Research Article

Hilbert Space Representations of Generalized Canonical Commutation Relations

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We consider Hilbert space representations of a generalization of canonical commutation relations (CCRs) with $n$ degrees of freedom ($n \in \mathbb{N} := \{1, 2, 3, \ldots\}$) of the following type:

\[ [X_j, X_k] = i \Theta_{jk} I \quad (j, k = 1, 2, \ldots, 2n), \tag{1} \]

where $X_j$'s are elements of an algebra with identity $I$, $[X_j, X_k] := X_j X_k - X_k X_j$, $i$ is the imaginary unit, and $\Theta_{jk} \in \mathbb{R}$ (the set of real numbers) with antisymmetry $\Theta_{jk} = -\Theta_{kj}$ ($j, k = 1, 2, \ldots, 2n$) such that, for some pair $(j, k)$, $\Theta_{jk} \neq 0$. For convenience, we call (1) the $\Theta$-GCCR with $n$ degrees of freedom and the $2n \times 2n$ matrix

\[ \Theta := (\Theta_{jk})_{j, k = 1, \ldots, 2n} \tag{2} \]

the noncommutative factor for $\{X_j\}_{j=1}^{2n}$.

Note that, in the case where $\Theta$ is equal to

\[ J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{3} \]

with $I_n$ being the $n \times n$ unit matrix, (1) becomes the CCR with $n$ degrees of freedom. Namely, if we put $Q_j := X_j$, $P_j := X_{m,j}$ ($j = 1, 2, \ldots, n$) in the present case, then we have

\[ [Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \]

\[ [Q_j, P_k] = i \delta_{jk} I \quad (j, k = 1, \ldots, n), \tag{4} \]

where $\delta_{jk}$ is the Kronecker delta. Thus, (1) is a natural generalization of the CCR with $n$ degrees of freedom.

The GCCR also includes some of non-commutative space times (e.g., [1–3]), non-commutative spaces (e.g., [4]), and non-commutative phase spaces (e.g., [5–11]). In fact, one of the motivations for the present work is to investigate general structures underlying those non-commutative objects. In this paper, however, we present only some fundamental aspects of Hilbert space representations of the GCCR. The main result is to establish a uniqueness theorem on Weyl type representations of the GCCR (for the definition, see Section 4).

In Section 2, we define Hilbert space representations of the GCCR and discuss some basic facts on them. It is shown that there exists a one-to-one correspondence between representations of the GCCR and the CCR with the same degrees of freedom. In Section 3, we introduce a Schrödinger-type representation of the GCCR, whose representation space...
is $L^2(\mathbb{R}^n)$ as in the case of the Schrödinger representation of the CCR with $n$ degrees of freedom. In Section 4, Weyl-type representations of the GCCR are defined by analogy with Weyl representations of CCR. In the last section, we prove the uniqueness theorem mentioned above. In Appendix, we present some basic properties of self-adjoint operators obeying generalized Weyl relations, which are used in the text.

2. Basic Facts on Hilbert Space Representations of the $\Theta$-GCCR

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (antilinear in the first variable and linear in the second one) and norm $\| \cdot \|$. For a linear operator $A$ on $\mathcal{H}$, we denote its domain by $D(A)$. For linear operators $A_1, \ldots, A_p$ on $\mathcal{H}$,

$$D \left( \sum_{i=1}^{p} A_i \right) := \cap_{i=1}^{p} D(A_i),$$

$$D(A_1A_2) := \{ \Psi \in D(A_2) \mid A_2\Psi \in D(A_1) \},$$

$$D(A_1 \cdots A_p) := D\left( (A_1 \cdots A_{p-1}) A_p \right) \quad (p \geq 3).$$

**Definition 1.** Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $X_j$, $j = 1, \ldots, 2n$, be symmetric (not necessarily essentially self-adjoint) operators on $\mathcal{H}$. Set $X := (X_1, \ldots, X_{2n})$. We say that the triple $(\mathcal{H}, \mathcal{D}, X)$ is a symmetric representation of the $\Theta$-GCCR with $n$ degrees of freedom if $\mathcal{D} \subset \cap_{j=1}^{2n} D(X_jX_j)$ and (1) holds on $\mathcal{D}$.

If all the $X_j$'s ($j = 1, \ldots, 2n$) are self-adjoint, we say that $(\mathcal{H}, \mathcal{D}, X)$ is a self-adjoint representation of the GCCR.

**Remark 2.** The concept of self-adjoint representation defined above is different from the one used in representation theory of *-algebra (e.g., [12, page 205]).

**Remark 3.** In each symmetric representation $(\mathcal{H}, \mathcal{D}, X)$ of the $\Theta$-GCCR, $\mathcal{H}$ is infinite dimensional (if $\mathcal{H}$ were finite dimensional, then, for $(j,k)$ such that $\Theta_{jk} \neq 0$, $0 = \text{trace of } [X_j, X_k] = i\Theta_{jk} \dim \mathcal{H} \neq 0$, and hence one is led to a contradiction).

**Remark 4.** It follows from a well-known fact on commutation properties of linear operators (e.g., [13, Theorem 1.2.3]) that, for $(j,k)$ with $\Theta_{jk} \neq 0$, at least one of $X_j$ and $X_k$ is unbounded. Hence, one has to be careful about domains of $X_j$'s.

**Remark 5.** In the case of Hilbert space representations of CCR, symmetric representations, but nonself-adjoint ones, also play important roles. For example, such representations appear in mathematical theories of time operators [14] (see also [15, 16] for investigations from purely operator-theoretic points of view). Thus, it is expected that, in addition to self-adjoint representations of the $\Theta$-GCCR, non-self-adjoint symmetric representations of it may have any importance in applications to quantum physics.

**Remark 6.** In the context of quantum mechanics, for a symmetric operator $A$ and a unit vector $\psi \in D(A)$, $(\Delta A)_{\psi} := \| (A - \langle \psi, A\psi \rangle) \psi \|$ is called the uncertainty of $A$ in the vector state $\psi$. Let $(\mathcal{H}, \mathcal{D}, X)$ be a symmetric representation of the $\Theta$-GCCR with $n$ degrees of freedom. Then, one has uncertainty relations of Robertson type [17]: for all unit vectors $\psi \in \mathcal{D}$ and $j, k = 1, \ldots, 2n$,

$$\| (\Delta X_j)_{\psi} (\Delta X_k)_{\psi} \| \geq \frac{1}{2} \left| \langle \psi, \Theta_{jk} \psi \rangle \right|.$$  \hspace{1cm} (6)

Let $(\mathcal{H}, \mathcal{D}, X)$ be a symmetric representation of the $\Theta$-GCCR as in Definition 1. We assume for simplicity the following:

**Assumption 1.** The noncommutative factor $\Theta$ is regular (invertible).

Under this assumption, $\Theta$ is a regular antisymmetric real matrix. Hence, by a well-known fact in the theory of linear algebra (e.g., [18, page 173, Problem 9]), the following fact holds.

**Lemma 7.** There exists a regular $2n \times 2n$ real matrix $T_0$ such that $T_0 \Theta T_0 = I$, where $T_0$ is the transposed matrix of $T_0$ and $I$ is defined by (3).

The matrix $T_0$ in Lemma 7 belongs to the set

$$M_\Theta := \{ T \mid T \text{ is a } 2n \times 2n \text{ real matrix such that } T \Theta T = I \}. $$ \hspace{1cm} (7)

It is easy to see that for each $T \in M_\Theta$, there exists a unique $2n \times 2n$ symplectic matrix $W$ (i.e., $W^TJW = J$) such that $T = T_0W$. Hence,

$$M_\Theta = \{ T_0W \mid W^TJW = J \}. $$ \hspace{1cm} (8)

For a $2n \times 2n$ real matrix $L = (L_{jk})_{j,k=1,\ldots,2n}$, we define

$$X_j^L := \sum_{k=1}^{2n} L_{jk} X_k, \quad j = 1, \ldots, 2n. $$ \hspace{1cm} (9)

We call the correspondence $X \mapsto X^L := (X_1^L, \ldots, X_{2n}^L)$ the $L$-transform of $X$.

Let $$\Theta_L := L \Theta L^T. $$ \hspace{1cm} (10)

**Proposition 8.** (i) For all $j = 1, \ldots, 2n$, $X_j^L$ is a symmetric operator on $\mathcal{H}$.

(ii) For all $j, k = 1, \ldots, 2n$,

$$[X_j^L, X_k^L] = i(\Theta_L)_{jk} $$ \hspace{1cm} (11)

on $\mathcal{D}$.

(iii) For each $T \in M_\Theta$ and $j, k = 1, \ldots, 2n$,

$$[X_j^T, X_k^T] = i\Theta_{jk} $$ \hspace{1cm} (12)

on $\mathcal{D}$. 

Proof. An easy exercise. □

Proposition 8-(i) and (ii) show that \((\mathcal{H}, \mathcal{D}, X^j)\) is a symmetric representation of the \(\Theta\)-GCCR with \(n\) degrees of freedom.

Proposition 8 (iii) implies the following.

Corollary 9. Let \(T \in M_\Theta\) and
\[ Q_j := X_j^T, \quad P_j := X_{n+j}^T \quad (j = 1, \ldots, n). \tag{13} \]
Then, \((\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)\) is a symmetric representation of the CCR with \(n\) degrees of freedom.

Corollary 9 means that for each \(T \in M_\Theta\), the \(T\)-transform of \(X\) gives a correspondence from a symmetric representation of the \(\Theta\)-GCCR with \(n\) degrees of freedom to a symmetric representation of the CCR with the same degrees of freedom.

One can easily see that (9) with \(L = T \in M_\Theta\) implies that
\[ X_j = \sum_{k=1}^{2n} (T^{-1})_{jk} X_k^T \tag{14} \]
on \(\mathcal{L}_{2n}(\mathbb{R}^n)\). Thus, every symmetric representation of the \(\Theta\)-GCCR with \(n\) degrees of freedom is constructed from a symmetric representation of the CCR with the same degrees of freedom via (14).

Conversely, if a symmetric representation \((\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)\) of the CCR with \(n\) degrees of freedom is given and let
\[ X_j \coloneqq \sum_{k=1}^{n} (T^{-1})_{jk} Q_k + \sum_{k=1}^{n} (T^{-1})_{(m+k)j} P_k \tag{15} \]
with \(Q := \{Q_1, \ldots, Q_n\}\) and \(P := \{P_1, \ldots, P_n\}\), then \((\mathcal{H}, \mathcal{D}, \mathcal{X}(Q, P; T))\) is a symmetric representation of the \(\Theta\)-GCCR and (9) holds with \(L = T\), \(X_j^T = Q_j\, X_{n+j}^T = P_j \quad (j = 1, \ldots, n)\) and \(X_j = X_j(Q, P; T) \quad (j = 1, \ldots, 2n)\). Hence, every symmetric representation of the CCR with \(n\) degrees of freedom is constructed from a symmetric representation of the \(\Theta\)-GCCR with the same degrees of freedom. Thus, for each \(T \in M_\Theta\), there exists a one-to-one correspondence between a symmetric representation of the \(\Theta\)-GCCR and a symmetric representation of the CCR with \(n\) degrees of freedom.

3. Representations of Schrödinger Type

Let \(T \in M_\Theta\). By the fact on \(\mathcal{X}(Q, P; T)\) stated in the preceding section, we can define a class of representations of the \(\Theta\)-GCCR. Let \((L^2(\mathbb{R}^n), C_{\alpha}^0(\mathbb{R}^n), \{q_j, p_j\}_{j=1}^n)\) be the Schrödinger representation of the CCR with \(n\) degrees of freedom, that is, \(q_j\) is the multiplication by the \(j\)th component \(x_j\) of \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(p_j := -iD_j\) with \(D_j\) being the generalized partial differential operator in \(x_j\), acting in \(L^2(\mathbb{R}^n)\). Let
\[ X_j(q, p; T) = \sum_{k=1}^{2n} (T^{-1})_{jk} \alpha_k + \sum_{n+k}^{n} (T^{-1})_{(n+k)j} \beta_k, \tag{16} \]
which is (15) with \(Q = q\) and \(P = p\). We denote the closure of \(X_j(q, p; T)\) by \(\overline{X_j(q, p; T)}\) and set
\[ \overline{X_j(q, p; T)} := (\overline{X_1(q, p; T)}, \ldots, \overline{X_{2n}(q, p; T)}). \tag{17} \]

We call the triple \(\pi_S^n := (L^2(\mathbb{R}^n), C_{\alpha}^0(\mathbb{R}^n), \overline{X(q, p; T)})\) the \(T\)-Schrödinger representation of the \(\Theta\)-GCCR.

It is easy to see that for all \(j = 1, \ldots, 2n\), \(X_j(q, p; T)\) is essentially self-adjoint on \(C_{\alpha}^0(\mathbb{R}^n)\) (apply, e.g., the Nelson commutator theorem [19, Theorem X.37] with dominating operator \(N = \sum_{j=1}^{2n} (q_j^2 + p_j^2) + 1\) (This can be proved also by applying Proposition 16). Hence \(\overline{X_j(q, p; T)}\) is self-adjoint. Thus, we obtain the following.

Proposition 10. For each \(T \in M_\Theta\), the \(T\)-Schrödinger representation \(\pi_S^n\) is a self-adjoint representation of the \(\Theta\)-GCCR.

4. Representations of Weyl Type

Based on an analogy with Weyl representations of CCR, we introduce a concept of Weyl representation for \(\Theta\)-GCCR.

Definition 11. Let \(\{X_j\}_{j=1}^{2n}\) be a set of self-adjoint operators on a Hilbert space \(\mathcal{H}\). We say that \(\{X_j\}_{j=1}^{2n}\) is a Weyl representation of the \(\Theta\)-GCCR with \(n\) degrees of freedom if for all \(s, t \in \mathbb{R}\) and \(j, k = 1, \ldots, 2n\),
\[ e^{itX_j}e^{isX_k} = e^{-i\theta_{jk}} e^{-itX_j}e^{-isX_k}e^{i\theta_{jk}}. \tag{18} \]
We call these relations the \(\Theta\)-Weyl relations.

For a linear operator \(A\) on a Hilbert space, we denote its spectrum by \(\sigma(A)\).

Proposition 12. Let \(\{X_j\}_{j=1}^{2n}\) be a Weyl representation of the \(\Theta\)-GCCR on \(\mathcal{H}\). Then, there is a dense subspace \(\mathcal{D}_0 \subset \mathcal{H}\) left invariant by each \(X_j\) \(j = 1, \ldots, 2n\) such that \((\mathcal{H}, \mathcal{D}_0, X)\) is a self-adjoint representation of the \(\Theta\)-GCCR. Moreover, for every pair \((X_j, X_k)\) such that \(\sigma_k \neq 0\), \(X_j\) and \(X_k\) are purely absolutely continuous with
\[ \sigma(X_j) = \sigma(X_k) = \mathbb{R}, \quad j = 1, \ldots, 2n. \tag{19} \]

Proof. By (18), we can apply the results described in the Appendix of the present paper. In the present context, we need only to take, in the notation in the Appendix, \(N = 2n\), \(a_j = \theta_{jk}\) and \(A_j = X_j\). By Proposition A.4-(iii) and Corollary A.5, there exists a dense subspace \(\mathcal{D}_0 \subset \mathcal{H}\) left invariant by \(X_j\) \(j = 1, \ldots, 2n\) and \([X_j, X_k] = i\theta_{jk}\) on \(\mathcal{D}_0\). Thus, the first half of the proposition is derived. The second half follows from Proposition A.1. □

Remark 13. As in the case of self-adjoint representations of CCR (e.g., [16, 20, 21]), the converse of Proposition 12 does not hold (i.e., a self-adjoint representation of the \(\Theta\)-GCCR is not necessarily a Weyl one).
We recall that a set \( \{Q_j, P_j\}_{j=1}^n \) of self-adjoint operators on \( \mathcal{H} \) is a Weyl representation of the CCR with \( n \) degrees of freedom if for all \( s, t \in \mathbb{R} \) and \( j, k = 1, \ldots, n \), the following Weyl relations hold:

\[
e^{iQ_j} e^{iP_k} = e^{-i \delta_{jk} \delta_{rs}} e^{iQ_k} e^{iP_j},
\]

\[
e^{iQ_j} e^{iP_k} = e^{iQ_k} e^{iP_j},
\]

\[
e^{iP_k} e^{iQ_k} = e^{iP_k} e^{iQ_k}.
\]

(20)

Remark 14. A set \( \{Q_j, P_j\}_{j=1}^n \) of self-adjoint operators on \( \mathcal{H} \) is a Weyl representation of the CCR with \( n \) degrees of freedom if and only if \( \{X_j\}_{j=1}^n \) with \( X_j := Q_j, X_{m,j} = P_j \) \( (j = 1, \ldots, n) \) is a Weyl representation of the \( J \)-GCCR, where \( J \) is given by (3).

Let \( T \in M_\Theta \) be arbitrarily fixed. The next proposition shows that the \( T \)-transform of each Weyl representation of the \( \Theta \)-GCCR is a Weyl representation of the \( \Theta \)-GCCR with \( n \) degrees of freedom.

**Proposition 15.** Let \( \{X_j\}_{j=1}^n \) be a Weyl representation of the \( \Theta \)-GCCR on \( \mathcal{H} \), and let \( X^T \) be the \( T \)-transform of \( X \). Then, each \( X^T_j \) is essentially self-adjoint, and \( \{X^T_j\}_{j=1}^n \) is a Weyl representation of the \( J \)-GCCR.

Proof. The essential self-adjointness of \( X^T_j \) follows from a simple application of Theorem 4.6 in Appendix. Corollary A.7 in Appendix and the relation \( \Theta^T = J \) imply that \( \{X^T_j\}_{j=1}^n \) satisfies the \( J \)-Weyl relations. \( \square \)

In the same way as in the proof of Proposition 15, we can prove the following proposition:

**Proposition 16.** Let \( \{Q_j, P_j\}_{j=1}^n \) be a Weyl representation of the CCR with \( n \) degrees of freedom on a Hilbert space \( \mathcal{H} \). Let \( X_j(Q, P; T) \) \( (j = 1, \ldots, 2n) \) be defined by (15). Then, each \( X_j(Q, P; T) \) is essentially self-adjoint and \( \{X_j(Q, P; T)\}_{j=1}^{2n} \) is a Weyl representation of \( \Theta \)-GCCR with \( n \) degrees of freedom.

This proposition shows that the converse of Proposition 15 holds too. Thus, for each \( T \in M_\Theta \), there exists a one-to-one correspondence between a Weyl representation of the CCR with \( n \) degrees of freedom and that of the \( \Theta \)-GCCR with the same degrees of freedom.

It is well known [22] that the Schrödinger representation \( \{q_j, p_j\}_{j=1}^n \) is a Weyl representation of the CCR with \( n \) degrees of freedom. Hence, we obtain the following result.

**Corollary 17.** For each \( T \in M_\Theta \), the \( T \)-Schrödinger representation \( \{X_j(q, p; T)\}_{j=1}^{2n} \) is a Weyl representation of the \( \Theta \)-GCCR.

We say that a Weyl representation \( \{X_j\}_{j=1}^{2n} \) of the \( \Theta \)-GCCR on \( \mathcal{H} \) is irreducible if every closed subspace \( \mathcal{M} \) of \( \mathcal{H} \) which is invariant under the action of \( e^{iX_j} \) \( (t \in \mathbb{R}, j = 1, \ldots, 2n) \) is \( \{0\} \) or \( \mathcal{H} \).

**Proposition 18.** Let \( T \in M_\Theta \). Then, the \( T \)-Schrödinger representation \( \{X_j(q, p; T)\}_{j=1}^{2n} \) as a Weyl representation of the \( \Theta \)-GCCR is irreducible.

Proof. Let \( \mathcal{M} \) be an invariant closed subspace of \( e^{iX_j(q, p; T)} \) \( (t \in \mathbb{R}, j = 1, \ldots, 2n) \). We have

\[
q_j = \sum_{k=1}^{2n} T_{kj} X_k(q, p; T),
\]

(21)

\[
p_j = \sum_{k=1}^{2n} T_{k(j+1)} X_k(q, p; T)
\]

on \( t \in \mathbb{R}, j = 1, \ldots, n \). Hence, by an application of Theorem A.6 in Appendix, \( e^{iX_j} \) and \( e^{iP_j} \) \( (t \in \mathbb{R}) \) can be written, respectively, as a scalar multiple of \( e^{iX_j(q, p; T)} \) or \( e^{iX_j(q, p; T)} \). Hence, \( \mathcal{M} \) is invariant under the action of \( e^{iX_j}, e^{iP_j} \) \( (t \in \mathbb{R}, j = 1, \ldots, n) \). It is well known that \( e^{iX_j}, e^{iP_j} \) \( (t \in \mathbb{R}, j = 1, \ldots, n) \) are irreducible. Thus, \( \mathcal{M} = \{0\} \) or \( \mathcal{H} \).

**5. Uniqueness Theorem on Weyl Representations of the \( \Theta \)-GCCR**

In this section, we prove the main result of the present paper, that is, a uniqueness theorem on Weyl representations of the \( \Theta \)-GCCR, which may be regarded as a GCCR version of the celebrated von Neumann uniqueness theorem of Weyl representations of CCR ([13, Theorem 4.11.1], [22], [23, Theorem VIII.14]).

**Theorem 19.** Let \( \{X_j\}_{j=1}^{2n} \) be a Weyl representation of the \( \Theta \)-GCCR on a separable Hilbert space \( \mathcal{H} \). Then, for each \( T \in M_\Theta \), there exist mutually orthogonal closed subspaces \( \mathcal{H}_\ell \) \( (\ell = 1, \ldots, N; N \in \mathbb{N} \) or co) such that the following (i)–(iii) hold.

(i) \( \mathcal{H} = \bigoplus_{\ell=1}^N \mathcal{H}_\ell \).

(ii) For each \( j = 1, \ldots, 2n \), \( X_j \) is reduced by each \( \mathcal{H}_\ell \) \( (\ell = 1, \ldots, N) \). We denote by \( X_j^{(\ell)} \) the reduced part of \( X_j \) to \( \mathcal{H}_\ell \).

(iii) For each \( \ell \), there exists a unitary operator \( U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n) \) such that

\[
U_\ell X_j^{(\ell)} U_\ell^{-1} = X_j(q, p; T), \quad j = 1, \ldots, 2n,
\]

(22)

where \( \{X_j(q, p; T)\}_{j=1}^{2n} \) is the \( T \)-Schrödinger representation of the \( \Theta \)-GCCR.

Proof. Let \( T \in M_\Theta \), \( X^T \) be the \( T \)-transform of \( X \) and \( \{Q_j, P_j\}_{j=1}^n \) be a Weyl representation of the CCR with \( n \) degrees of freedom. Hence, by the von Neumann uniqueness theorem mentioned above, there exist mutually orthogonal closed subspaces \( \mathcal{H}_\ell \) such that (i) given above and the following (a) and (b) hold.
(a) For each \( j = 1, \ldots, n \) and all \( t \in \mathbb{R} \), \( e^{iH_Q} \) and \( e^{iH_T} \) leave each \( \mathbb{H}_\ell \) invariant (\( \ell = 1, \ldots, N \)).

(b) For each \( \ell \), there exists a unitary operator \( U \in \mathbb{H}_\ell : \mathbb{H}_\ell \to L^2(\mathbb{R}^n) \) such that
\[
U e^{iH_T} U^{-1} = e^{i\hat{a}_j}, \quad j = 1, \ldots, n,
\]
\[
U e^{iH_Q} U^{-1} = e^{i\hat{p}_j}, \quad t \in \mathbb{R}, \quad j = 1, \ldots, n.
\]

By (14), we have \( X_j = X_j(Q, P; T) \) on \( \cap_{j=1}^N \mathbb{D}(X_j) \). Hence, \( X_j \subset X_j(Q, P; T) \). By Proposition 16, \( X_j(Q, P; T) \) is self-adjoint. Hence, \( X_j = X_j(Q, P; T) \). Therefore, by Theorem A.6 in Appendix, we obtain
\[
e^{iX_j} = e^{\frac{i}{2} \sum_{j=1}^N (T^{-1})^j (T^j)_{m_j} / 2} e^{iQ_j} \cdots e^{iQ_j} e^{iP_j} \cdots e^{iP_j},
\]
\[
 j = 1, \ldots, 2n.
\]

Hence, each \( e^{iX_j} \) leaves \( \mathbb{H}_\ell \) invariant (\( \ell = 1, \ldots, N \)). Therefore, \( X_j \) is reduced by each \( \mathbb{H}_\ell \) by \( X_j^{(l)} \). Then, we have by (23)
\[
U e^{iX_j} U^{-1} = e^{\frac{i}{2} \sum_{j=1}^N (T^{-1})^j (T^j)_{m_j} / 2} e^{iQ_j} \cdots e^{iQ_j} e^{iP_j} \cdots e^{iP_j},
\]
\[
 j = 1, \ldots, n.
\]

Thus, (22) follows.

Theorem 19 tells us that every Weyl representation of the \( \Theta \)-GCCR on a separable Hilbert space is unitarily equivalent to a direct sum of the \( T \)-Schrödinger representation of the \( \Theta \)-GCCR, where \( T \in M_{\Theta} \) is arbitrary.

The next corollary immediately follows from Theorem 19.

**Corollary 20.** Let \( \{ X_j^{(l)} \}_{j=1}^{2n} \) be an irreducible Weyl representation of the \( \Theta \)-GCCR on a separable Hilbert space \( \mathbb{H} \). Then, for each \( T \in M_{\Theta} \), there exists a unitary operator \( U : \mathbb{H} \to L^2(\mathbb{R}^n) \) such that
\[
UX_j U^{-1} = X_j(Q, P; T), \quad j = 1, \ldots, 2n.
\]

The following result shows that the arbitrariness of the choice of \( T \) in the \( T \)-Schrödinger representation of the \( \Theta \)-GCCR is implemented by unitary operators.

**Corollary 21.** Let \( S, T \in M_{\Theta} \). Then, there exists a unitary operator \( V \) on \( L^2(\mathbb{R}^n) \) such that
\[
VX_j S V^{-1} = X_j(Q, P; T), \quad j = 1, \ldots, 2n.
\]

**Proof.** We need only to apply Corollary 20 to the case where \( X_j = X_j(Q, P; S) \).

**Remark 22.** As in the case of non-Weyl representations of CCR, for non-Weyl representations of the \( \Theta \)-GCCR, the conclusion of Theorem 19 does not hold in general. Examples of such representations of the \( \Theta \)-GCCR can be constructed from non-Weyl representations of CCR (e.g., [15, 16, 20, 21]). A detailed description of some examples is given in [5].

### Appendix

**Some Properties of Self-Adjoint Operators Satisfying Relations of Weyl Type**

Let \( N \geq 2 \) be an integer, and let \( A_j (j = 1, \ldots, N) \) be self-adjoint operators on a Hilbert space \( \mathbb{H} \) satisfying relations of Weyl type:
\[
e^{iA_j} e^{itA_k} = e^{-itA_k} e^{itA_j}, \quad t, s \in \mathbb{R}, \quad j, k = 1, \ldots, N, \quad (A.1)
\]
where \( a_j \)'s are real constants. It follows that \( a_j \) is antisymmetric in \( (j, k) \):
\[
a_jk = -a_kj, \quad j, k = 1, \ldots, N. \quad (A.2)
\]
The unitarity of \( e^{iA_j} \) and functional calculus imply that
\[
\exp \left( is^{iA_j} e^{-itA_j} \right) = \exp \left( is \left( A_k - ta_k \right) \right), \quad s, t \in \mathbb{R}. \quad (A.3)
\]
Hence, we have the operator equality
\[
e^{iA_j} A_k e^{-itA_j} = A_k - ta_k, \quad t \in \mathbb{R}, \quad j, k = 1, \ldots, N. \quad (A.4)
\]

For a linear operator \( A \) on a Hilbert space, we denote the spectrum of \( A \) by \( \sigma(A) \).

**Proposition A.1.** Suppose that there exists a pair \( (j, k) \) such that \( a_jk \neq 0 \) (hence, \( j \neq k \)). Then,
\[
\sigma \left( A_j \right) = \mathbb{R}, \quad \sigma \left( A_k \right) = \mathbb{R}. \quad (A.5)
\]
Moreover, \( A_j \) and \( A_k \) are purely absolutely continuous.

**Proof.** By (A.4) and the unitary invariance of spectrum, we have \( \sigma(A_k) = \sigma(A_k - ta_j) \) for all \( t \in \mathbb{R} \). Since \( a_jk \neq 0 \), this implies the second equation of (A.5). By (A.2), we have \( a_jk \neq 0 \). Hence, by considering the case of \( (j, k) \) replaced by \( (k, j) \), we obtain the first equation of (A.5).

Relation (A.4) means that \( (A_k, A_j) \) is a weak Weyl representation of the CCR with one degree of freedom [14, 15, 24]. Hence \( A_j \) is purely absolutely continuous [14, 15]. Similarly, we can show that \( A_k \) is purely absolutely continuous.

**Proposition A.2.** Let \( j \) and \( k \) be fixed. Then, for all \( \psi \in D(A_j) \cap D(A_k) \), \( \psi \) is in \( D(A_j A_k) \) and
\[
A_j A_k \psi = ia_jk \psi. \quad (A.6)
\]

**Proof.** An easy exercise (use (A.4)).

For each function \( f \in C^\infty_c(\mathbb{R}^N) \) and each vector \( \psi \in \mathbb{H} \), we define a vector \( \psi_f \) by
\[
\psi_f := \int_{\mathbb{R}^N} f(t) e^{iA_j} e^{iA_k} \psi dt, \quad (A.7)
\]
where \( t = (t_1, \ldots, t_N) \in \mathbb{R}^N \) and the integral on the right-hand side is taken in the strong sense. We introduce

\[
\mathcal{D}_0 := \operatorname{Span}\left\{ \psi_f \mid \psi \in \mathcal{H}, f \in C^0_c \left( \mathbb{R}^N \right) \right\},
\]

(A.8)

where \( \operatorname{Span}\left\{ \cdot \right\} \) denotes the subspace algebraically spanned by the vectors in the set \( \left\{ \cdot \right\} \). It is easy to see that \( \mathcal{D}_0 \) is dense in \( \mathcal{H} \).

For \( f : \mathbb{R}^N \to \mathbb{C} \) (the set of complex numbers), we set \( \| f \|_1 := \int_{\mathbb{R}^N} |f(t)| \, dt \).

**Lemma A.3.** Let \( f_n, f \in C^0_c (\mathbb{R}^N) \) such that \( \| f_n - f \|_1 \to 0 \) \((n \to \infty)\). Then, \( \| \psi_{f_n} - \psi_f \| \to 0 \) \((n \to \infty)\).

**Proof.** Since \( \mathcal{D}_0 \) is an invariant (i.e., \( A_j \mathcal{D}_0 \subset \mathcal{D}_0 \)) and \( \| \psi_{f_n} - \psi_f \| \) is unitary, we have

\[
\| \psi_{f_n} - \psi_f \|_1 \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence, \( \| \psi_{f_n} - \psi_f \| \to 0 \) \((n \to \infty)\).

It is easy to see that \( f(t) \) is in \( C^0_c (\mathbb{R}^N) \). Hence, \( \psi_{f(t)} \in \mathcal{D}_0 \).

Thus, \( e^{itA_j} \) leaves \( \mathcal{D}_0 \) invariant.

(ii) By (A.14), we have for all \( t \in \mathbb{R} \setminus \{0\} \), \( \| (e^{itA_j} - 1) \psi_{f(t)} \| = \| \psi_{f(t) - \frac{t}{2}} \| \)\( t \to \infty \). It is easy to see that \( \| (f(t) - f) \|_1 \to 0 \) \((t \to 0)\). Hence, by Lemma A.3,

\[
\lim_{t \to 0} \frac{(e^{itA_j} - 1) \psi_f}{t} = \psi_{F(f)}.
\]

(A.16)

Therefore, \( \psi_f \) is in \( D(A_j) \) and \( iA_j \psi_f = \psi_{F(f)} \). Hence, (A.10) with \( \ell = 1 \) holds. Then, one can prove (A.10) by induction.

(iii) This easily follows from (ii).

Propositions A.2 and A.4 immediately yield the following result.

**Corollary A.5.** For all \( j, k = 1, \ldots, N \), \( [A_j, A_k] = iA_{jk} \) on \( \mathcal{D}_0 \).

**Theorem A.6.** For all \( c_j \in \mathbb{R} \), \( j = 1, \ldots, N \), \( \sum_{j=1}^N c_j A_j \) is essentially self-adjoint on \( \mathcal{D}_0 \) and

\[
\psi_{\sum_{j=1}^N c_j A_j} = e^{i\sum_{j=1}^N c_j x_j A_j} e^{i\sum_{j=1}^N x_j A_j},
\]

(A.17)

where for a closable operator \( C \), \( \overline{C} \) denotes the closure of \( C \).

**Proof.** For each \( t \in \mathbb{R} \), we define an operator \( U(t) \) by

\[
U(t) := e^{i\sum_{j=1}^N c_j x_j A_j} e^{i\sum_{j=1}^N x_j A_j}, \quad t \in \mathbb{R}.
\]

(A.18)

By using (A.1), one can show that \( \{U(t)\}_{t \in \mathbb{R}} \) is a strongly continuous one-parameter unitary group. Hence, by the Stone theorem, there exists a unique self-adjoint operator \( A \) on \( \mathcal{H} \) such that \( U(t) = e^{itA}, \quad t \in \mathbb{R} \). By Proposition A.4, \( U(t) \) leaves \( \mathcal{D}_0 \) invariant and strongly differentiable on \( \mathcal{D}_0 \) with

\[
\frac{dU(t) \psi}{dt} \bigg|_{t=0} = \sum_{j=1}^N c_j A_j \psi, \quad \psi \in \mathcal{D}_0.
\]

(A.19)

Hence, \( \mathcal{D}_0 \) is a core of \( A \) (e.g., [23, Theorem VIII.10]). Hence \( A \psi = \sum_{j=1}^N c_j A_j \psi, \quad \psi \in \mathcal{D}_0 \). Thus, the desired result follows.

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References


