of class \( \langle S_\lambda \rangle \) and satisfies (P2). Let \( \Lambda \) be a positive number. Assume that \( Tx \geq 0 \) implies \( x = 0 \), \( T \) satisfies \( (S_\lambda) \) and
\[
(\mathcal{P}1) \text{ there exists } \alpha > 0 \text{ and } \lambda \in (0, \Lambda] \text{ such that }
\]
\[
|Tx + \lambda Px| \geq \alpha, \quad x \in D(T + P).
\]

Then the nonzero eigenvectors of problem (34) form a continuous branch of infinite length with corresponding eigenvalues \( \lambda \in (0, \Lambda] \).

**Proof.** Let \( r_0 > 0 \) be given. Let \( \epsilon_0 > 0 \) be so small that \( \epsilon_0 r_0 < \alpha \). Then
\[
|Tx + \lambda Px + \psi x| \geq |Tx + \lambda Px| - |\psi x|
\]
\[
\geq \alpha - \epsilon_0 r_0,
\]
where the last inequality follows from Theorem 3, (i) of [9]. Consequently, the inclusion \( H(t, x) \geq 0 \) has a solution in \( \Omega \) for each \( t \in [0, 1] \). In particular, this says that \( Tx + \lambda Px + \epsilon f_p x \geq 0 \) has a solution in \( \Omega \) for every \( \lambda > 0 \). This of course contradicts condition \( (\mathcal{P}) \) and finishes the proof of (i).

(ii) Let \( \lambda_n \in (0, \Lambda], x_n \in D(P) \cap \partial \Omega \) be such that, for some \( u_n^* \in TX_n \),
\[
u_n^* + \lambda_n Px_n + \left\langle \left( \frac{1}{n} \right) I_p x_n, x_n \right\rangle = 0.
\]
again, \( \lambda_n = 0 \) for any \( n \) is not possible. Since \( \lambda_n > 0 \), we have by property (P2) that
\[
\langle u_n^*, x_n \rangle = -\lambda_n \langle Px_n, x_n \rangle - \left\langle \left( \frac{1}{n} \right) I_p x_n, x_n \right\rangle \\
\leq \lambda_n \phi \left( \|x_n\| \right) - \frac{1}{n} \psi \left( \|x_n\| \right) \|x_n\| \\
\leq \Lambda \phi \left( S_\lambda \right) + \psi \left( S_\lambda \right) S_\lambda,
\]
\[ x \in D(T + P), \text{ which implies that the inclusion } \]
\[ Tx + \lambda Px + eJx \ni 0 \quad (37) \]
has no solution \( x \in D(T + P) \cap B_r(0) \) for any \( \epsilon \in (0, \epsilon_0] \). Since \( 0 \not\in T(\partial B_{\epsilon_0}(0)) \) and \( T \) is of type \((S_1)\), Theorem 8 implies the existence of a solution \( x_{\lambda_n} \in D(T + P) \cap \partial B_r(0) \), for some \( \lambda_n \in (0, \Lambda] \). The same argument can be repeated for any \( r > r_0 \). This complete the proof.

Remark 11. This result is also true when \( P : X \supset D(P) \to 2^{X^*} \) is assumed to be multivalued. We have the following theorem without proof.

**Theorem 12.** Assume that the operator \( T : X \supset D(T) \to 2^{X^*} \) is maximal monotone with \( 0 \in D(T) \) and \( 0 \in T(0) \). Assume that the operator \( P : X \supset D(P) \to 2^{X^*} \) with \( L \subset D(P) \) is of class \((S_q)\), and satisfies \((P2)\). Let \( \Lambda \) be a positive number. Assume that \( Tx \ni 0 \) implies \( x = 0 \), \( T \) satisfies \((S_q)\), and \((P1)\) there exists \( \alpha > 0 \) and \( \lambda \in (0, \Lambda] \) such that
\[ |Tx + \lambda Px| \geq \alpha, \quad x \in D(T + P). \quad (38) \]
Then the nonzero eigenvectors of problem (34) form a continuous branch of infinite length with corresponding eigenvalues \( \lambda \in (0, \Lambda] \).

In the following result, we assume that the operator \( T \) is defined and bounded on all of \( X \). In this case we demonstrate the fact that the assumption \( |Tx + Px| \geq 0 \) may be replaced by the assumption \(||Px|| \geq 0 \) on \( D(P) \).

**Theorem 13.** Assume that the operator \( T : X \supset D(T) \to 2^{X^*} \) is maximal monotone and bounded with \( 0 \in D(T) \) and \( 0 \in T(0) \). Assume that the operator \( P : X \supset D(P) \to X^* \), with \( L \subset D(P) \) satisfies condition \((S_q)\) and \((P2)\). Assume that \( Tx \ni 0 \) implies \( x = 0 \), \( T \) satisfies \((S_q)\), and there exist \( \alpha > 0 \) such that
\[ \|Px\| \geq \alpha, \quad x \in D(P). \quad (39) \]
Then the nonzero eigenvectors of the problem (34) form a continuous branch of infinite length.

**Proof.** We show that the problem (34) has eigenvectors on the set \( \partial B_r(0) \) for every \( r > 0 \). To this end, let \( \epsilon > 0 \) and show the existence of \( \tilde{\lambda} > 0 \) such that
\[ d(T + \tilde{\lambda} P + eJ, B_r(0), 0) = 0. \quad (40) \]
If this is not true, then there exist a sequence \( \{\lambda_n\} \subset (0, \infty) \) such that \( \lambda_n \to \infty \) and one of the following is true:

(i) the degree \( d(T + \lambda_n P + eJ, B_r(0), 0) \) is not well-defined;
(ii) \( d(T + \lambda_n P + eJ, B_r(0), 0) \neq 0 \).

In the case (i) there exist eigenvectors \( x_n \in \partial B_r(0) \) such that
\[ Tx_n + \lambda_n Px_n + eJx_n \ni 0. \quad (41) \]
In the case (ii) there exist eigenvectors \( x_n \in B_r(0) \) such that (41) holds. Thus in either case, there exist a sequence \( \{x_n\} \subset B_r(0) \) such that (41) is true. But this leads to a contradiction because \(|\lambda_n Px_n + eJx_n| \geq \alpha \lambda_n - \epsilon r \to \infty\), while the set \( T x_n \) lie in a bounded set. Thus, (40) is true for some \( \tilde{\lambda} > 0 \). We consider the homotopy
\[ H(t, x) \equiv Tx + t\tilde{\lambda} Px + eJx, \quad t \in [0, 1], \quad x \in D(H(t, \cdot)), \]
(42)
either there exist \( t_0 \in [0, 1] \) and \( x_{t_0} \in \partial B_r(0) \) such that
\[ Tx_{t_0} + t_0\tilde{\lambda} Px_{t_0} + eJx_{t_0} \ni 0, \quad (43) \]
or
\[ d(H(t, \cdot)) = d(H(1, \cdot), 0) = d(H(0, \cdot)) = 1, \quad t \in [0, 1] \]
(44)
that is a contradiction. The last equality in (44) follows from Theorem 3, (i) in [2]. It follows that (43) is true. Naturally, we must have \( t_0 \neq 0 \) in (43) because otherwise \( 0 \in (T + eJ)(\partial B_r(0)) \). This cannot happen because we already have \( 0 \in (T + eJ)(0) \) and \( T + eJ \) is one-to-one. From (43) we obtain sequences \( \lambda_n \in (0, \infty) \), \( \{x_n\} \subset \partial B_r(0) \) such that
\[ Tx_n + \lambda_n Px_n + \left(\frac{1}{n}\right)Jx_n \ni 0. \quad (45) \]
We may assume that \( x_n \to x_0 \). Again, \( \{\lambda_n\} \) cannot contain a subsequence \( \{\lambda_{n_k}\} \) such that \( \lambda_{n_k} \to \infty \) as \( k \to \infty \) because the sequence \( Tx_{n_k} + (1/n_k)Jx_{n_k} \) lies in a bounded set and \( \lambda_{n_k} ||Px_{n_k}|| \to \infty \) as \( k \to \infty \). Also we have that the sequence \( \{x_{n_k}\} \) is bounded. Thus we may assume that \( Px_n \to p^* \in X^* \). Since the sequence \( \{\lambda_n\} \) is bounded, we may assume that \( \lambda_n \to \lambda_0 \). Again, \( \lambda_0 \neq 0 \) otherwise \( \lambda_n Px_n + (1/n)x_n \to 0 \) and the \((S_q)\) property of \( T \) would imply that \( x_n \to x_0 \in \partial \Omega \). Since \( T \) is demiclosed, Lemma 6 would imply \( T x_0 \ni 0 \). This is a contradiction to our assumption that \( Tx \ni 0 \) implies \( x = 0 \). It follows that \( \lambda_0 > 0 \) and we may also assume that \( \lambda_0 > 0 \) for all \( n \).

Now proceeding as in the proof of Theorem 8, it is easy to see that
\[ \lim_{n \to \infty} \sup_{\epsilon > 0} \langle \lambda_n Px_n + \lambda_0 p^*, x_n \rangle \leq 0, \]
\[ \lim_{n \to \infty} \langle \lambda_n Px_n + \lambda_0 p^*, v \rangle = 0, \quad \forall v \in U_{n=1}^\infty F_n. \]
(46)
Now since \( P \) is of class \((S_1)\), we conclude that \( x_n \to x_0, x_0 \in D(P) \) and \( P x_0 = p^* \). Thus since also \( y_n \to x_n \to x_0 \) and the demiclosedness of \( T \), we obtain \( x_0 \in D(P) \cap \partial B_r(0) \) and \( T x_0 + \lambda_0 P x_0 \ni 0 \). Since \( r > 0 \) is arbitrary, the nonzero eigenvectors of problem (34) form a continuous branch of infinite length.

\[ \text{References} \]


