Research Article

On Schauder Frames in Conjugate Banach Spaces

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Received 31 August 2012; Accepted 17 November 2012

Academic Editor: Ding-Xuan Zhou

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Weak*-Schauder frame in conjugate Banach spaces has been introduced and studied. A sufficient condition for the existence of weak*-Schauder frame in the conjugate space of a separable Banach space has been given. It has been shown that ℓ_∞ has weak*-Schauder frame. Finally, a sufficient condition for the existence of a Schauder frame sequence has been given.

1. Introduction

Frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [1]. Later, Daubechies et al. [2] found new applications to wavelets in which frames played an important role. Frames are main tools for use in signal and image processing, compression, sampling theory, optics, filter banks, signal detection, and so forth. In order to have more applications of frames, several notions generalizing the concept of frames have been introduced and studied, namely, pseudoframes [3], oblique frames [4], frames of subspaces (fusion frames) [5], G-frames [6], orthogonal frames [7, 8], and so forth.

Feichtinger and Gröchenig [9] extended the notion of atomic decomposition to Banach spaces. Gröchenig [10] introduced a more general concept for Banach spaces called Banach frame. Banach frames and atomic decompositions were further studied in [11–14]. Han and Larson [15] defined Schauder frame for a Banach space. In [16], Casazza, et al. gave various definitions of frames for Banach spaces including that of Schauder frame. In 2008, Casazza et al. [17] studied the coefficient quantization of Schauder frames in Banach spaces. Liu [18] gave the concepts of minimal and maximal associated bases with respect to Schauder frames and closely connected them with the duality theory. In [19], Liu and Zheng gave a characterization of Schauder frames which are near-Schauder bases. In fact, they generalized some results due to Holub [20]. Beanland et al. [21] proved that the upper and lower estimates theorems for finite dimensional

decompositions of Banach spaces can be extended and modified to Schauder frames and gave a complete characterization of duality for Schauder frames. Φ -Schauder frames were introduced and studied by Vashisht [22]. Recently, Liu [23] associated an operator with a Schauder frame and called it Hilbert-Schauder frame operator.

In the present paper, we introduce the concept of weak*-Schauder frame and weak-Schauder frame in conjugate Banach spaces. A sufficient condition for the existence of weak*-Schauder frame in a conjugate Banach space of a separable Banach space has been given. Also, an example of a conjugate space of a nonseparable Banach space which has no weak*-Schauder frame is given. Further, it has been shown that ℓ_∞ has weak*-Schauder frame. Finally, a sufficient condition for the existence of a Schauder frame sequence has been given.

2. Preliminaries

Throughout this paper, E will denote a Banach space, H will denote a Hilbert space, let E^* the dual space of E , $[x_n]$ be the closed linear span of $\{x_n\}$ in the norm topology of E , and let π be the canonical mapping of E into E^{**} . A series $\sum_{n=1}^\infty f_i$ in a conjugate Banach space E^* is called weak-convergent to f if it converges in $\sigma(E^*, E^{**})$ -topology. In this case, we write $f \stackrel{w}{=} \sum_{n=1}^\infty f_n$. A series $\sum_{n=1}^\infty f_i$ in a conjugate Banach space E^* is called weak*-convergent to f if it converges in $\sigma(E^*, E)$ -topology. In this case, we write $f \stackrel{w^*}{=} \sum_{n=1}^\infty f_n$.

Definition 1. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset H$ is called a frame for H if there exist A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \quad x \in H. \quad (1)$$

The positive constants A and B , respectively, are called lower and upper frame bounds of the frame $\{x_n\}_{n \in \mathbb{N}}$. The inequality (1) is called the frame inequality.

Definition 2. Let E be a Banach space. A sequence $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ ($\{x_n\} \subset E, \{f_n\} \subset E^*$) is called a Schauder frame for E if

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \quad x \in E. \quad (2)$$

Definition 3. Let E be a Banach space. A sequence $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ ($\{x_n\} \subset E, \{f_n\} \subset E^*$) is called a Schauder frame sequence for E if $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ is a Schauder frame for $[x_n]$.

Next, we give the definition of a retro Banach frame introduced in [24].

Definition 4. Let E be a Banach space and let E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar-valued sequences associated with E^* , indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \rightarrow E^*$ be given. The pair $(\{x_n\}, T)$ is called a retro Banach frame for E^* with respect to $(E^*)_d$ if

- (i) $\{f(x_n)\} \in (E^*)_d, f \in E^*$,
- (ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad f \in E^*, \quad (3)$$

- (iii) T is a bounded linear operator such that

$$T(\{f(x_n)\}) = f, \quad f \in E^*. \quad (4)$$

The positive constants A and B , respectively, are called lower and upper frame bounds of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \rightarrow E^*$ is called the reconstruction operator (or the preframe operator). The inequality (3) is called the retro frame inequality. A retro Banach frame $(\{x_n\}, T)$ ($\{x_n\} \subset E, T : (E^*)_d \rightarrow E^*$) for E with respect to $(E^*)_d$ with bounds A, B is said to be tight, if it is possible to choose $A = B$, normalized tight, if $A = B = 1$, and exact, if there exists no reconstruction operator T_0 such that $(\{x_n\}_{n \neq j}, T_0)$ ($j \in \mathbb{N}$) is a retro Banach frame E^* .

Finally, we give the following results which will be used in the subsequent results.

Theorem 5 (see [25]). If $\{g_n\} \subset (\ell_\infty)^*$ and $\lim_{n \rightarrow \infty} g_n(x) = 0$, for all $x \in \ell_\infty$, then $\lim_{n \rightarrow \infty} \Phi(g_n) = 0$ for all $\Phi \in E^{**}$.

Theorem 6 (see [24]). Let $(\{x_n\}, T)$ ($\{x_n\} \subset E, T : (E^*)_d \rightarrow E^*$) be a retro Banach frame for E^* with respect to $(E^*)_d$. Then, $(\{x_n\}, T)$ is exact if and only if $x_n \notin [x_i]_{i \neq n}$ for all $n \in \mathbb{N}$.

In view of Theorem 6, one may observe that if $(\{x_n\}, T)$ is an exact retro Banach frame for E^* , then there exists a sequence $\{g_n\}$ in E^* , called an admissible sequence to the retro Banach frame $(\{x_n\}, T)$, such that $g_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

3. Main Results

We begin with the following definitions of weak*-Schauder frame and weak-Schauder frame in E^* .

Definition 7. Let E be a Banach space, let $\{f_n\}$ be a sequence in E^* and let $\{\Phi_n\}$ be a sequence in E^{**} , and let M be a subset of E^* . Then, $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is said to be

- (a) weak-Schauder frame for E^* with respect to M if

$$f \stackrel{w}{=} \sum_{n=1}^{\infty} \Phi_n(f) f_n, \quad f \in M, \quad (5)$$

- (b) weak*-Schauder frame for E^* with respect to M if

$$f \stackrel{w^*}{=} \sum_{n=1}^{\infty} \Phi_n(f) f_n, \quad f \in M. \quad (6)$$

In particular, if $M = E^*$, then $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is simply called a weak-Schauder frame and weak*-Schauder frame for E^* , respectively. Further, one may observe that if $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a Schauder frame for E^* , then it is also a weak*-Schauder frame (weak-Schauder frame) for E^* .

Now, we give an example of a conjugate space of a nonseparable Banach space which has no weak*-Schauder frame.

Example 8. ℓ_∞^* has no weak*-Schauder frame. Assume on the contrary that ℓ_∞^* has a weak*-Schauder frame $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ ($\{f_n\} \subset \ell_\infty^*, \{\Phi_n\} \subset \ell_\infty^{**}$). Then, we have

$$f(x) = \sum_{i=1}^{\infty} \Phi_i(f) f_i(x), \quad x \in E, f \in \ell_\infty^*, \quad (7)$$

that is,

$$\left| f(x) - \sum_{i=1}^n \Phi_i(f) f_i(x) \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (8)$$

So, by Theorem 5, we have

$$\lim_{n \rightarrow \infty} \Phi \left(f - \sum_{i=1}^n \Phi_i(f) f_i \right) = 0, \quad \Phi \in \ell_\infty^{**}. \quad (9)$$

This gives ℓ_∞^* is weak-separable. This is not possible. Hence, ℓ_∞^* has no weak*-Schauder frame.

In view of Example 8 we have the following problem.

Problem 1. Does the conjugate space E^* of a separable Banach space E possess a weak*-Schauder frame?

In this direction, we have the following result.

Proposition 9. *If a Banach space E has a Schauder frame, then its conjugate Banach space has a weak*-Schauder frame.*

Proof. Let $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ be a Schauder frame for E ; then

$$x = \sum_{i=1}^{\infty} f_i(x) x_i, \quad x \in E. \quad (10)$$

Therefore, for each $f \in E^*$, we have

$$\begin{aligned} f(x) &= f\left(\sum_{i=1}^{\infty} f_n(x) x_n\right) \\ &= \sum_{n=1}^{\infty} \pi(x_n)(f) f_n(x), \quad x \in E. \end{aligned} \quad (11)$$

Corollary 10. ℓ_{∞} has weak*-Schauder frame.

Proof. It follows from Proposition 9. $\square$

Note. ℓ_{∞} does not have a Schauder frame.

Towards the converse of Proposition 9, we have the following result.

Theorem 11. *Let E be a Banach space. Let $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ be a weak*-Schauder frame for E^* such that each Φ_n is weak*-continuous. Then, there exists a sequence $\{x_n\} \subset E$ such that $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ is a Schauder frame for E .*

Proof. For each $f \in E^*$, we have

$$\sum_{n=1}^{\infty} \Phi_n(f) f_n \stackrel{w^*}{=} f. \quad (12)$$

Since each Φ_n is weak*-continuous, there exists a sequence $\{x_n\} \subset E$ such that $\Phi_n = \pi(x_n)$, $n \in \mathbb{N}$. So, we have

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \pi(x_n)(f) f_n(x) \\ &= f\left(\sum_{n=1}^{\infty} f_n(x) x_n\right), \quad x \in E, \quad f \in E^*. \end{aligned} \quad (13)$$

This gives

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \quad x \in E. \quad (14)$$

In the following result, we characterize weak*-Schauder frame in terms of weak-Schauder frame with respect to a subset.

Theorem 12. *Let E be a Banach space. Let $\{f_n\}$ be a sequence in E^* and let $\{\Phi_n\}$ be a sequence in E^{**} . Then, $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak*-Schauder frame for E^* if and only if $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak-Schauder frame for E^* with respect to $\pi(E)$.*

Proof. $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak*-Schauder frame for E^* if and only if

$$(\pi(x))(f) = \sum_{n=1}^{\infty} \Phi_n(f) \pi(x)(f_n), \quad f \in E^*, \quad \pi(x) \in \pi(E) \quad (15)$$

if and only if $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak-Schauder frame for E^* with respect to $\pi(E)$. $\square$

Remark 13. Let E be a reflexive Banach space. Let $\{f_n\}$ be a sequence in E^* and $\{\Phi_n\}$ let be a sequence in E^{**} . Then, by Theorem 12, $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak*-Schauder frame for E^* if and only if $\{(f_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a weak-Schauder frame for E^* .

Finally, we give a sufficient condition for the existence of a Schauder frame sequence in a Banach space.

Theorem 14. *Let E be a Banach space and let $(\{x_n\}, T)$ be an exact retro Banach frame for E^* with admissible sequence $\{f_n\} \subset E^*$. Then, $\{(x_n, f_n)\}_{n \in \mathbb{N}}$ is a Schauder-frame sequence in E .*

Proof. For each $n \in \mathbb{N}$, define $S_n : E \rightarrow E$ by

$$S_n(x) = \sum_{i=1}^n f_i(x) x_i, \quad x \in E. \quad (16)$$

Let S_n^* be the adjoint operator to S_n . Then

$$\begin{aligned} (S_n^*(g))(x) &= g\left(\sum_{i=1}^n f_i(x) x_i\right) = \left(\sum_{i=1}^n g(x_i) f_i\right)(x), \\ x &\in E, \quad g \in E^*, \quad n = 1, 2, 3, \dots \end{aligned} \quad (17)$$

This gives

$$S_n^*(g) = \sum_{i=1}^n g(x_i) f_i, \quad g \in E^*, \quad n = 1, 2, 3, \dots \quad (18)$$

Thus, for every finite linear combination $g = \sum_{i=1}^m \beta_i f_i$, we have

$$\begin{aligned} S_n^*(g) &= \sum_{i=1}^n g(x_i) f_i \\ &= \sum_{i=1}^m \beta_i f_i \\ &= g, \quad n = m, m+1, \dots \end{aligned} \quad (19)$$

Let $f \in [f_n]$ and $\varepsilon > 0$ be given. Then, there exists a finite linear combination $g = \sum_{j=1}^{m_\varepsilon} \beta_j^{(\varepsilon)} f_j$ such that $\|f - g\| < \varepsilon/(v+1)$, where $v = \sup_{1 \leq n < \infty} \|S_n\| < \infty$. Also,

$$\begin{aligned} \|S_n^*(f) - f\| &\leq \|S_n^*(f) - S_n^*(g)\| + \|S_n^*(g) - g\| + \|g - f\| \\ &< \frac{v\varepsilon}{v+1} + \frac{\varepsilon}{v+1} \\ &= \varepsilon, \quad n = m_\varepsilon, m_\varepsilon + 1, \dots \end{aligned} \quad (20)$$

$\square$

Acknowledgment

The authors thank the referee(s) for providing valuable comments and useful suggestions for the improvement of the paper.

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