Research Article

Energy- and Regularity-Dependent Stability Estimates for Near-Field Inverse Scattering in Multidimensions

M. I. Isaev

Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France

Correspondence should be addressed to M. I. Isaev; isaev.m.i@gmail.com

Received 20 November 2012; Revised 17 December 2012; Accepted 17 December 2012

Abstract We prove new global Hölder-logarithmic stability estimates for the near-field inverse scattering problem in dimension \(d \geq 3\). Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In addition, a global logarithmic stability estimate for this inverse problem in dimension \(d = 2\) is also given.

1. Introduction

We consider the Schrödinger equation:

\[ L\psi = E\psi, \quad L = -\Delta + \nu(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \tag{1} \]

where

\[ \nu \text{ is real-valued, } \quad \nu \in \mathcal{L}^\infty(\mathbb{R}^d), \]

\[ \nu(x) = O(|x|^{-d-\varepsilon}), \quad |x| \to \infty, \quad \text{for some } \varepsilon > 0. \tag{2} \]

We consider the resolvent \(R(E)\) of the Schrödinger operator \(L\) in \(\mathcal{L}^2(\mathbb{R}^d)\):

\[ R(E) = (L - E)^{-1}, \quad E \in \mathbb{C} \setminus \sigma(L), \tag{3} \]

where \(\sigma(L)\) is the spectrum of \(L\) in \(\mathcal{L}^2(\mathbb{R}^d)\). We assume that \(R(x, y, E)\) denotes the Schwartz kernel of \(R(E)\) as of an integral operator. We consider also

\[ R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \quad E \in \mathbb{R}_+, \tag{4} \]

We recall that in the framework of (1) the function \(R^+(x, y, E)\) describes scattering of the spherical waves,

\[ R_0^+(x, y, E) = -i \left( \frac{\sqrt{E}}{2\pi|x-y|} \right)^{(d-2)/2} H_{(d-1)/2}^{(1)}(\sqrt{E}|x-y|), \tag{5} \]

generated by a source at \(y\) (where \(H_{\mu}^{(1)}\) is the Hankel function of the first kind of order \(\mu\)). We recall also that \(R^+(x, y, E)\) is the Green function for \(L - E, E \in \mathbb{R}_+, \) with the Sommerfeld radiation condition at infinity.

In addition, the function

\[ S^+(x, y, E) = R^+(x, y, E) - R_0^+(x, y, E), \quad x, y \in \partial B_r, \quad E \in \mathbb{R}_+, \quad r \in \mathbb{R}_+ \tag{6} \]

is considered as near-field scattering data for (1), where \(B_r\) is the open ball of radius \(r\) centered at 0.

We consider, in particular, the following near-field inverse scattering problem for (1).

Problem 1. Given \(S^+\) on \(\partial B_r \times \partial B_r\), for some fixed \(r, E \in \mathbb{R}_+, \) find \(\nu\) on \(B_r\).

This problem can be considered under the assumption that \(\nu\) is a priori known on \(\mathbb{R}^d \setminus B_r\). Actually, in the present paper we consider Problem 1 under the assumption that \(\nu \equiv 0\) on \(\mathbb{R}^d \setminus B_r\), for some fixed \(r \in \mathbb{R}_+.\) Below in this paper we always assume that this additional condition is fulfilled.

It is well known that the near-field scattering data of Problem 1 uniquely and efficiently determine the scattering amplitude \(f\) for (1) at fixed energy \(E\), see [1]. Therefore, approaches of [2–12] can be applied to Problem 1 via this reduction.
In addition, it is also known that the near-field data of Problem 1 uniquely determine the Dirichlet-to-Neumann map in the case when $E$ is not a Dirichlet eigenvalue for operator $L$ in $B_r$, see [8, 13]. Therefore, approaches of [3, 8, 14–24] can be also applied to Problem 1 via this reduction.

However, in some case it is much more optimal to deal with Problem 1 directly, see, for example, logarithmic stability results of [25] for Problem 1 in dimension $d = 3$. A principal improvement of estimates of [25] was given recently in [26]: stability of [26] efficiently increases with increasing regularity of $v$.

Problem 1 can be also considered as an example of ill-posed problem: see [27, 28] for an introduction to this theory.

In the present paper we continue studies of [25, 26]. We give new global Hölder-logarithmic stability estimates for Problem 1 in dimension $d \geq 3$, see Theorem 1. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Results of such a type for the Gel’fand inverse problem were obtained recently in [15] for $d = 3$.

In addition, we give also global logarithmic stability estimates for Problem 1 in dimension $d = 2$, see Theorem 2.

2. Stability Estimates

We recall that if $v$ satisfies (2) and $\text{supp} \, v \in B_{r_1}$ for some $r_1 > 0$, then

$$ S^+ (E) \text{ is bounded in } L^2 (\partial B_r \times \partial B_r) \text{ for any } r > r_1, $$

where $S^+ (E)$ is the near-field scattering data of $v$ for (1) with $E > 0$, for more details see, for example, Section 2 of [25].

2.1. Estimates for $d \geq 3$. In this subsection we assume for simplicity that

$$ v \in W^{m,1} (\mathbb{R}^d) \text{ for some } m > d, \quad \text{and } v \text{ is real-valued}, $$

supp $v \subset B_{r_1}$ for some $r_1 > 0$, where

$$ W^{m,1} (\mathbb{R}^d) = \{ v: \partial^J v \in L^1 (\mathbb{R}^d), \, |J| \leq m \}, \quad m \in \mathbb{N} \cup 0, $$

where

$$ J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^{d} J_i, $$

$$ \partial^J v (x) = \frac{\partial^{\lfloor J \rfloor} v (x)}{\partial x_1^{J_1} \cdots \partial x_d^{J_d}}. $$

Let

$$ \|v\|_{m,1} = \max_{|J| \leq m} \| \partial^J v \|_{L^1 (\mathbb{R}^d)}. $$

Note that (8)$\Rightarrow$(2).

**Theorem 1.** Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d \geq 3$ and potentials $v_1, v_2$ satisfy (8). Let $\|v\|_{m,1} \leq N$, for some $N > 0$. Let $S^+_1 (E)$ and $S^+_2 (E)$ denote the near-field scattering data for $v_1$ and $v_2$, respectively. Then for $r \in (0,1)$ and any $s \in [0, s^*)$ the following estimate holds:

$$ \|v_2 - v_1\|_{L^\infty (B_r)} \leq C_1 (1 + E)^{s/2} + C_2 (1 + E)^{(s-s^*)/2} (\ln (3 + \delta^{-1}))^{-s}, $$

where $s^* = (m - d)/d$, $\delta = \|S^+_1 (E) - S^+_2 (E)\|_{L^1 (\partial B_{r_1} \times \partial B_{r_1})}$ and constants $C_1, C_2 > 0$ depend only on $N, m, d, r, r'$.

Proof of Theorem 1 is given in Section 5. This proof is based on results presented in Sections 3 and 6.

2.2. Estimates for $d=2$. In this subsection we assume for simplicity that

$$ v \text{ is real-valued, } \quad v \in C^2 (\overline{B_{r_1}}), $$

supp $v \subset B_{r_1}$ for some $r_1 > 0$. Note also that (13)$\Rightarrow$(2).

**Theorem 2.** Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d = 2$ and potentials $v_1, v_2$ satisfy (13). Let $\|v\|_{C^2 (B_r)} \leq N$, for some $N > 0$. Let $S^+_1 (E)$ and $S^+_2 (E)$ denote the near-field scattering data for $v_1$ and $v_2$, respectively. Then

$$ \|v_1 - v_2\|_{L^\infty (B_r)} \leq C_3 (\ln (3 + \delta^{-1}))^{-3/4} (\ln (3 \ln (3 + \delta^{-1})))^{1/4}, $$

where $\delta = \|S^+_1 (E) - S^+_2 (E)\|_{L^1 (\partial B_{r_1} \times \partial B_{r_1})}$ and constant $C_3 > 0$ depends only on $N, m, r$.

Proof of Theorem 2 is given in Section 7. This proof is based on results presented in Sections 3 and 6.

2.3. Concluding Remarks

**Remark 3.** The logarithmic stability estimates for Problem 1.1 of [25, 26] follow from estimate (12) for $d = 3$ and $s = s^*$. Apparently, using the methods of [19, 20] it is possible to improve estimate (12) for $s^* = m - d$.

**Remark 4.** In the same way as in [25, 26] for dimension $d = 3$, using estimates (12) and (14), one can obtain logarithmic stability estimates for the reconstruction of a potential $v$ from the inverse scattering amplitude $f$ for any $d \geq 2$. 
Remark 5. Actually, in the proof of Theorem 1 we obtain the following estimate (see formula (57)):

\[
\|v_1 - v_2\|_{L^\infty(B_r)} \leq C_4 (1 + E)^2 \sqrt{E + \rho^2} e^{2\rho(r+1)} \delta + C_5 (E + \rho^2)^{(m-d)/2d},
\]

where constants \(C_4, C_5 > 0\) depend only on \(N, m, d, r\) and the parameter \(\rho > 0\) is such that \(E + \rho^2\) is sufficiently large:

\[
E + \rho^2 \geq C_6 (N, r, m).
\]

Estimate of Theorem 1 follows from estimate (15).

3. Alessandrini-Type Identity for Near-Field Scattering

In this section we always assume that assumptions of Theorems 1 and 2 are fulfilled (in the cases of dimension \(d \geq 3\) and \(d = 2\), resp.).

Consider the operators \(\tilde{R}_j, j = 1, 2\), defined as follows

\[
(\tilde{R}_j \phi)(x) = \int_{\partial B_r} R_j^+(x, y, E) \phi(y) \, dy, \quad x \in \partial B_r, \quad j = 1, 2.
\]

(16)

Note that

\[
\|\tilde{R}_1 - \tilde{R}_2\|_{L^2(\partial B_r)} \leq \|S_1^+(E) - S_2^+(E)\|_{L^2(\partial B_r) \times L^2(\partial B_r)}.
\]

(17)

We recall that (see [25]) for any functions \(\phi_1, \phi_2 \in C(\mathbb{R}^d)\), sufficiently regular in \(\mathbb{R}^d \setminus \partial B_r\), and satisfying

\[
-\Delta \phi + v(x) \phi = E \phi, \quad \text{in } \mathbb{R}^d \setminus \partial B_r,
\]

(18)

\[
\lim_{|x| \to \infty} |x|^{(d-1)/2} \left( \frac{\partial}{\partial |x|} \phi - i \sqrt{E} \phi \right) = 0,
\]

with \(v = v_1\) and \(v = v_2\), respectively, the following identity holds:

\[
\int_{\partial B_r} (v_2 - v_1) \phi_1 \phi_2 \, dx
\]

\[
= \int_{\partial B_r} \left( \frac{\partial \phi_1}{\partial v_+} - \frac{\partial \phi_1}{\partial v_-} \right) \left( \frac{\partial \phi_2}{\partial v_+} - \frac{\partial \phi_2}{\partial v_-} \right) \, dx,
\]

(19)

where \(v_+\) and \(v_-\) are the outward and inward normals to \(\partial B_r\), respectively.

Remark 6. The identity (19) is similar to the Alessandrini identity (see Lemma 1 of [14]), where the Dirichlet-to-Neumann maps are considered instead of operators \(\tilde{R}_j\).

To apply identity (19) to our considerations, we use also the following lemma.

Lemma 7. Let \(E, r > 0\) and \(d \geq 2\). Then, there is a positive constant \(C_7\) (depending only on \(r\) and \(d\)) such that for any \(\phi \in C(\mathbb{R}^d \setminus B_r)\) satisfying

\[
-\Delta \phi = E \phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B_r},
\]

\[
\lim_{|x| \to \infty} |x|^{(d-1)/2} \left( \frac{\partial}{\partial |x|} \phi - i \sqrt{E} \phi \right) = 0,
\]

(20)

the following inequality holds:

\[
\left\| \frac{\partial \phi}{\partial v_+} \right\|_{L^2(\partial B_r)} \leq C_7 (1 + E) \left\| \phi \right\|_{H^1(\partial B_r)},
\]

(21)

where \(H^1(\partial B_r)\) denotes the standard Sobolev space on \(\partial B_r\).

The proof of Lemma 7 is given in Section 8.

4. Faddeev Functions

In dimension \(d \geq 3\), we consider the Faddeev functions \(h, \psi, G\) (see [6, 8, 30, 31]):

\[
h(k, l) = (2\pi)^d \int_{\mathbb{R}^d} e^{-ik \cdot x} \psi(x, k) \, dx,
\]

(22)

where \(k, l \in \mathbb{C}^d, k^2 = k^2, 1 \mathrm{m} k = 1 \mathrm{m} l \neq 0, \psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) \, dy,
\]

(23)

\[
G(x, k) = e^{ikx} g(x, k),
\]

(24)

\[
g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} e^{\xi_k x} d\xi,
\]

where \(x \in \mathbb{R}^d, k \in \mathbb{C}^d, 1 \mathrm{m} k \neq 0, d \geq 3, \mathrm{One\ can\ consider\ (22),\ (23)\ assuming\ that\ \psi\ is\ a\ sufficiently\ regular\ function\ on\ \mathbb{R}^d\ with\ sufficient\ decay\ at\ infinity.\}

(25)

For example, in connection with Theorem 1, we consider (22), (23) assuming that

\[
v \in L^\infty(B_r), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus B_r.
\]

(26)

We recall that (see [6, 8, 30, 31])

\[
(\Delta + k^2) G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, k \in \mathbb{C}^d \setminus \mathbb{R}^d,
\]

(27)

formula (23) at fixed \(k\) is considered as an equation for

\[
\psi = e^{ikx} \mu(x, k),
\]

(28)

where \(\mu\) is sought in \(L^\infty(\mathbb{R}^d)\); as a corollary of (23), (24), and (27), \(\psi\) satisfies (1) for \(E = k^2; h\) of (22) is a generalized “scattering” amplitude.
In addition, \( h, \psi, G \) in their zero energy restriction, that is for \( E = 0 \) were considered for the first time in [32]. The Faddeev functions \( h, \psi, G \) were, actually, rediscovered in [32]. Let

\[
\Sigma_E = \{ k \in \mathbb{C}^d : k^2 = k_1^2 + \cdots + k_d^2 = E \},
\]

\[
\Theta_E = \{ k \in \Sigma_E, l \in \Sigma_E : |l| = 1 \},
\]

\[
|k| = (|\text{Re} \, k|^2 + |\text{Im} \, k|^2)^{1/2}.
\]

Let

\[
\nu \text{ satisfy } (8), \quad \| \nu \|_{m,1} \leq N, \quad \nu(l) = \frac{(2\pi)^{-d}}{E} \int_{\mathbb{R}^d} e^{ipx} \xi(x) \, dx, \quad p \in \mathbb{R}^d,
\]

then we have that

\[
\mu(x,k) \rightarrow 1 \text{ as } |k| \rightarrow \infty \quad (31)
\]

and, for any \( \sigma > 1 \),

\[
|\mu(x,k)| + |\nabla \mu(x,k)| \leq \sigma \quad \text{for } |k| \geq \lambda_1 (N,m,d,r,\sigma), \quad (32)
\]

where \( x \in \mathbb{R}^d, k \in \Sigma_E; \)

\[
|\tilde{\nu}(p) - h(k,l)| \leq \frac{c_1 (m,d,r) N^2}{(E + \rho^2)^{1/2}}, \quad \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad (33)
\]

\[
|\text{Im} \, k| = |\text{Im} \, l| = \rho, \quad E + \rho^2 \geq \lambda_2 (N,m,d,r), \quad \rho \leq 4 \left( E + \rho^2 \right). \quad (34)
\]

Results of the type (31), (32) go back to [32]. For more information concerning (32) see estimate (4.11) of [33]. Results of the type (33), (34) (with less precise right-hand side in (34)) go back to [6]. Estimate (34) follows, for example, from formulas (23), (22), and the estimate

\[
\| \Lambda^{-s} g(k) \Lambda^{-s} \|_{L^2(\mathbb{R}^d)} \rightarrow L^2(\mathbb{R}^d) \quad (35)
\]

for \( s > 1/2 \), where \( g(k) \) denotes the integral operator with the Schwartz kernel \( g(x-y, k) \) and \( \Lambda \) denotes the multiplication operator by the function \((1 + |x|^2)^{1/2}\). Estimate (35) was formulated, first, in [34] for \( d \geq 3 \). Concerning proof of (35), see [35].

In addition, we have that

\[
\begin{align*}
& h_2 (k,l) - h_1 (k,l) \\
& \quad = (2\pi)^{-d} \int_{\mathbb{R}^d} \psi_1 (x-l) (v_2 (x) - v_1 (x)) \psi_2 (x, k) \, dx
\end{align*}
\]

for \( (k,l) \in \Theta_E \), \( |l| = |l| \neq 0 \), and \( v_1, v_2 \) satisfying (25), (36)

and, under assumptions of Theorem 1,

\[
\begin{align*}
& \| \tilde{v}_1 (p) - \tilde{v}_2 (p) - h_1 (k,l) + h_2 (k,l) \|_{L^\infty(\mathbb{R}^d)} \\
& \quad \leq c_2 (m,d,r) N \| v_1 - v_2 \|_{L^\infty(\mathbb{R}^d)}
\end{align*}
\]

for \( (k,l) \in \Theta_E \), \( p = k - l \), \( |l| = |l| = \rho \), \( E + \rho^2 \geq \lambda_3 (N,m,d,r), \quad p^2 \leq 4 \left( E + \rho^2 \right), \)

where \( h_j, \psi_j \) denote \( h \) and \( \psi \) of (22) and (23) for \( \nu \) satisfying (25), (36)

and, under assumption of Theorem 1,

\[
\begin{align*}
& \| v_1 - v_2 \|_{L^\infty(B_N)} \\
& \quad \leq \| v_1 - v_2 \|_{L^\infty(\mathbb{R}^d)} \| w_1 - w_2 \|_{L^\infty(\mathbb{R}^d)}
\end{align*}
\]

for \( (k,l) \in \Theta_E \), \( p = k - l \), \( |l| = |l| = \rho \), \( E + \rho^2 \geq \lambda_4 (N,m,d,r), \quad p^2 \leq 4 \left( E + \rho^2 \right), \)

where \( h_j, \psi_j \) denote \( h \) and \( \psi \) of (22) and (23) for \( v = v_j, j = 1, 2 \).

Formula (36) was given in [36]. Estimate (37) was given for example in [15].

5. Proof of Theorem 1

Let

\[
\mathbb{L}^\infty_\nu (\mathbb{R}^d) = \{ u \in \mathbb{L}^\infty (\mathbb{R}^d) : \| u \|_\nu < +\infty \},
\]

\[
\| u \|_\nu = \text{ess sup}_{p \in \mathbb{R}^d} (1 + |p|^2) |u(p)|, \quad \mu > 0. \quad (38)
\]

Note that

\[
w \in \mathbb{W}^{m,1} (\mathbb{R}^d) \quad \Longrightarrow \quad \tilde{w} \in \mathbb{L}^\infty_\nu (\mathbb{R}^d) \cap C(\mathbb{R}^d), \quad \| \tilde{w} \|_\nu \leq c_3 (m,d) \| w \|_{m,1}, \quad \text{for } \mu = m, \quad (39)
\]

where \( \mathbb{W}^{m,1} \), \( \mathbb{L}^\infty_\nu \) are the spaces of (9), (38),

\[
\tilde{w}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} w(x) \, dx, \quad p \in \mathbb{R}^d. \quad (40)
\]

Using the inverse Fourier transform formula

\[
w(x) = \int_{\mathbb{R}^d} \frac{e^{-ipx}}{2\pi} \tilde{w}(p) \, dp, \quad x \in \mathbb{R}^d, \quad (41)
\]

we have that

\[
\| v_1 - v_2 \|_{L^\infty(B_N)} \leq \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ipx} (\tilde{v}_2 (p) - \tilde{v}_1 (p)) \, dp \right| \leq I_1 (x) + I_2 (x), \quad \text{for any } x > 0, \quad (42)
\]

where

\[
\begin{align*}
I_1 (x) &= \int_{|p| > x} |\tilde{v}_2 (p) - \tilde{v}_1 (p)| \, dp, \\
I_2 (x) &= \int_{|p| \leq x} |\tilde{v}_2 (p) - \tilde{v}_1 (p)| \, dp.
\end{align*}
\]

(43)
Using (39), we obtain that
\[ |\vec{v}_2(p) - \vec{v}_1(p)| \leq 2c_3(m,d)N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \]  
(44)

Let
\[ c_4 = \int_{\mathbb{R}^d \setminus |p| = 1} dp. \]  
(45)

Combining (43), (44), we find that, for any \( \kappa > 0 \),
\[ I_2(\kappa) \leq 2c_3(m,d)Nc_4 \int_{|t| = 1} dt \leq \frac{2c_3(m,d)Nc_4}{m - d} \]  
(46)

Due to (37), we have that
\[ |\vec{v}_2(p) - \vec{v}_1(p)| \leq |h_2(k,l) - h_1(k,l)| + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{L^\infty(B_r)}}{(E + \rho^2)^{1/2}} \]  
(47)

for \( (k,l) \in \Theta_E, \quad \rho = k - l, \quad |\text{Im } k| = |\text{Im } l| = \rho, \quad E + \rho^2 \geq \lambda_3(N,m,d,r), \quad \rho^2 \leq 4(E + \rho^2). \)

Let
\[ \delta = \|S^*_1(E) - S^*_1(E)\|_{L^2(\partial B_r \setminus \partial B_r)} \]  
(48)

Combining (17), (19), and (36), we get that
\[ |h_2(k,l) - h_1(k,l)| \leq \delta \|\frac{\partial \phi_j}{\partial v_+} - \frac{\partial \phi_j}{\partial v_-}\|_{L^2(B_r \setminus B_{r_0})}, \quad j = 1, 2, \]  
(49)

where \( \phi_j, j = 1, 2 \), denotes the solution of (18) with \( v = v_j \), satisfying
\[ \phi_j(x) = \psi_j(x,k), \quad x \in \overline{B}_r. \]  
(50)

Using (21), (32), and the fact that \( C^1(\partial B_r) \subset \mathcal{H}^1(\partial B_r) \), we find that
\[ \|\frac{\partial \phi_j}{\partial v_+} - \frac{\partial \phi_j}{\partial v_-}\|_{L^2(B_r \setminus B_{r_0})} \leq \sigma c_5(r,d)(1 + E) \exp(|\text{Im } k| (r + 1)), \]  
(51)

\( k \in \Sigma_E, \quad |k| \geq \lambda_1(N,m,d,r,s), \quad j = 1, 2. \)

Here and below in this section the constant \( \sigma \) is the same that in (32).

Combining (49) and (51), we obtain that
\[ |h_2(k,l) - h_1(k,l)| \leq c_3^2 \sigma^2(1 + E)(E + \rho^2)^{1/2} k \]  
(52)

for \( (k,l) \in \Theta_E, \quad \rho = |\text{Im } k| = |\text{Im } l|, \quad E + \rho^2 \geq \lambda_2^2(N,m,d,r,s). \)

Using (47), (52), we get that
\[ |\vec{v}_2(p) - \vec{v}_1(p)| \leq c_2(m,d,r) N \|v_1 - v_2\|_{L^\infty(B_r)} \]  
(53)

\[ + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{L^\infty(B_r)}}{(E + \rho^2)^{1/2}}, \quad p \in \mathbb{R}^d, \quad \rho^2 \leq 4(E + \rho^2), \quad E + \rho^2 \geq \lambda_2^2(N,m,d,r,s). \]

Let
\[ \epsilon = \frac{1}{2c_2(m,d,r) Nc_6}, \quad c_6 = \int_{\mathbb{R}^d \setminus |p| = 1} dp, \]  
(54)

and \( \lambda_4(N,m,d,r,s) > 0 \) be such that
\[ E + \rho^2 \geq \lambda_4(N,m,d,r,s), \quad \lambda_4(N,m,d,r,s) \]  
(55)

Using (43), (53), we get that
\[ I_1(\kappa) \leq c_6 \kappa^d \left(c_2^2 \sigma^2(1 + E)(E + \rho^2)^{1/2} k \right) \]  
(56)

\[ + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{L^\infty(B_r)}}{(E + \rho^2)^{1/2}} \]  
(57)

\( \kappa > 0, \quad \kappa^2 \leq 4(E + \rho^2), \)  
\[ E + \rho^2 \geq \lambda_4(N,m,d,r,s). \]

Combining (42), (46), and (56) for \( \kappa = \epsilon(E + \rho^2)^{-1/2} \) and (55), we get that
\[ \|v_1 - v_2\|_{L^\infty(B_r)} \leq c_7(N,m,d,r,s)(1 + E)^2 \sqrt{E + \rho^2} e^{2\rho^2 |r + 1|} \]  
(58)

\[ + c_8(N,m,d)(E + \rho^2)^{-|m-d|/2d} \]  
(59)

\[ \kappa^2 \leq 4(E + \rho^2), \quad E + \rho^2 \geq \lambda_4(N,m,d,r,s). \]

Let \( \beta = \frac{1 - r}{2(r + 1)}, \quad \rho = \beta \ln \left(3 + \delta^{-1}\right), \)
and \( \delta_1 = \delta_1(N, m, d, \sigma, r, \tau') > 0 \) be such that
\[
E + (\beta \ln(3 + \delta^{-1}))^2 \geq \lambda_4(N, m, d, r, \sigma),
\]
\[
E + (\beta \ln(3 + \delta^{-1}))^2 \leq (1 + E)(\beta \ln(3 + \delta^{-1}))^2.
\]
(59)

Then for the case when \( \delta \in (0, \delta_1) \), due to (57), we have that
\[
\frac{1}{2} \|v_1 - v_2\|_{L^\infty(B_r)}^2 \leq c_7(1 + E)^2 \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (3 + \delta^{-1})^{2(r+1)} \delta
\]
\[
+ c_9 \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-(m-d)/2d}
\]
\[
= c_7(1 + E)^2 \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (1 + 3\delta)^{1-r} \delta^r
\]
\[
+ c_9 \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-(m-d)/2d}.
\]
(60)

Combining (59) and (60), we obtain that for \( s \in [0, s^*] \), \( r \in (0, \tau) \) and \( \delta \in (0, \delta_1) \) the following estimate holds:
\[
\|v_2 - v_1\|_{L^\infty(B_r)} \leq c_4(1 + E)^{s^*/2} \delta^s
\]
\[
+ c_{10} (1 + E)^{(r-s)/2} (\ln(3 + \delta^{-1}))^{-s},
\]
(61)
where \( s^* = (m-d)/d \) and \( c_9, c_{10} > 0 \) depend only on \( N, m, d, r, \sigma, \tau \), and \( r \).

Estimate (61) in the general case (with modified \( c_9 \) and \( c_{10} \)) follows from (61) for \( \delta \leq \delta_1(N, m, d, \sigma, r, \tau') \) and the property that
\[
\|v_j\|_{L^\infty(B_r)} \leq c_{11}(m, d) N.
\]
(62)

This completes the proof of (12).

6. Buckhgeim-Type Analogs of the Faddeev Functions

Let us identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) and use coordinates \( z = x_1 + ix_2 \), \( \bar{z} = x_1 - ix_2 \), where \((x_1, x_2) \in \mathbb{R}^2\). Following [21–23, 37], we consider the functions \( G_{z_0}, \psi_{z_0}, \bar{\psi}_{z_0}, \delta h_{z_0} \) going back to Buckhgeim’s paper [3] and being analogs of the Faddeev functions:
\[
\psi_{z_0}(z, \lambda) = e^{i(z - z_0)^2},
\]
\[
\bar{\psi}_{z_0}(z, \lambda) = e^{i(z - z_0)^2},
\]
\[
G_{z_0}(z, \xi) = \int_{B_r} G_{z_0}(z, \xi, \lambda) v(\xi) \, d\xi d\lambda,
\]
\[
\bar{\psi}_{z_0}(z, \lambda) = \int_{B_r} \bar{G}_{z_0}(z, \xi, \lambda) v(\xi) \, d\xi d\lambda,
\]
(63)

where \( v \) satisfies (13),
\[
\delta h_{z_0}(\lambda) = \int_{B_r} \bar{G}_{z_0,1}(z, -\lambda) \left( \psi_{z_0}(z) - \psi_{z_0}(z) \right) \, d\lambda,
\]
(65)
where \( v_1, v_2 \) satisfy (13) and \( \bar{G}_{z_0,1}, \psi_{z_0,2} \) denote \( \bar{G}_{z_0}, \psi_{z_0} \) of (63) for \( v = v_1 \) and \( v = v_2 \), respectively.

We recall that (see [21, 22]):

(i) the function \( G_{z_0} \) satisfies the equations
\[
\frac{\partial^2}{\partial z \partial \bar{z}} G_{z_0}(z, \xi, \lambda) = \delta(z - \xi),
\]
\[
\frac{\partial^2}{\partial \xi \partial \bar{\xi}} G_{z_0}(z, \xi, \lambda) = \delta(z - \xi),
\]
(66)
where \( z, \xi, \lambda \in B_r, \lambda \in \mathbb{C} \), and \( \delta \) is the Dirac delta function;

(ii) formulas (63) at fixed \( z_0 \) and \( \lambda \) are considered as equations for \( \psi_{z_0}, \bar{\psi}_{z_0} \) in \( L^\infty(B_r) \);

(iii) as a corollary of (63), (64), (66), the functions \( \psi_{z_0}, \bar{\psi}_{z_0} \) satisfy (1) in \( B_r \) for \( E = 0 \) and \( d = 2 \);

(iv) the function \( \delta h_{z_0} \) is similar to the right side of (36).

Let potentials \( v, v_1, v_2 \in C^2(\overline{B_r}) \), and
\[
\|v\|_{C^2(\overline{B_r})} \leq N, \quad \|v_j\|_{C^2(\overline{B_r})} \leq N, \quad j = 1, 2,
\]
\[
(v_1 - v_2)_{\partial B_r} = 0, \quad \frac{\partial}{\partial n}(v_1 - v_2)_{\partial B_r} = 0,
\]
(67)
then we have that
\[ \psi_{z_0}(z, \lambda) = e^{(z-z_0)^2} \mu_{z_0}(z, \lambda), \]
\[ \tilde{\psi}_{z_0}(z, \lambda) = e^{\overline{(z-z_0)^2}} \tilde{\mu}_{z_0}(z, \lambda), \]
for \( |\lambda| \rightarrow 1, \quad \tilde{\mu}_{z_0}(z, \lambda) \rightarrow 1 \) as \( |\lambda| \rightarrow \infty \) (69)
and, for any \( \sigma > 1, \)
\[ |\mu_{z_0}(z, \lambda)| + |\nabla \mu_{z_0}(z, \lambda)| \leq \sigma, \quad \tilde{\mu}_{z_0}(z, \lambda) + |\nabla \tilde{\mu}_{z_0}(z, \lambda)| \leq \sigma, \] (70a)
(70b)
where \( \nabla = \left( \partial/\partial x_1, \partial/\partial x_2 \right), z = x_1 + i x_2, z_0 \in B_r, \lambda \in \mathbb{C}, \)
\[ |\lambda| \geq \rho_1(N_r, \sigma); \]
\[ v_2(z_0) - v_1(z_0) = \lim_{\lambda \to \infty} \frac{1}{2\pi} |\lambda| \partial h_{z_0}(\lambda) \quad \text{for any } z_0 \in B_r, \]
\[ |v_2(z_0) - v_1(z_0) - \frac{2}{\pi} |\lambda| \partial h_{z_0}(\lambda)| \leq \epsilon_{12}(N, r, \sigma) \left( \ln(3|\lambda|)^2/|\lambda|^{3/4} \right), \]
for \( z_0 \in B_r, \quad |\lambda| \geq \rho_2(N_r, \sigma). \) (71)
Formulas (68) can be considered as definitions of \( \mu_{z_0}, \tilde{\mu}_{z_0}. \)
Formulas (69), (71) were given in [21, 22] and go back to [3].
Estimates (70a) and (70b) were proved in [33]. Estimate (72) was obtained in [21, 37].

7. Proof of Theorem 2

We suppose that \( \psi_{z_1}(\cdot, \lambda), \psi_{z_2}(\cdot, \lambda), \partial h_{z_0}(\lambda) \) are defined as in Section 6 but with \( u_j - E \) in place of \( u_j, \lambda, \) \( j = 1, 2. \) Note that functions \( \psi_{z_1}(\cdot, \lambda), \psi_{z_2}(\cdot, \lambda) \) satisfy (1) in \( B_r \) with \( \lambda = v_j, \)
\( j = 1, 2, \) respectively. We also use the notation \( N_E = N + E. \) Then, using (72), we have that
\[ |v_2(z_0) - v_1(z_0) - \frac{2}{\pi} |\lambda| \partial h_{z_0}(\lambda)| \leq \epsilon_{12}(N_E, r, \sigma) \left( \ln(3|\lambda|)^2/|\lambda|^{3/4} \right), \]
for \( z_0 \in B_r, \quad |\lambda| \geq \rho_2(N_E, r). \) (73)
Let
\[ \delta = \| S^+_1(E) - S^+_2(E) \|_{L^2(\partial B_r \cap \partial B_\delta)}; \] (74)
Combining (17), (19), and (65), we get that
\[ |\partial h_{z_0}(\lambda)| \leq \delta \left( \frac{\partial \phi_1}{\partial \nu_+} - \frac{\partial \phi_2}{\partial \nu_+} \right) \left( \frac{\partial \phi_1}{\partial \nu_-} - \frac{\partial \phi_2}{\partial \nu_-} \right), \]
(75)
where \( \phi_j, j = 1, 2, \) denotes the solution of (18) with \( \nu = u_j, \)
\[ \phi_1(x) = \psi_{z_1}(x, -\lambda), \]
\[ \phi_2(x) = \psi_{z_2}(x, \lambda), \quad \text{for } x \in \overline{E_r}. \] (76)
Using (21), (70a), and (70b) and the fact that \( C^1(\partial B_r) \subset \mathcal{H}^1(\partial B_r), \) we find that
\[ \left\| \frac{\partial \phi_j}{\partial \nu_+} - \frac{\partial \phi_j}{\partial \nu_-} \right\|_{L^2(\partial B_r)} \leq \sigma c_{13} (r^2 + 4r) \delta \| \lambda \| \delta \lambda \delta \lambda \]
and
\[ \lambda \in \mathbb{C}, \quad |\lambda| \geq \rho_1(N_E, r, \sigma), \quad j = 1, 2. \] (77)
Here and below in this section the constant \( \sigma \) is the same that in (70a) and (70b).
Combining (75), (77), we obtain that
\[ \left| \delta h_{z_0}(\lambda) \right| \leq c_{14} \left( E, r, \sigma \right) \exp \left( \| \lambda \| \left( 8r^2 + 8r \right) \right) \delta, \]
(78)
\[ \lambda \in \mathbb{C}, \quad \| \lambda \| \geq \rho_1(N_E, r, \sigma). \]
Using (73) and (78), we get that
\[ |v_2(z_0) - v_1(z_0)| \leq c_{14} \left( E, r, \sigma \right) \exp \left( \| \lambda \| \left( 8r^2 + 8r \right) \right) \delta \]
\[ + c_{12}(N_E, r) \left( \ln(3|\lambda|) \right)^2, \]
(79)
We fix some \( \tau \in (0, 1) \) and let
\[ \beta = \frac{1 - \tau}{8r^2 + 8r}, \quad \lambda = \beta \ln \left( 3 + \delta^{-1} \right), \]
(80)
where \( \delta \) is so small that \( |\lambda| \geq \rho_3(N_E, r, \sigma). \) Then due to (79), we have that
\[ \left\| v_1 - v_2 \right\|_{L^\infty(B_r)} \leq c_{14} \left( E, r, \sigma \right) \left( 3 + \delta^{-1} \right)^{\beta(8r^2 + 8r)} \delta \]
\[ + c_{12}(N_E, r) \left( \ln(3|\lambda|) \right)^2 \]
\[ + c_{12}(N_E, r) \beta^{-3/4} \left( \ln(3|\lambda|) \right)^2, \]
(81)
where \( r, \beta, \) and \( \delta \) are the same as in (80).
Using (81), we obtain that
\[ \left\| v_1 - v_2 \right\|_{L^\infty(B_r)} \leq c_{15} \left( N, E, r, \sigma \right) \left( \ln(3 + \delta^{-1}) \right)^{-3/4} \]
(82)
for \( \delta = \| S_1^\gamma (E) - S_2^\gamma (E) \|_{L^2(\partial B_r \times \partial B_r)} \leq \delta_2 (N_N, r, \sigma) \), where \( \delta_2 \) is a sufficiently small positive constant. Estimate (82) in the general case (with modified \( c_1 \)) follows from (82) for \( \delta \leq \delta_2 (N_N, \eta, \rho) \) and the property that \( \| u_j \|_{L^p(B_r)} \leq N \).

This completes the proof of (14).

8. Proof of Lemma 7

In this section we assume for simplicity that \( r = 1 \) and therefore \( \partial B_r = S^{d-1} \).

We fix an orthonormal basis in \( L^2(\partial B_r) \):

\[
\{ f_{jp} : j \geq 0; 1 \leq p \leq p_j \},
\]

(83)

where \( p_j \) is the dimension of the space of spherical harmonics of order \( j \),

\[
p_j = \left( \frac{j + d - 1}{d - 1} \right) - \left( \frac{j + d - 3}{d - 1} \right),
\]

(84)

where

\[
\binom{n}{k} = \frac{n! (n - k)!}{k!}, \quad \text{for } n \geq 0,
\]

(85)

\[
\binom{n}{k} = 0, \quad \text{for } n < 0.
\]

The precise choice of \( f_{jp} \) is irrelevant for our purposes. Besides orthonormality, we only need \( f_{jp} \) to be the restriction of a homogeneous harmonic polynomial of degree \( j \) to the sphere \( \partial B_r \) and so \( |x| f_{jp}(x \mid |x|) \) is harmonic on \( \mathbb{R}^d \). In the Sobolev spaces \( H^s(\partial B_r) \) the norm is defined by

\[
\left\| \sum_{j,p} c_{jp} f_{jp} \right\|_{H^s(\partial B_r)}^2 = \sum_{j,p} (1 + j)^{2s} |c_{jp}|^2.
\]

(86)

The solution \( \phi \) of the exterior Dirichlet problem

\[
-\Delta \phi = E \phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B},
\]

\[
\lim_{|x| \to +\infty} |x|^{(d-1)/2} \left( \frac{\partial}{\partial |x|} \phi - i \sqrt{E} \phi \right) = 0,
\]

(87)

\[
\phi|_{\partial B_r} = u \in H^1(\partial B_r),
\]

can be expressed in the following form (see, e.g., [1, 38]):

\[
\phi = \sum_{j,p} c_{jp} \phi_{jp},
\]

(88)

where \( c_{jp} \) are expansion coefficients of \( u \) in the basis \( \{ f_{jp} : j \geq 0; 1 \leq p \leq p_j \} \), and

\[
\phi_{jp} \text{ denotes the solution of (89) with } u = f_{jp},
\]

(89)

\[
\phi_{0,j}^0(x) = |x|^{-j-d+2} f_{jp} \left( \frac{x}{|x|} \right),
\]

(90)

Note that \( \phi_{0,j}^0 \) is harmonic in \( \mathbb{R}^d \setminus \{0\} \) and

\[
\lim_{|x| \to +\infty} |x|^{(d-1)/2} \left( \frac{\partial}{\partial |x|} \phi_{0,j}^0 - i \sqrt{E} \phi_{0,j}^0 \right) = 0, \quad \text{for } j + \frac{d - 3}{2} > 0.
\]

(91)

Using the Green formula and the radiation condition for \( \phi_{jp}, \phi_{0,j}^0 \), we get that

\[
\int_{\mathbb{R}^d \setminus B_r} E \phi_{jp} \phi_{0,j}^0 dx = \int_{\mathbb{R}^d \setminus B_r} \left( \Delta \phi_{jp} - \Delta \phi_{0,j}^0 \phi_{0,j}^0 \right) dx = 0, \quad j + \frac{d - 3}{2} > 0.
\]

(92)

Due to (89) and (90), we have that

\[
\int_{\partial B_r} \frac{\partial \phi_{jp}}{\partial \nu^*} \phi_{0,j}^0 dx = (j + d - 2) \int_{\partial B_r} f_{jp}^2 dx = j + d - 2.
\]

(93)

Using also the following property of the Hankel function of the first kind (see, e.g., [39]):

\[
|H^{(1)}_\mu(x)| \text{ is a decreasing function of } x, \quad \text{for } x \in \mathbb{R}_+, \mu \in \mathbb{R},
\]

(94)
we get that
\[
\left| \int_{\mathbb{R}^d \setminus B_r} \phi_j \phi_j^* d\lambda \right| = \left| \int_1^{\infty} t^{-j-d+2} h_{jj}(t) t^{d-1} dt \right|
\]
\[
\leq \int_1^{\infty} t^{-j-d+2} dt = \frac{1}{j + (d/2) - 1}
\]
\[
\leq 2, \quad \text{for } j + \frac{d-3}{2} > 0.
\]

Combining (89), (90), (92), (93), and (95), we obtain that
\[
\left| \frac{\partial \phi_j^0}{\partial \nu_x} \right| = \left| \frac{h'_{jj}(r)}{h_{jj}(r)} \right| \leq j + d - 2 + 2E, \quad \text{for } j + \frac{d-3}{2} > 0.
\]

Let consider the cases when \( j + ((d-3)/2) \leq 0 \).

\textbf{Case 1} \( (j = 0, d = 2) \). Using the property \( dH_{0}^{(1)}(t)/dt = -H_{0}^{(1)}(t) \), we get that
\[
\frac{h'_{jj}(r)}{h_{jj}(r)} = \sqrt{E} H_{0}^{(1)}(\sqrt{E})).
\]

We recall that functions \( H_{0}^{(1)} \) and \( H_{1}^{(1)} \) have the following asymptotic forms (see, e.g. [39]):
\[
H_{0}^{(1)}(t) \sim \frac{2i}{\pi t} \ln \left( \frac{t}{2} \right), \quad \text{as } t \to +0,
\]
\[
H_{0}^{(1)}(t) \sim \frac{2i}{\pi t} e^{i(-\pi/4)}, \quad \text{as } t \to +\infty,
\]
\[
H_{1}^{(1)}(t) \sim -\frac{i}{\pi} \left( \frac{2}{t} \right), \quad \text{as } t \to +0,
\]
\[
H_{1}^{(1)}(t) \sim \frac{2i}{\pi} e^{i(-2\pi/4)}, \quad \text{as } t \to +\infty.
\]

Using (94) and (98), we get that for some \( c > 0 \)
\[
\frac{H_{1}^{(1)}(t)}{H_{0}^{(1)}(t)} \leq c \left( 1 + \frac{1}{t} \right).
\]

Combining (97) and (99), we obtain that for \( j = 0, d = 2 \)
\[
\left| \frac{h'_{jj}(r)}{h_{jj}(r)} \right| \leq c \left( 1 + \sqrt{E} \right).
\]

\textbf{Case 2} \( (j = 0, d = 3) \). We have that
\[
H_{j+((d-2)/2)}^{(1)}(r) = \sqrt{\frac{2}{\pi t e^{i(-\pi/2)}}}.
\]

Using (89) and (101), we get that for \( j = 0, d = 3 \)
\[
\frac{h'_{jj}(r)}{h_{jj}(r)} = -1 + i \sqrt{E}.
\]

Combining (86)–(89), (96), (100), and (102), we get that for some constant \( c' = c'(d) > 0 \):
\[
\left\| \frac{\partial \phi_j}{\partial \nu_x} \right\|_{L^2(B_r)}^2 = \sum_{j,p} \left| \frac{h_{jj}(r)}{h_{jj}(r)} \right|^2 \leq c' (1 + E)^2 \sum_{j,p} \left( 1 + j \right)^2 c_{jp}.
\]

Using (86) and (103), we obtain (21).

\textbf{Acknowledgment}

This work was partially supported by FCP Kadry no. 14.A18.21.0866.

\textbf{References}


[8] R. G. Novikov, "A multidimensional inverse spectral problem for the equation \( -\Delta \psi + (\xi(x) - E\xi(x))\psi = 0 \), Funktsional'nyi


