Hindawi Publishing Corporation
Journal of Mathematics
Volume 2013, Article ID 316576, 8 pages
http://dx.doi.org/10.1155/2013/316576

Research Article

Positive Solutions for a Fourth-Order Boundary Value Problem

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Received 24 August 2012; Accepted 29 October 2012

Academic Editor: Andrea Pascucci

This paper deals with the existence and multiplicity of positive solutions for the fourth-order boundary value problem $u^{(4)} = f(t, u, u', -u'', u''')$, $u(0) = u'(1) = u'''(0) = u''(1) = 0$. Here $f \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)(\mathbb{R}_+ := [0, +\infty))$. We use fixed point index theory to establish our main results based on a priori estimates achieved by utilizing some integral identities and integral inequalities.

1. Introduction

The purpose of this paper is to study the existence and multiplicity of positive solutions for the fourth-order boundary value problem

$$u^{(4)} = f\left(t, u, u', -u'', u'''\right), \quad u(0) = u'(1) = u'''(0) = u''(1) = 0, \tag{1}$$

where $f \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)$. By a positive solution of (1), we mean a function $u \in C^4([0,1], \mathbb{R}_+)$ that solves (1) and satisfies $u(t) > 0$ for all $t \in (0, 1]$.

In recent years, fourth-order boundary value problems for nonlinear ordinary differential equations have been extensively studied by using diverse methods, including fixed point theorems on cones [1–6], the method of lower and upper solutions [7–10], the iterative method [11–14], critical point theory [15], and the shooting method [16]. It should be remarked that none of the results cited above involves derivatives of all orders in their nonlinearities. In [9], Ma and Yang studied the fourth-order four-point boundary value problem

$$u^{(4)} = f\left(t, u, -u''\right), \quad u(0) = 0, \quad u(1) = au(\eta), \quad u''(0) = 0, \quad u''(1) = bu(\xi), \tag{2}$$

where $\eta, \xi \in (0, 1)$, $a, b \geqslant 0$. The authors used the method of upper and lower solutions and a new maximum principle to establish their existence results. In [14], Pei and Chang studied a class of fourth-order boundary value problem

$$y^{(4)} = f(t, y), \quad y(0) = y(1) = y'(0) = y'(1) = 0. \tag{3}$$

By using the monotone iterative technique, the authors proved that problem (3) has at least one symmetric positive solution under certain conditions. In [17], Pang et al. studied the existence and multiplicity of nontrivial solutions for the fourth-order boundary value problem

$$x^{(4)} = f\left(x, -x''\right), \quad x(0) = x(1) = x''(0) = x''(1) = 0, \tag{4}$$

where $f \in C(\mathbb{R}^2, \mathbb{R})$. Making use of the theory of Leray-Schauder degree, under appropriate conditions on the nonlinearity f, the authors proved that problem (4) has at least six different nontrivial solutions, including two positive ones and two negative ones.

In [18], Yang and Sun studied the fourth-order boundary value problem

$$u^{(4)} = f\left(t, u, u', -u'', -u'''\right), \quad u(0) = u'(1) = u''(0) = u'''(1) = 0, \tag{5}$$

where $f \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)$. By using fixed point index theory based on a priori estimates, the authors established the main results on the existence, multiplicity and uniqueness of positive solutions for (5).

Motivated by [18], in this paper, we will discuss the existence and multiplicity of positive solutions for problem (1). As in [18], we first use the method of order reduction to transform (1) into a boundary value problem for a second-order integro-differential equation and then seek the existence and multiplicity of positive solutions for the resultant problem. To overcome the difficulties stemming from the presence of derivatives of all orders and the difference between (1) and (5) in their boundary value conditions, we have to prove that the maximum of every nonnegative concave function v can be dominated by the integrals

$$\begin{aligned} &\int_0^1 v(t)(1-t)e^{1-t}\mathrm{d}t, \\ &\int_0^1 \left((B_1 v)(t) + 2(B_1 v)'(t)\right)(1-t)e^{1-t}\mathrm{d}t \end{aligned} \tag{6}$$

(see Lemmas 5 and 6 below for more details). Based on a priori estimates achieved by utilizing some integral identities and integral inequalities, we use fixed point index theory to prove the existence and multiplicity of positive solutions for (1).

This paper is organized as follows. In Section 2, we transform (1) into a boundary value problem for a second-order integro-differential equation and then establish some basic integral identities and integral inequalities that are useful in deriving the priori estimates in the next section. Our main results, namely Theorems 10–12, are stated and proved in Section 3.

2. Preliminaries

Let $E := C^1([0,1], \mathbb{R})$ and $\|u\| := \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 := \max\{|u(t)| : t \in [0,1]\}$. Furthermore, let

$$P := \left\{u \in E : u(t) \geqslant 0,\ u'(t)\ 0,\ \forall t \in [0,1]\right\}. \tag{7}$$

Clearly $(E, \|\cdot\|)$ is a real Banach space and P is a cone in E.

Let

$$\begin{aligned} k_1(t,s) &:= \min\{t,s\}, \\ k_2(t,s) &:= \min\{1-s, 1-t\}. \end{aligned} \tag{8}$$

Define the linear integral operators B_1 and B_2 by

$$(B_i v)(t) := \int_0^1 k_i(t,s)\, v(s)\,\mathrm{d}s, \quad v \in P\ (i = 1,2). \tag{9}$$

Substituting $v(t) := -u''(t)$ into (1), we transform (1) into the following second-order boundary value problem for the integro-differential equation

$$\begin{aligned} -v''(t) &= f\left(t, (B_1 v)(t), (B_1 v)'(t), v(t), -v'(t)\right), \\ v'(0) &= v(1) = 0, \end{aligned} \tag{10}$$

which is equivalent to the nonlinear integral equation

$$v(t) = \int_0^1 k_2(t,s)\, f\left(s, (B_1 v)(s), (B_1 v)'(s), v(s), -v'(s)\right)\mathrm{d}s. \tag{11}$$

Define the operator $A : P \to P$ by

$$(Av)(t) := \int_0^1 k_2(t,s)\, f\left(s, (B_1 v)(s), (B_1 v)'(s), v(s), -v'(s)\right)\mathrm{d}s, \quad v \in P. \tag{12}$$

Now $f \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)$ implies that A is a completely continuous operator. In our setting, the existence of positive solutions for (1) is equivalent to that of positive fixed points of $A : P \to P$.

To establish the priori estimates of positive solutions for some problems associated with (11), we need several integral identities and integral inequalities below.

Lemma 1. *If $v \in P \cap C^2[0,1]$, $v'(0) = v(1) = 0$, then*

$$\int_0^1 -v''(t)(1-t)e^{1-t}\mathrm{d}t = \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t. \tag{13}$$

Proof. Integrating by parts and using $v'(0) = v(1) = 0$, we have

$$\begin{aligned} &\int_0^1 \left(-v''(t)\right)(1-t)\ e^{1-t}\mathrm{d}t \\ &\quad = \int_0^1 v'(t)(t-2)e^{1-t}\mathrm{d}t \\ &\quad = \int_0^1 \left(-v'(t)\right)te^{1-t}\mathrm{d}t - 2\int_0^1 v'(t)(1-t)e^{1-t}\mathrm{d}t, \end{aligned} \tag{14}$$

$$\int_0^1 -v'(t)\,te^{1-t}\mathrm{d}t = \int_0^1 v(t)(1-t)e^{1-t}\mathrm{d}t, \tag{15}$$

from which (13) follows. This completes the proof. □

Lemma 2. *If $v \in C([0,1], \mathbb{R}_+)$, then*

$$\int_0^1 \left((B_1 v)(t) + 2(B_1 v)'(t)\,te^t\mathrm{d}t\right) = \int_0^1 v(t)\,te^t\mathrm{d}t. \tag{16}$$

Proof. Notice $(B_1 v)(0) = 0$. Integrating by parts, we obtain

$$\int_0^1 (B_1 v)(t)\,te^t\mathrm{d}t = \int_0^1 (1-t)e^t(B_1 v)'(t)\,\mathrm{d}t, \tag{17}$$

so that

$$\int_0^1 \left((B_1 v)(t) + 2(B_1 v)'(t)\right)te^t\mathrm{d}t = \int_0^1 (1+t)e^t(B_1 v)'(t)\,\mathrm{d}t. \tag{18}$$

Notice $(B_1 v)'(1) = 0$ and $(B_1 v)''(t) = -v(t)$. Integrating the right hand of the above by parts again gives (16). This completes the proof. □

If both h_1 and h_2 are continuous and decreasing on $[0,1]$, then Chebyshev's inequality for h_1 and h_2 is given by

$$\int_0^1 h_1(t) h_2(t)\,dt \geqslant \int_0^1 h_1(t)\,dt \int_0^1 h_2(t)\,dt. \quad (19)$$

In addition, if both h_1 and h_2 are continuous, with one of them decreasing on $[0,1]$ and the other increasing on $[0,1]$, then Chebyshev's inequality for h_1 and h_2 becomes

$$\int_0^1 h_1(t) h_2(t)\,dt \leqslant \int_0^1 h_1(t)\,dt \int_0^1 h_2(t)\,dt. \quad (20)$$

Lemma 3. *If $v \in P, v(1) = 0$, and v' is decreasing on $[0,1]$, then*

$$\int_0^1 v(t)(1-t)e^{1-t}dt \geqslant (e-2)\int_0^1 -v'(t)(1-t)e^{1-t}dt. \quad (21)$$

Proof. Notice $v(1) = 0$ and $v' \leqslant 0$. Since $-v'$ is increasing on $[0,1]$ and $(1-t)e^{1-t}$ is decreasing on $[0,1]$, it follows from Chebyshev's inequality that

$$\begin{aligned}\int_0^1 -v'(t)(1-t)e^{1-t}dt &\leqslant \int_0^1 -v'(t)\,dt \int_0^1 (1-t)e^{1-t}dt \\ &= v(0).\end{aligned} \quad (22)$$

Notice that te^{1-t} is increasing on $[0,1]$. Chebyshev's inequality again implies

$$\int_0^1 -v'(t)te^{1-t}dt \geqslant \int_0^1 -v'(t)\,dt \int_0^1 te^{1-t}dt = (e-2)v(0). \quad (23)$$

Note that (15) is guaranteed by $v \in P, v(1) = 0$. By virtue of (15), we have

$$\int_0^1 v(t)(1-t)e^{1-t}dt \geqslant (e-2)v(0). \quad (24)$$

Combining this with (22) yields (21). This complete the proof. □

Lemma 4. *If $v \in P, v(1) = 0$, then*

$$v(0) \leqslant \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)e^{1-t}dt. \quad (25)$$

Proof. By (15), we have

$$\begin{aligned}\int_0^1 &\left(v(t) - 2v'(t)\right)(1-t)e^{1-t}ds \\ &= \int_0^1 -v'(t)(2-t)e^{1-t}dt \\ &\geqslant \int_0^1 -v'(t)\,dt = v(0).\end{aligned} \quad (26)$$

This completes the proof. □

Lemma 5. *If $v \in P \cap C^2[0,1]$ and $v'' \leqslant 0$, then*

$$(e-2)\|v\|_0 \leqslant \int_0^1 v(t)(1-t)e^{1-t}dt \leqslant e\|v\|_0. \quad (27)$$

Proof. It is easy to see that $\int_0^1 v(t)te^t dt \leqslant e\|v\|_0$. The condition $v' \leqslant 0$ implies $\|v\|_0 = v \leqslant (0)$. By the concavity of v, we have

$$\begin{aligned}\int_0^1 v(t)(1-t)e^{1-t}dt &\geqslant \int_0^1 (1-t)e^{1-t}v(t\cdot 1 + (1-t)\cdot 0)\,dt \\ &\geqslant v(0)\int_0^1 (1-t)^2 e^{1-t}dt = (e-2)\|v\|_0.\end{aligned} \quad (28)$$

This completes the proof. □

Lemma 6. *If $v \in P \cap C^2[0,1]$ and $v'' \leqslant 0$, then we have*

$$\begin{aligned}\frac{3-e}{6}\|v\|_0 &\leqslant \int_0^1 \left((B_1 v)(t) + 2(B_1 v)'(t)\right)(1-t)e^{1-t}dt \\ &\leqslant 3e\|v\|_0.\end{aligned} \quad (29)$$

Proof. By the concavity of $B_1 v$ and v, we have

$$\begin{aligned}\int_0^1 &\left((B_1 v)(t) + 2(B_1 v)'(t)\right)(1-t)e^{1-t}dt \\ &\geqslant \int_0^1 (B_1 v)(t)(1-t)e^{1-t}dt \\ &= \int_0^1 (B_1 v)(t\cdot 1 + (1-t)\cdot 0)(1-t)e^{1-t}dt \\ &\geqslant (B_1 v)(1)\int_0^1 t(1-t)e^{1-t}dt \\ &= (3-e)\int_0^1 sv(s)\,ds \\ &\geqslant (3-e)\int_0^1 s(1-s)v(0)\,ds \\ &= \frac{3-e}{6}\|v\|_0, \\ \int_0^1 &\left((B_1 v)(t) + 2(B_1 v)'(t)\right)(1-t)e^{1-t}dt \\ &\leqslant e\int_0^1 (B_1 v)(t) + 2(B_1 v)'(t)\,dt \\ &= e\int_0^1 \left(\int_0^1 k_1(t,s)v(s)\,ds + 2\int_0^1 v(s)\,ds\right)dt \\ &\leqslant 3e\|v\|_0.\end{aligned} \quad (30)$$

This completes the proof. □

Lemma 7 (see [19]). *Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set and that*

$T : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If there exists $\omega_0 \in P \setminus \{0\}$ such that

$$\omega - T\omega \neq \lambda\omega_0, \quad \forall \lambda \geqslant 0, \ \omega \in \partial\Omega \cap P, \tag{31}$$

then $i(T, \Omega \cap P, P) = 0$, where i indicates the fixed point index on P.

Lemma 8 (see [19]). *Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If*

$$\omega - \lambda T\omega \neq 0, \quad \forall \lambda \in [0, 1], \ \omega \in \partial\Omega \cap P, \tag{32}$$

then $i(T, \Omega \cap P, P) = 1$.

3. Existence and Multiplicity of Positive Solutions for (1)

For simplicity, we denote by $x := (x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4$ and $I_\rho := [0, \rho]$ for $\rho > 0$ in this section. Now we list our hypotheses.

(H1) $f \in C([0, 1] \times \mathbb{R}_+^4, \mathbb{R}_+)$.

(H2) There exist $a_1 \geqslant 0$, $b_1 \geqslant 0$, $c > 0$, such that $d_1 := a_1 (3 - e(e - 2))/6e^2 + b_1 > 1$, and

$$f(t, x) \geqslant a_1 (x_1 + 2x_2) + b_1 (x_3 + 2x_4) - c \tag{33}$$

holds for all $(t, x) \in [0, 1] \times \mathbb{R}_+^4$.

(H3) There exists a function $\Phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, x) \leqslant \Phi(x_4) \tag{34}$$

for all $x \in [0, 1] \times I_m^3 \times \mathbb{R}_+$ and

$$\int_0^\infty \frac{\xi \mathrm{d}\xi}{\Phi(\xi) + 2me} > m. \tag{35}$$

where $m := c/(d_1 - 1) > 0$ and c, d_1 are given as in (H2).

(H4) There exist $a_2 \geqslant 0$, $b_2 \geqslant 0$, $r > 0$ such that $d_2 := 3a_2 e/(e - 2) + b_2 < 1$, and

$$f(t, x) \leqslant a_2 (x_1 + 2x_2) + b_2 (x_3 + 2x_4) \tag{36}$$

holds for all $(t, x) \in [0, 1] \times I_r^3$.

(H5) There exist $a_3 \geqslant 0$, $b_3 \geqslant 0$, $c > 0$ such that $d_3 := 3a_3 e/(e - 2) + b_2 < 1$, and

$$f(t, x) \leqslant a_3 (x_1 + 2x_2) + b_3 (x_3 + 2x_4) + c \tag{37}$$

holds for all $(t, x) \in [0, 1] \times \mathbb{R}_+^4$.

(H6) There are nonnegative constants $a_4 \geqslant 0$, $b_4 \geqslant 0$, $r > 0$, such that $d_4 := a_4(3 - e)(e - 2)/6e^2 + b_4 > 1$, and

$$f(t, x) \geqslant a_4 (x_1 + 2x_2) + b_4 (x_3 + 2x_4) \tag{38}$$

holds for all $(t, x) \in [0, 1] \times I_r^4$.

(H7) There is a constant $\omega > 0$ such that $f(t, x)$ is increasing on I_ω^4 in x, and

$$\int_0^1 f(s, \omega, \omega, \omega, \omega) \, \mathrm{d}s < \omega. \tag{39}$$

Remark 9. $f(t, x)$ is said to be increasing in x if $f(t, x) \leqslant f(t, y)$ holds for every pair $x, y \in \mathbb{R}_+^4$ with $x \leqslant y$ and for all $t \in [0, 1]$, where the partial ordering in $\leqslant \mathbb{R}^4$ is understood componentwise.

We denote $\Omega_\rho := \{v \in E : \|v\| < \rho\}$ for $\rho > 0$ in the sequel.

Theorem 10. *If (H1)–(H4) hold, then* (1) *has at least one positive solution.*

Proof. Let

$$\mathscr{M}_1 := \{v \in P : v = Av + \mu\varphi, \ \mu \geqslant 0\}, \tag{40}$$

where $\varphi(t) := (1 - t)e^{t-1} \in P$. We are now going to prove that $\mathscr{M}_1$ is bounded. Indeed, if $v \in \mathscr{M}_1$, then there exist $\mu \geqslant 0$ such that $v = Av + \mu\varphi$, which can be written in the form

$$\begin{aligned} &v(t) \\ &= \int_0^1 k_2(t, s) f\left(s, (B_1 v)(s), (B_1 v)'(s), v(s), -v'(s)\right) \mathrm{d}s \\ &\quad + \mu\varphi(t), \end{aligned} \tag{41}$$

which is equivalent to

$$\begin{aligned} -v''(t) = f&\left((t), (B_1 v)(t), (B_1 v)'(t), v(t), -v'(t)\right) \\ &+ \mu(1 + t)e^{t-1}. \end{aligned} \tag{42}$$

By (H2), we have

$$\begin{aligned} -v''(t) \geqslant a_1 &\left((B_1 v)(t) + 2(B_1 v)'(t)\right) \\ &+ b_1 \left(v(t) - 2v'(t)\right) - c. \end{aligned} \tag{43}$$

Note (27) and (29). Multiply the above by $(1-t)e^{1-t}$, integrate over $[0,1]$, and use Lemmas 3, 5, and 6 to obtain

$$\begin{aligned}
&\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\geqslant a_1\int_0^1 \left((B_1v)(t)+2(B_1v)'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\qquad + b_1\int_0^1 \left(v(t)2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t - c \\
&\quad\geqslant \frac{a_1(3-e)}{6}\|v\|_0 \\
&\qquad + b_1\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t - c \\
&\quad\geqslant \frac{a_1(3-e)}{6e}\int_0^1 v(t)(1-t)e^{1-t}\mathrm{d}t \\
&\qquad + b_1\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t - c \\
&\quad\geqslant d_1\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t - c,
\end{aligned} \tag{44}$$

so that

$$\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \leqslant m, \quad \forall v\in\mathscr{M}_1. \tag{45}$$

Now Lemma 4 implies

$$\|v\|_0 = v(0) \leqslant m, \quad \forall v\in\mathscr{M}_1, \tag{46}$$

where m is given in (H3). Furthermore, this estimate leads to

$$\begin{aligned}
\|(B_1v)(t)\|_0 &= (B_1v)(1) = \int_0^1 sv(s)\,\mathrm{d}s \leqslant m, \\
\left\|(B_1v)'(t)\right\|_0 &= (B_1v)'(0) = \int_0^1 v(s)\,\mathrm{d}s \leqslant m
\end{aligned} \tag{47}$$

for all $v\in\mathscr{M}_1$. Let

$$\begin{aligned}
\Lambda := \{\mu \geqslant 0 : &\text{ there exist some } v\in P \\
&\text{ such that } v = Av+\mu\varphi\}.
\end{aligned} \tag{48}$$

Now (46) implies that $\mu \leqslant me$ for all $\mu\in\Lambda$. By (H3), there is a function $\Phi\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$\begin{aligned}
&f\left(t,(B_1v)(t),(B_1v)'(t),v(t),-v'(t)\right) \\
&\quad\leqslant \Phi\left(-v'(t)\right), \quad \forall v\in\mathscr{M}_1.
\end{aligned} \tag{49}$$

Thus

$$\begin{aligned}
-v''(t) &= f\left(t,(B_1v)(t),(B_1v)'(t),v(t),-v'(t)\right) \\
&\quad + \mu(1+t)e^{t-1} \\
&\leqslant \Phi\left(-v'(t)\right) + \mu(1+t)e^{t-1} \\
&\leqslant \Phi\left(-v'(t)\right) + 2me, \quad \forall v\in\mathscr{M}_1,\ \mu\in\Lambda, \\
&\int_0^{-v'(1)} \frac{\xi\mathrm{d}\xi}{\Phi(\xi)+2me} \\
&\quad\leqslant \int_0^1 -v'(t)\,\mathrm{d}t = v(0) \leqslant m, \quad \forall v\in\mathscr{M}_1.
\end{aligned} \tag{50}$$

Combining the last inequality with (35), we see that there exists $M>0$ such that $-v(1)\leqslant M$ for all $v\in\mathscr{M}_1$. This means that $\mathscr{M}_1$ is bounded. Taking $R > \sup\{\|v\| : v\in\mathscr{M}_1\}$, we have

$$v \neq Av+\mu\varphi, \quad \forall v\in\Omega_R\cap P,\ \mu\geq 0. \tag{51}$$

Now Lemma 7 yields

$$i\left(A,\Omega_R\cap P,P\right) = 0. \tag{52}$$

Let

$$\mathscr{M}_2 := \left\{v\in\overline{\Omega}_r\cap P : v = \lambda Av,\ \lambda\in[0,1]\right\}. \tag{53}$$

Now we want to prove that $\mathscr{M}_2 = \{0\}$. In fact, if $v\in\mathscr{M}_2$, then $v\in\overline{\Omega}_r\cap P$ and there is $\lambda\in[0,1]$ such that $v=\lambda Av$, that is,

$$\begin{aligned}
v(t) &= \lambda\int_0^1 k_2(t,s) \\
&\quad\times f\left(s,(B_1v)(s),(B_1v)'(s),v(s),-v'(s)\right)\mathrm{d}s,
\end{aligned} \tag{54}$$

which can be written in the form

$$-v''(t) = \lambda f\left(t,(B_1v)(t),(B_1v)'(t),v(t),-v'(t)\right). \tag{55}$$

By (H4), we have

$$-v''(t) \leqslant a_2\left((B_1v)(t)+2(B_1v)'(t)\right) + b_2\left(v(t)-2v'(t)\right). \tag{56}$$

Note (27) and (29). Multiply (56) by $(1-t)e^{1-t}$, integrate over $[0,1]$, and use Lemmas 5 and 6 to obtain

$$\begin{aligned}
&\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\leqslant a_2\int_0^1 \left((B_1v)(t)+2(B_1v)'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\qquad + b_2\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\leqslant 3ea_2\|v\|_0 + b_2\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\leqslant \frac{3ea_2}{e-2}\int_0^1 v(t)(1-t)e^{1-t}\mathrm{d}t \\
&\qquad + b_2\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\leqslant d_2\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t,
\end{aligned} \tag{57}$$

so that $\int_0^1(v(t)-2v'(t))(1-t)e^{1-t}=0$, whence $v(t)\equiv 0$ and $\mathscr{M}_2=\{0\}$, as required. A consequence of this is

$$v\neq\lambda Av,\quad \forall v\in\partial\Omega_r\cap P,\ \lambda\in[0,1]. \tag{58}$$

Now Lemma 8 yields

$$i(A,\Omega_r\cap P,P)=1. \tag{59}$$

This together with (52) implies

$$i\left(A,\left(\Omega_R\setminus\overline{\Omega}_r\right)\cap P,P\right)=0-1=-1. \tag{60}$$

Therefore A has at least one fixed point on $(\Omega_R\setminus\overline{\Omega}_r)\cap P$ and thus (1) has at least one positive solution. This completes the proof. □

Theorem 11. *If (H1), (H5), and (H6) hold, then* (1) *has at least one positive solution.*

Proof. Let

$$\mathscr{M}_3 := \{v\in P: v=\lambda Av,\ \lambda\in[0,1]\}. \tag{61}$$

We now assert that $\mathscr{M}_3$ is bounded. Indeed, if $v\in\mathscr{M}_3$, then there is $\lambda\in[0,1]$ such that $v=\lambda Av$, which is equivalent to

$$-v''(t)=\lambda f\left(t,(B_1v)(t),(B_1)'(t),v(t),-v'(t)\right). \tag{62}$$

Now (H5) implies

$$\begin{aligned}
-v''(t)&\leqslant a_3\left((B_1v)(t)+2(B_1v)'(t)\right)\\
&\qquad + b_3\left(v(t)-2v'(t)\right)+c.
\end{aligned} \tag{63}$$

Note (27) and (29). Multiply the above by $(1-t)e^{1-t}$, integrate over $[0,1]$, and use Lemmas 5 and 6 to obtain

$$\begin{aligned}
&\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\quad\leqslant a_3\int_0^1 \left((B_1v)(t)+2(B_1v)'(t)\right)(1-t)e^{1-t}\mathrm{d}t \\
&\qquad + b_3\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t+c \\
&\quad\leqslant 3ea_3\|v\|_0 + b_3\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t+c \\
&\quad\leqslant d_3\int_0^1 \left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t+c,
\end{aligned} \tag{64}$$

so that $\int_0^1(v(t)-2v'(t))(1-t)e^{1-t}\mathrm{d}t\leqslant c_1$ for all $v\in\mathscr{M}_3$, where $c_1:=c/(1-d_3)>0$. By Lemma 4, we obtain

$$\begin{aligned}
\|v\|_0=v(0)&\leqslant\int_0^1\left(v(t)-2v'(t)\right)(1-t)e^{1-t}\mathrm{d}t\\
&\leqslant c_1,\quad \forall v\in\mathscr{M}_3.
\end{aligned} \tag{65}$$

Furthermore, this estimate leads to

$$\begin{aligned}
\|B_1v\|_0&=(B_1v)(1)=\int_0^1 sv(s)\,\mathrm{d}s\leqslant c_1,\\
\left\|(B_1v)'\right\|_0&=(B_1v)'(0)=\int_0^1 v(s)\,\mathrm{d}s\leqslant c_1
\end{aligned} \tag{66}$$

for all $v\in\mathscr{M}_3$. By (63), we have

$$-v''(t)\leqslant(3a_3+b_3)c_1-2b_3v'(t)+c,\quad\forall v\in\mathscr{M}_3. \tag{67}$$

Note $v(1)=v'(0)=0$ and $v''\leqslant 0$. Integrating the above over $[0,1]$, we obtain

$$\begin{aligned}
\left\|v'\right\|_0&=-v'(1)\leqslant(3a_3+b_3)c_1+2b_3v(0)+c\\
&\leqslant 3(a_3+b_3)c_1+c.
\end{aligned} \tag{68}$$

This proves the boundedness of $\mathscr{M}_3$. Taking $R>\sup\{\|v\|: v\in\mathscr{M}_3\}$, we have

$$v\neq\lambda Av,\quad\forall v\in\partial\Omega_R\cap P,\ \lambda\in[0,1]. \tag{69}$$

Now Lemma 8 yields

$$i(A,\Omega_R\cap P,P)=1. \tag{70}$$

On the other hand, let

$$\mathscr{M}_4:=\left\{v\in\overline{\Omega}_r\cap P: v=Av+\lambda\varphi,\ \lambda\geqslant 0\right\}, \tag{71}$$

where $\varphi\in P\setminus\{0\}$ is given by $\varphi(t):=(1-t)e^{t-1}$. Now we want to prove that $\mathscr{M}_4\subset\{0\}$. Indeed, if $v\in\mathscr{M}_4$, then there exists $\lambda\geqslant 0$ such that $v=Av+\lambda\varphi$, which can be written as

$$\begin{aligned}
-v''(t)&=f\left(t,(B_1v)(t),(B_1v)'(t),v(t),-v'(t)\right)\\
&\qquad+\lambda(t+1)e^{t-1}.
\end{aligned} \tag{72}$$

By (H6), we have

$$-v''(t)\,a_4 \geqslant \left(\left(B_1 v\right)(t) + 2\left(B_1 v\right)'(t)\right) + b_4\left(v(t) - 2v'(t)\right). \tag{73}$$

Note (27) and (29). Multiply the above by $(1-t)e^{1-t}$, integrate over $[0,1]$, and use Lemmas 3, 5, and 6 to obtain

$$\begin{aligned}
&\int_0^1 \left(v(t) - 2v'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t \\
&\quad\geqslant a_4 \int_0^1 \left(\left(B_1 v\right)(t) + 2\left(B_1 v\right)'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t \\
&\qquad + b_4 \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t, \\
&\quad\geqslant \frac{a_4(3-e)}{6e}\|v\|_0 + b_4 \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t, \\
&\quad\geqslant \frac{a_4(3-e)}{6e}\int_0^1 v(t)(1-t)\,e^{1-t}\mathrm{d}t \\
&\qquad + b_4 \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t, \\
&\quad\geqslant d_4 \int_0^1 \left(v(t) - 2v'(t)\right)(1-t)\,e^{1-t}\mathrm{d}t,
\end{aligned} \tag{74}$$

so that $\int_0^1 (v(t)-2v'(t))(1-t)e^{1-t}\mathrm{d}t = 0$, whence $v(t) \equiv 0$. This proves $\mathscr{M}_4 \subset \{0\}$, as required. As a result of this, we obtain

$$v \neq Av + \lambda\varphi, \quad \forall v \in \partial\Omega_r \cap P,\ \lambda \geqslant 0. \tag{75}$$

Lemma 7 implies

$$i\left(A, \Omega_r \cap P, P\right) = 0. \tag{76}$$

Now (70) and (76) combined lead to

$$i\left(A, \left(\Omega_R \setminus \overline{\Omega}_r\right) \cap P, P\right) = 1 - 0 = 1. \tag{77}$$

Consequently, A has at least one fixed point on $(\Omega_R \setminus \overline{\Omega}_r) \cap P$. Thus (1) has at least one positive solution. This completes the proof. □

Theorem 12. *If (H1), (H2), (H3), (H6), and (H7) hold, then* (1) *has at least two positive solutions.*

Proof. If $v \in P$ and $\|v\| \leqslant \omega$, then we have

$$\begin{aligned}
&\|B_1 v\|_0 = \left(B_1 v\right)(1) = \int_0^1 s v(s)\,\mathrm{d}s \leqslant \omega, \left\|\left(B_1 v\right)'\right\|_0 = \left(B_1 v\right)(0) \\
&\quad = \int_0^1 v(s)\,\mathrm{d}s \leqslant \omega.
\end{aligned} \tag{78}$$

By (H7), we have

$$f(t, x) \leqslant f(t, \omega, \omega, \omega, \omega), \quad \forall t \in [0,1],\ x \in I_\omega^4. \tag{79}$$

Consequently, for all $v \in \partial\Omega_\omega \cap P$, we have

$$\begin{aligned}
\|Av\|_0 &= (Av)(0) \\
&= \int_0^1 (1-s)\, f\left(s, \left(B_1 v\right)(s), \left(B_1 v\right)'(s)\, v(s), -v'(s)\right)\mathrm{d}s \\
&\leqslant \int_0^1 f(s, \omega, \omega, \omega, \omega)\,\mathrm{d}s < \omega = \|v\|,
\end{aligned} \tag{80}$$

$$\begin{aligned}
&\left\|(Av)'\right\|_0 \\
&\quad = -(Av)'(1) \\
&\quad = \int_0^1 f\left(s, \left(B_1 v\right)(s), \left(B_1 v\right)'(s), v(s), -v'(s)\right)\mathrm{d}s \\
&\quad \leqslant \int_0^1 f(s, \omega, \omega, \omega, \omega)\,\mathrm{d}s < \omega = \|v\|.
\end{aligned} \tag{81}$$

The preceding inequalities imply

$$\|Av\| < \omega = \|v\|, \quad \forall v \in \partial\Omega_\omega \cap P \tag{82}$$

and thus

$$v \neq \lambda Av, \quad \forall v \in \partial\Omega_\omega \cap P,\ 0 \leqslant \lambda \leqslant 1. \tag{83}$$

Now Lemma 5 yields

$$i\left(A, \Omega_\omega \cap P, P\right) = 1. \tag{84}$$

By (H2), (H3), and (H6), we know that (52) and (76) hold (see the proofs of Theorems 10 and 11). Note that we can choose $R > \omega > r$ in (52) and (76). Combining (52), (76), and (84), we obtain

$$\begin{aligned}
&i\left(A, \left(\Omega_R \setminus \overline{\Omega}_\omega\right) \cap P, P\right) = 0 - 1 = -1, \\
&i\left(A, \left(\Omega_\omega \setminus \overline{\Omega}_r\right) \cap P, P\right) = 1 - 0 = 1.
\end{aligned} \tag{85}$$

Therefore, A has at least two fixed points, with one on $(\Omega_R \setminus \overline{\Omega}_\omega) \cap P$ and the other on $(\Omega_\omega \setminus \overline{\Omega}_r) \cap P$. Hence (1) has at least two positive solutions. This completes the proof. □

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