Research Article

Totally Contact Umbilical Lightlike Hypersurfaces of Indefinite $\mathcal{S}$ -Manifolds

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We study totally contact umbilical lightlike hypersurfaces of indefinite $\mathcal{S}$ -manifolds and prove the nonexistence of totally contact umbilical lightlike hypersurface in indefinite $\mathcal{S}$ -space form.

1. Introduction

The general theory of nondegenerate submanifolds of Riemannian or semi-Riemannian manifolds is one of the most important topics of differential geometry [1, 2]. But the theory of degenerate or lightlike submanifolds of semi-Riemannian manifolds is relatively new [3] and different due to the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study of lightlike submanifolds becomes more difficult than the study of nondegenerate submanifolds and one cannot use, in the usual way, the classical theory of submanifold to define induced objects on a lightlike submanifold. The degenerate geometry rises within the semi-Riemannian context, due to the existence of causal character of geometrical objects: their spacelike, timelike, or lightlike nature implies the existence of three types of hypersurfaces and submanifolds.

A lightlike framed hypersurface of a Lorentz C -manifold, with an induced metric connection, is a Killing horizon and a globally hyperbolic spacetime; a de Sitter spacetime can carry a framed structure [3, 4]. Moreover, the contact geometry has a significant use in differential equations, optics, and phase spaces of dynamical systems [5–7]. All these motivated us to study lightlike hypersurfaces of indefinite globally framed f -manifold, in particular indefinite $\mathcal{S}$ -manifolds.

In this paper, we study totally contact umbilical lightlike hypersurfaces of indefinite $\mathcal{S}$ -manifolds and prove the

nonexistence of totally contact umbilical lightlike hypersurface in indefinite $\mathcal{S}$ -space form.

2. Preliminaries

A manifold $\overline{M}$ of dimension $2n + r$ is called a globally framed f -manifold ($g.f.f$ -manifold) if it is endowed with a nowhere vanishing $(1, 1)$ -tensor field $\overline{\phi}$ of constant rank, such that $\ker \overline{\phi}$ is parallelizable; that is, there exist global vector fields $\overline{\zeta}_\alpha$, $\alpha \in \{1, \dots, r\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying

$$\overline{\phi}^2 = -I + \overline{\eta}^\alpha \otimes \overline{\zeta}_\alpha, \quad \overline{\eta}^\alpha(\overline{\zeta}_\beta) = \delta_\beta^\alpha. \quad (1)$$

If the metric $\overline{g}$ is a semi-Riemannian metric with index ν , $0 < \nu < 2n + r$ such that

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) - \sum_{\alpha=1}^r \epsilon_\alpha \overline{\eta}^\alpha(X) \overline{\eta}^\alpha(Y), \quad (2)$$

for any $X, Y \in \Gamma(T\overline{M})$, $\epsilon_\alpha = \pm 1$ accordingly $\overline{\zeta}_\alpha$ is either spacelike or timelike, then the $g.f.f$ -manifold $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\zeta}_\alpha, \overline{\eta}^\alpha)$ is called an indefinite metric $g.f.f$ -manifold. Then clearly $\overline{\eta}^\alpha(X) = \epsilon_\alpha \overline{g}(X, \overline{\zeta}_\alpha)$, for any $\alpha \in \{1, \dots, r\}$. An indefinite metric $g.f.f$ -manifold is called an indefinite $\mathcal{S}$ -manifold if it is normal that is, the tensor field $N = N_{\overline{\phi}} + 2d\overline{\eta}^\alpha \otimes \overline{\zeta}_\alpha$

vanishes, where $N_{\bar{\phi}}$ is Nijenhuis tensor of $\bar{\phi}$ and $d\bar{\eta}^\alpha = \Phi$, for any $\alpha \in \{1, \dots, r\}$, where $\Phi(X, Y) = \bar{g}(X, \bar{\phi}Y)$, for any $X, Y \in \Gamma(T\bar{M})$. The Levi-Civita connection of an indefinite $\mathcal{S}$ -manifold satisfies

$$(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(\bar{\phi}X, \bar{\phi}Y)\bar{\zeta} + \bar{\eta}(Y)\bar{\phi}^2(X), \quad (3)$$

where $\bar{\zeta} = \sum_{\alpha=1}^r \bar{\zeta}_\alpha$ and $\bar{\eta} = \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha$. Also

$$\bar{\nabla}_X \bar{\zeta}_\alpha = -\epsilon_\alpha \bar{\phi}X, \quad (4)$$

and $\ker \bar{\phi}$ is integrable flat distribution, since $\bar{\nabla}_{\bar{\zeta}_\alpha} \bar{\zeta}_\beta = 0$, for detail see [8].

We recall notations and fundamental equations for lightlike hypersurfaces, which are due to Duggal and Bejancu [3].

Let $(\bar{M}, \bar{g})$ be a $(2n+1)$ -dimensional semi-Riemannian manifold with index s , $0 < s < 2n+1$, and let (M, g) be a hypersurface of $\bar{M}$, with $g = g|_M$. M is a lightlike hypersurface of $\bar{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^\perp$ is a distribution of rank 1 on M . A complementary bundle of $TM^\perp$ in TM is a rank $2n-1$ nondegenerate distribution over M . It is denoted by $S(TM)$ and known as a screen distribution. Therefore we have

$$\begin{aligned} TM &= S(TM) \perp TM^\perp, \\ T\bar{M}|_M &= S(TM) \perp S(TM)^\perp, \end{aligned} \quad (5)$$

where $S(TM)^\perp$ is a orthogonal complementary vector bundle of $S(TM)$ in $T\bar{M}|_M$. The following theorem has important roles in studying the geometry of lightlike hypersurface.

Theorem 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a unique vector bundle $\text{tr}(TM)$ of rank 1 over M , such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $\text{tr}(TM)$ on $\mathcal{U}$ satisfying*

$$\begin{aligned} \bar{g}(\xi, N) &= 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0, \\ \forall W &\in \Gamma(S(TM)|_{\mathcal{U}}). \end{aligned} \quad (6)$$

Hence, we have the following decompositions of $T\bar{M}|_M$:

$$\begin{aligned} T\bar{M}|_M &= S(TM) \perp (TM^\perp \oplus \text{tr}(TM)) \\ &= TM \oplus \text{tr}(TM). \end{aligned} \quad (7)$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then using the decompositions in (7), Gauss and Weingarten formulae are given as

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \bar{\nabla}_X N &= -A_N X + \nabla_X^\perp N, \end{aligned} \quad (8)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(\text{tr}(TM))$, where $\nabla_X Y$ and $A_N X$ belong to $\Gamma(TM)$ while $h(X, Y)$ and $\nabla_X^\perp N$ belong to $\Gamma(\text{tr}(TM))$. Here ∇ is a torsion free linear connection

on M , h is a $\Gamma(\text{tr}(TM))$ -valued symmetric bilinear form on $\Gamma(TM)$ and known as the second fundamental form. A_N is a linear operator on $\Gamma(TM)$ and known as the shape operator of lightlike immersion and ∇^t is a linear connection on $\Gamma(\text{tr}(TM))$.

Locally, for the pair $\{\xi, N\}$, following the Duggal and Bejancu's notation [3], we recall local second fundamental form B and 1-form τ as

$$\begin{aligned} B(X, Y) &= \bar{g}(h(X, Y), \xi), \\ \text{therefore } h(X, Y) &= B(X, Y)N, \\ \tau(X) &= \bar{g}(\nabla_X^t N, \xi) \end{aligned} \quad (9)$$

$$\text{therefore } \nabla_X^t N = \tau(X)N.$$

Hence locally, (8) becomes

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \\ \bar{\nabla}_X N &= -A_N X + \tau(X)N. \end{aligned} \quad (10)$$

If P denotes the projection morphism of TM on $S(TM)$ then from (5), we have

$$\begin{aligned} \nabla_X PY &= \nabla_X^* PY + h^*(X, PY), \\ \nabla_X \xi &= -A_\xi^* X + \nabla_X^{*t} \xi, \end{aligned} \quad (11)$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$, where $\nabla_X^* PY$ and $A_\xi^* X$ belong to $\Gamma(S(TM))$ while $h^*(X, PY)$ and $\nabla_X^{*t} \xi$ belong to $\Gamma(TM^\perp)$. Here h^* and A_ξ^* are called the second fundamental form and the shape operator of the screen distribution, respectively. It should be noted that the induced linear connection ∇ is not a metric connection as it satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad (12)$$

where θ is a differential 1 form locally defined on M by $\theta(\cdot) = \bar{g}(N, \cdot)$. By direct calculation, we have

$$\begin{aligned} g(A_N Y, PX) &= \bar{g}(N, h^*(Y, PX)), & \bar{g}(A_N Y, N) &= 0, \\ g(A_\xi^* X, PY) &= \bar{g}(\xi, h(X, PY)), & \bar{g}(A_\xi^* X, N) &= 0, \end{aligned} \quad (13)$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$, and $N \in \Gamma(\text{tr}(TM))$. According to Duggal and Bejancu's notation [3], locally we have know

$$\begin{aligned} C(X, PY) &= \bar{g}(h^*(X, PY), N) \\ \text{therefore } h^*(X, PY) &= C(X, PY)\xi, \\ \epsilon(X) &= \bar{g}(\nabla_X^{*t} \xi, N) \\ \text{therefore } \nabla_X^{*t} \xi &= \epsilon(X)\xi. \end{aligned} \quad (14)$$

Then (11) become

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \quad (15)$$

$$\nabla_X \xi = -A_\xi^* X + \epsilon(X)\xi = -A_\xi^* X - \tau(X)\xi \quad (16)$$

and also give

$$\begin{aligned} g(A_N Y, PW) &= C(Y, PW), & \bar{g}(A_N Y, N) &= 0, \\ g(A_\xi^* X, PY) &= B(X, PY), & \bar{g}(A_\xi^* X, N) &= 0. \end{aligned} \quad (17)$$

From (9), it is clear that

$$B(X, \xi) = 0, \quad (18)$$

that is, the second fundamental form of a lightlike hypersurface is degenerate. Following the notations of Duggal and Bejancu [3], for the curvature tensor $\bar{R}$ of $\bar{M}$, we have $\bar{R}(X, Y, Z, W) = \bar{g}(\bar{R}(X, Y)Z, W)$, for any $X, Y, Z, W \in \Gamma(T\bar{M})$.

3. Lightlike Hypersurface of Indefinite $\mathcal{S}$ -Manifolds

Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite $g.f.f$ -manifold $(\bar{M}^{2n+r}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$, and $\alpha \in \{1, \dots, r\}$ such that the characteristic vector fields $\bar{\xi}_\alpha, \alpha \in \{1, \dots, r\}$ are tangent to M . As the ambient manifold $\bar{M}$ has an additional geometric structure $\bar{\phi}$, we must look for a particular screen distribution on M . Since $\bar{g}(\bar{\phi}\xi, \xi) = 0$, for any $\xi \in \Gamma(TM^\perp)$ therefore $\bar{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $TM^\perp \cap \bar{\phi}(TM^\perp) = \{0\}$. Moreover $\bar{g}(\bar{\phi}N, \xi) = -\bar{g}(N, \bar{\phi}\xi) = 0$, for any $N \in \Gamma(\text{tr}(TM))$, therefore $\bar{\phi}N$ is tangent to M . Since $\bar{g}(\bar{\phi}N, N) = 0$; that is, the component of $\bar{\phi}N$ with respect to ξ vanishes; this implies $\bar{\phi}N \in \Gamma(S(TM))$. As ξ and N are null vector fields satisfying $\bar{g}(\xi, N) = 1$, therefore from (2), we deduce that $\bar{\phi}\xi$ and $\bar{\phi}N$ are null vector fields satisfying $\bar{g}(\bar{\phi}\xi, \bar{\phi}N) = 1$. Hence $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(\text{tr}(TM))$ is a vector subbundle of $S(TM)$ of rank 2. It is known [9] that if structure vector fields $\bar{\xi}_\alpha, \alpha \in \{1, \dots, r\}$ are tangent to M then $\bar{\xi}_\alpha$ belong to $S(TM)$. Therefore $\bar{g}(\bar{\phi}\xi, \bar{\xi}_\alpha) = \bar{g}(\bar{\phi}N, \bar{\xi}_\alpha) = 0$ implies that there exists a nondegenerate distribution D_0 of rank $2n - 4$ on M such that

$$S(TM) = \{\bar{\phi}(\text{tr}(TM)) \oplus \bar{\phi}TM^\perp\} \perp D_0 \perp \{\bar{\xi}_\alpha\}, \quad (19)$$

therefore

$$TM = \{\bar{\phi}(\text{tr}(TM)) \oplus \bar{\phi}TM^\perp\} \perp D_0 \perp \{\bar{\xi}_\alpha\} \perp TM^\perp. \quad (20)$$

Let $D = TM^\perp \perp \bar{\phi}TM^\perp \perp D_0$, and let $D' = \bar{\phi}\text{tr}(TM)$ then D clearly is invariant and D' is anti-invariant under $\bar{\phi}$, and we have

$$TM = D' \oplus D \perp \{\bar{\xi}_\alpha\}. \quad (21)$$

Consider the local lightlike vector fields as

$$\begin{aligned} U &= -\bar{\phi}N \in \bar{\phi}(\text{tr}(TM)) = D', \\ V &= -\bar{\phi}\xi \in D. \end{aligned} \quad (22)$$

Using the decompositions in (21), any $X \in \Gamma(TM)$ can be written as

$$X = SX + QX + \sum_{\alpha=1}^r \bar{\eta}^\alpha(X) \bar{\xi}_\alpha, \quad (23)$$

where S and Q are the projection morphisms of TM into D and D' , respectively, and where $QX = u(X)U$, and u is a local 1-form on M defined by $u(X) = \bar{g}(V, X)$. Therefore

$$u(U) = 1, \quad u(X) = 0, \quad \forall X \in \Gamma(D), \quad \bar{\phi}^2 N = -N. \quad (24)$$

Applying $\bar{\phi}$ to (23) and then using (24), we obtain $\bar{\phi}X = \bar{\phi}(SX) + u(X)N$. By assuming $\bar{\phi}(SX) = \phi X$ for any $X \in \Gamma(TM)$, we obtain a tensor field ϕ of type $(1, 1)$ on M and given by

$$\bar{\phi}X = \phi X + u(X)N. \quad (25)$$

Applying $\bar{\phi}$ to (25) and then using the definition of ϕX , we obtain

$$\phi^2 X = -X + u(X)U + \sum_{\alpha=1}^r \bar{\eta}^\alpha(X) \bar{\xi}_\alpha. \quad (26)$$

Moreover, since $\phi X = \bar{\phi}(SX)$ therefore

$$\phi U = 0, \quad \bar{\eta}^\alpha \circ \phi = 0, \quad u(\phi X) = 0, \quad (27)$$

for any $X \in \Gamma(TM)$. Thus we have the following theorem analogous to a theorem proved in [10].

Theorem 2. Let $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ be an indefinite $g.f.f$ -manifold and let $(M, g, S(TM))$ be a lightlike hypersurface of $\bar{M}$; then $(M, \phi, \bar{\xi}_\alpha, U, \bar{\eta}^\alpha, u)$ is also a $g.f.f$ -manifold.

Since $\phi U = 0$ therefore applying ϕ to (26), we get $\phi^3 + \phi = 0$, this implies that ϕ is an f structure of constant rank on M . Now using (2) and (25), we obtain

$$\begin{aligned} \bar{g}(\phi X, \phi Y) &= \bar{g}(X, Y) - u(X)v(Y) \\ &\quad - u(Y)v(X) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(X) \bar{\eta}^\alpha(Y), \end{aligned} \quad (28)$$

for any $X, Y \in \Gamma(T\bar{M})$, where v is a 1 form locally defined on M by $v(\cdot) = \bar{g}(U, \cdot)$. Replace X by ϕX in (28) and using (26) and (27), we derive

$$\begin{aligned} \bar{g}(\phi X, Y) + \bar{g}(X, \phi Y) \\ = -u(X)\theta(Y) - u(Y)\theta(X). \end{aligned} \quad (29)$$

Let M be a lightlike hypersurface of an indefinite $\mathcal{S}$ -manifold $\bar{M}$; then by using (4), (10), and (25), we obtain

$$\nabla_X \bar{\xi}_\alpha + B(X, \bar{\xi}_\alpha)N = -\epsilon_\alpha \phi X - \epsilon_\alpha u(X)N, \quad (30)$$

for any $X \in \Gamma(T\bar{M})$ then comparing the tangential and transversal components, we get

$$\nabla_X \bar{\zeta}_\alpha = -\epsilon_\alpha \phi X, \quad (31)$$

$$B(X, \bar{\zeta}_\alpha) = -\epsilon_\alpha u(X), \quad (32)$$

this implies $h(X, \bar{\zeta}_\alpha) = -\epsilon_\alpha u(X) N$.

Using (15) and (31), we get $\nabla_X^* \bar{\zeta}_\alpha + C(X, \bar{\zeta}_\alpha) \xi = -\epsilon_\alpha \phi X$, taking inner product with $N \in \Gamma(\text{tr}(TM))$, we get $C(X, \bar{\zeta}_\alpha) = -\epsilon_\alpha \bar{g}(\phi X, N)$, and then using (29), we obtain

$$\begin{aligned} C(X, \bar{\zeta}_\alpha) &= \epsilon_\alpha g(X, \phi N) + u(X) \theta(N) - u(N) \theta(X) \\ &= -\epsilon_\alpha \bar{g}(X, U) \\ &= -\epsilon_\alpha v(X). \end{aligned} \quad (33)$$

Using (3), for any $N \in \Gamma(\text{tr}(TM))$, we get $\bar{\nabla}_X \bar{\phi} N = \bar{\phi} \bar{\nabla}_X N + \bar{g}(X, N) \bar{\xi}$, therefore

$$\begin{aligned} B(X, U) &= \bar{g}(h(X, U), \xi) \\ &= -\bar{g}(\bar{\nabla}_X \bar{\phi} N, \xi) \\ &= \bar{g}(\bar{\nabla}_X, \phi \xi) \\ &= g(A_N X, V) \\ &= C(X, V). \end{aligned} \quad (34)$$

4. Totally Contact Umbilical Lightlike Hypersurface of Indefinite $\mathcal{S}$ -Manifolds

Here we will follow Bejancu's definition [11] of totally contact umbilical submanifolds of Sasakian manifolds to state totally contact umbilical lightlike hypersurfaces of indefinite $\mathcal{S}$ -manifolds.

Definition 3. A lightlike hypersurface M of an indefinite $\mathcal{S}$ -manifold is said to be totally contact umbilical lightlike hypersurface if the second fundamental form h of M satisfies

$$\begin{aligned} h(X, Y) &= \left\{ g(X, Y) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(X) \bar{\eta}^\alpha(Y) \right\} H \\ &+ \sum_{\alpha=1}^r \bar{\eta}^\alpha(X) h(Y, \bar{\zeta}_\alpha) \\ &+ \sum_{\alpha=1}^r \bar{\eta}^\alpha(Y) h(X, \bar{\zeta}_\alpha), \end{aligned} \quad (35)$$

for any $X, Y \in \Gamma(TM)$, where H is a transversal vector field on M ; that is, $H = \lambda N$, where λ is a smooth function on $\mathcal{U} \subset M$.

Remark 4. We can also write (35) as

$$\begin{aligned} h(X, Y) &= B(X, Y) N \\ &= \{B_1(X, Y) + B_2(X, Y)\} N, \end{aligned} \quad (36)$$

where

$$B_1(X, Y) = \lambda \left\{ g(X, Y) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(X) \bar{\eta}^\alpha(Y) \right\}, \quad (37)$$

and using (32)

$$B_2(X, Y) = -\bar{\eta}(X) u(Y) - \bar{\eta}(Y) u(X), \quad (38)$$

for any $X, Y \in \Gamma(TM)$. If $\lambda = 0$, that is, $B_1 = 0$; then lightlike hypersurface M is said to be totally contact geodesic.

Let $\bar{R}$ be the curvature tensor fields of $\bar{M}$; then for an indefinite $\mathcal{S}$ -space form $\bar{M}(c)$, we have (see [8])

$$\begin{aligned} \bar{R}(X, Y, W, Z) &= \frac{c+3\epsilon}{4} \{ \bar{g}(\bar{\phi} Y, \bar{\phi} Z) \bar{g}(\bar{\phi} X, \bar{\phi} W) \\ &\quad - \bar{g}(\bar{\phi} X, \bar{\phi} Z) \bar{g}(\bar{\phi} Y, \bar{\phi} W) \} \\ &+ \frac{c-\epsilon}{4} \{ \Phi(W, X) \Phi(Z, Y) - \Phi(Z, X) \Phi(W, Y) \\ &\quad + 2\Phi(X, Y) \Phi(W, Z) \} \\ &+ \{ \bar{\eta}(W) \bar{\eta}(X) \bar{g}(\bar{\phi} Z, \bar{\phi} Y) - \bar{\eta}(W) \bar{\eta}(Y) \bar{g}(\bar{\phi} Z, \bar{\phi} X) \\ &\quad + \bar{\eta}(Y) \bar{\eta}(Z) \bar{g}(\bar{\phi} W, \bar{\phi} X) \\ &\quad - \bar{\eta}(Z) \bar{\eta}(X) \bar{g}(\bar{\phi} W, \bar{\phi} Y) \}, \end{aligned} \quad (39)$$

for any $X, Y, Z, W \in \Gamma(T\bar{M})$, where $\Phi(X, Y) = \bar{g}(X, \bar{\phi} Y)$ and $\epsilon = \sum_{\alpha=1}^r \epsilon_\alpha$. Using (39), we obtain

$$\begin{aligned} \bar{R}(X, Y, W, \xi) &= \frac{c-\epsilon}{4} \{ \bar{g}(V, X) \bar{g}(W, \bar{\phi} Y) \\ &\quad - \bar{g}(V, Y) \bar{g}(W, \bar{\phi} X) \\ &\quad + 2\bar{g}(V, W) \bar{g}(X, \bar{\phi} Y) \}. \end{aligned} \quad (40)$$

Also, for the pair $\{\xi, N\}$ on $\mathcal{U} \subset M$, from (3.8) of page no. 94 of [3], we have

$$\begin{aligned} \bar{R}(X, Y, W, \xi) &= (\nabla_X B)(Y, W) - (\nabla_Y B)(X, W) \\ &\quad + \tau(X) B(Y, W) - \tau(Y) B(X, W), \end{aligned} \quad (41)$$

$X, Y, W \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, where

$$\begin{aligned} (\nabla_X B)(Y, W) &= X \cdot B(Y, W) - B(\nabla_X Y, W) \\ &\quad - B(Y, \nabla_X W). \end{aligned} \quad (42)$$

Hence using (40) and (41), we obtain

$$\begin{aligned} & (\nabla_X B)(Y, W) - (\nabla_Y B)(X, W) \\ &= \frac{c-\epsilon}{4} \left\{ \bar{g}(V, X) \bar{g}(W, \bar{\phi}Y) - \bar{g}(V, Y) \bar{g}(W, \bar{\phi}X) \right. \\ & \quad \left. + 2\bar{g}(V, W) \bar{g}(X, \bar{\phi}Y) \right\} \\ & \quad - \tau(X) B(Y, W) + \tau(Y) B(X, W), \end{aligned} \quad (43)$$

for any $X, Y, W \in \Gamma(T\bar{M})$.

Lemma 5. Let (M, g) be a totally contact umbilical lightlike hypersurface of an indefinite $\mathcal{S}$ -manifold $(\bar{M}, \bar{g})$; then for any $X, Y, W \in \Gamma(TM)$, one has

$$\begin{aligned} (\nabla_X B_1)(Y, W) &= (X \cdot \lambda) \left\{ g(Y, W) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(Y) \bar{\eta}^\alpha(W) \right\} \\ & \quad + \lambda \{ B(X, Y) \theta(W) + B(X, W) \theta(Y) \} \\ & \quad + \lambda \bar{\eta}(W) \{ \bar{g}(\phi X, Y) + u(X) \theta(Y) \} \\ & \quad + \lambda \bar{\eta}(Y) \{ \bar{g}(\phi X, W) + u(X) \theta(W) \}. \\ (\nabla_X B_2)(Y, W) &= \epsilon u(W) \{ \bar{g}(Y, \phi X) + u(X) \theta(Y) \} \\ & \quad + \epsilon u(Y) \{ \bar{g}(W, \phi X) + u(X) \theta(W) \} \\ & \quad + \bar{\eta}(W) \{ B(X, \phi Y) + \tau(X) u(Y) \} \\ & \quad + \bar{\eta}(Y) \{ B(X, \phi W) + \tau(X) u(W) \}. \end{aligned} \quad (44)$$

Proof. By virtue of (37) and (42), we obtain

$$\begin{aligned} (\nabla_X B_1)(Y, W) &= (X \cdot \lambda) \left\{ g(Y, W) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(Y) \bar{\eta}^\alpha(W) \right\} \\ & \quad - \lambda \{ g(\nabla_X Y, W) + g(Y, \nabla_X W) \} \\ & \quad + \lambda \sum_{\alpha=1}^r \epsilon_\alpha \{ \bar{\eta}^\alpha(\nabla_X Y) \bar{\eta}^\alpha(W) \\ & \quad \quad + \bar{\eta}^\alpha(\nabla_X W) \bar{\eta}^\alpha(Y) \}. \end{aligned} \quad (45)$$

Now using (12) in (45), we have

$$\begin{aligned} (\nabla_X B_1)(Y, W) &= (X \cdot \lambda) \left\{ g(Y, W) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(Y) \bar{\eta}^\alpha(W) \right\} \\ & \quad - \lambda \{ X(g(Y, W)) - B(X, Y) \theta(W) \\ & \quad \quad - B(X, W) \theta(Y) \} \\ & \quad + \lambda \sum_{\alpha=1}^r \left\{ g(\nabla_X Y, \bar{\zeta}_\alpha) \bar{\eta}^\alpha(W) \right. \\ & \quad \quad \left. + g(\nabla_X W, \bar{\zeta}_\alpha) \bar{\eta}^\alpha(Y) \right\}. \end{aligned} \quad (46)$$

Again using (12), we have

$$\begin{aligned} g(\nabla_X Y, \bar{\zeta}_\alpha) &= X(g(Y, \bar{\zeta}_\alpha)) + \epsilon_\alpha g(Y, \phi X) \\ & \quad + \epsilon_\alpha u(X) \theta(Y). \end{aligned} \quad (47)$$

Thus from (46) and (47), the first expression of the theorem follows. Next, using (38) and (42), we obtain

$$\begin{aligned} (\nabla_X B_2)(Y, W) &= \sum_{\alpha=1}^r \epsilon_\alpha \{ \bar{\eta}^\alpha(\nabla_X Y) u(W) + \bar{\eta}^\alpha(\nabla_X W) u(Y) \\ & \quad + \bar{\eta}^\alpha(W) u(\nabla_X Y) \\ & \quad + \bar{\eta}^\alpha(Y) u(\nabla_X W) \}. \end{aligned} \quad (48)$$

Using (31) and (32), we get

$$\bar{\eta}^\alpha(\nabla_X Y) = g(Y, \phi X) + u(X) \theta(Y), \quad (49)$$

and from (2), (3), (10), (16), (17), (18), and (25), we obtain

$$u(\nabla_X Y) = B(X, \phi Y) + \tau(X) u(Y). \quad (50)$$

Thus by substituting (49) and (50) in (48), we obtain (44). $\square$

Theorem 6. Let $\bar{M}(c)$ be an indefinite $\mathcal{S}$ -space form and M be a totally contact umbilical lightlike hypersurface of $\bar{M}(c)$. Then $c = -3\epsilon$, where $\epsilon = \sum_{\alpha=1}^r \epsilon_\alpha$ and λ satisfies the following differential equations:

$$\begin{aligned} \xi \cdot \lambda + \lambda \tau(\xi) - \lambda^2 &= 0, \\ PX \cdot \lambda + \lambda \tau(PX) &= 0, \end{aligned} \quad (51)$$

for any $X \in \Gamma(TM)$.

Proof. Let M be a totally contact umbilical lightlike hypersurface of an indefinite $\mathcal{S}$ -space form $\bar{M}(c)$ of constant $\bar{\phi}$ -sectional curvature c . Then using (35), we have $B(X, \phi Y) = \lambda \bar{g}(X, \phi Y)$. Substituting (29), (44) in (43), we obtain

$$\begin{aligned} & (X \cdot \lambda) \left\{ \bar{g}(Y, W) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(Y) \bar{\eta}^\alpha(W) \right\} \\ & \quad - (Y \cdot \lambda) \left\{ \bar{g}(X, W) - \sum_{\alpha=1}^r \epsilon_\alpha \bar{\eta}^\alpha(X) \bar{\eta}^\alpha(W) \right\} \\ & \quad + \lambda \{ B(X, W) \theta(Y) - B(Y, W) \theta(X) \} \\ & \quad + 2\lambda \{ \bar{g}(\phi X, Y) + u(X) \theta(Y) \} \bar{\eta}(W) \\ & \quad + \lambda \{ (\bar{g}(W, \phi X) \bar{\eta}(Y) - \bar{g}(W, \phi Y) \bar{\eta}(X)) \} \\ & \quad + \lambda \{ u(X) \bar{\eta}(Y) - u(Y) \bar{\eta}(X) \} \theta(W) \\ & \quad + 2\epsilon \{ \bar{g}(\phi X, Y) + u(X) \theta(Y) \} u(W) \\ & \quad + \{ \lambda \bar{g}(X, \phi Y) + \tau(X) u(Y) \\ & \quad \quad - \lambda \bar{g}(Y, \phi X) - \tau(Y) u(X) \} \bar{\eta}(W) \end{aligned}$$

$$\begin{aligned}
& + \lambda \{ \bar{g}(X, \phi W) \bar{\eta}(Y) - \bar{g}(Y, \phi W) \bar{\eta}(X) \} \\
& + \{ \tau(X) \bar{\eta}(Y) - \tau(Y) \bar{\eta}(X) \} u(W) \\
& + \epsilon \{ \bar{g}(W, \phi X) u(Y) - \bar{g}(W, \phi Y) u(X) \} \\
& = \frac{c-\epsilon}{4} \{ \bar{g}(V, X) \bar{g}(W, \bar{\phi} Y) \\
& \quad - \bar{g}(V, Y) \bar{g}(W, \bar{\phi} X) \\
& \quad + 2 \bar{g}(V, W) \bar{g}(X, \bar{\phi} Y) \} \\
& \quad + \tau(Y) B(X, W) - \tau(X) B(Y, W),
\end{aligned} \tag{52}$$

for any $X, Y, W \in \Gamma(\overline{TM})$. Put $X = \xi$ in (52); we get

$$\begin{aligned}
& (\xi \cdot \lambda) \left\{ \bar{g}(Y, W) - \sum_{\alpha=1}^r \epsilon_{\alpha} \bar{\eta}^{\alpha}(Y) \bar{\eta}^{\alpha}(W) \right\} \\
& - \lambda B(Y, W) - 2 \lambda u(Y) \bar{\eta}(W) \\
& - \lambda u(W) \bar{\eta}(Y) - 2 \epsilon u(Y) u(W) \\
& + \tau(\xi) u(Y) \bar{\eta}(W) + 2 \lambda u(Y) \bar{\eta}(W) \\
& + \tau(\xi) u(W) \bar{\eta}(Y) - \epsilon u(W) u(Y) \\
& + \lambda u(W) \bar{\eta}(Y) \\
& = \frac{c-\epsilon}{4} \{ u(Y) u(W) + 2 u(W) u(Y) \} \\
& \quad - \tau(\xi) B(Y, W).
\end{aligned} \tag{53}$$

Put $Y = W = U$ in (53) and then using $g(U, U) = 0$, $u(U) = 1$, $\bar{\eta}^{\alpha}(U) = 0$; we obtain

$$-\lambda B(U, U) - 3\epsilon = 3 \frac{(c-\epsilon)}{4} - \tau(\xi) B(U, U), \tag{54}$$

since $B(U, U) = 0$, we get $c = -3\epsilon$. Moreover, by putting $Y = V$ and $W = U$ in (52), we obtain

$$\xi \cdot \lambda + \lambda \tau(\xi) - \lambda^2 = 0. \tag{55}$$

Finally, putting $X = PX$, $Y = PY$, and $W = PW$ in (52) with $c = -3\epsilon$ and using that $S(TM)$ is nondegenerate, we obtain

$$\begin{aligned}
& \{PX \cdot \lambda + \lambda \tau(PX)\} \left(PY - \sum_{\alpha=1}^r \bar{\eta}^{\alpha}(PY) \bar{\zeta}_{\alpha} \right) \\
& = \{PY \cdot \lambda + \lambda \tau(PY)\} \left(PX - \sum_{\alpha=1}^r \bar{\eta}^{\alpha}(PX) \bar{\zeta}_{\alpha} \right).
\end{aligned} \tag{56}$$

Putting $PX = \bar{\zeta}_{\alpha}$ in (56), we get

$$\{ \bar{\zeta}_{\alpha} \cdot \lambda + \lambda \tau(\bar{\zeta}_{\alpha}) \} \left(PY - \sum_{\alpha=1}^r \bar{\eta}^{\alpha}(PY) \bar{\zeta}_{\alpha} \right) = 0. \tag{57}$$

By taking $Y = V$, we obtain

$$\bar{\zeta}_{\alpha} \cdot \lambda + \lambda \tau(\bar{\zeta}_{\alpha}) = 0. \tag{58}$$

Since $PX \in \Gamma(S(TM))$ therefore using (19), we can write

$$\begin{aligned}
PX &= u(PX)U + v(PX)V + \sum_{i=1}^{2n-4} \beta_i F_i \\
&+ \sum_{\alpha=1}^r \epsilon_{\alpha} \bar{\eta}^{\alpha}(PX) \bar{\zeta}_{\alpha} \\
&= PX' + \sum_{\alpha=1}^r \epsilon_{\alpha} \bar{\eta}^{\alpha}(PX) \bar{\zeta}_{\alpha},
\end{aligned} \tag{59}$$

where $\{F_i\}_{1 \leq i \leq 2n-4}$ is an orthogonal basis of D_0 , then using (58), we have

$$\begin{aligned}
PX \cdot \lambda + \lambda \tau(PX) &= \left(PX' + \sum_{\alpha=1}^r \epsilon_{\alpha} \bar{\eta}^{\alpha}(PX) \bar{\zeta}_{\alpha} \right) \cdot \lambda \\
&+ \lambda \tau \left(PX' + \sum_{\alpha=1}^r \epsilon_{\alpha} \bar{\eta}^{\alpha}(PX) \bar{\zeta}_{\alpha} \right) \\
&= PX' \cdot \lambda + \lambda \tau(PX'),
\end{aligned} \tag{60}$$

which leads to get from (56)

$$\{PX' \cdot \lambda + \lambda \tau(PX')\} PY' = \{PY' \cdot \lambda + \lambda \tau(PY')\} PX'. \tag{61}$$

□

Now, assume that there exists a vector field X_0 on some neighborhood of M such that $PX'_0 \cdot \lambda + \lambda \tau(PX'_0) \neq 0$ at some point p in the neighborhood. Then from (61), it is clear that all the vectors of the fiber $(S(TM) - \{\bar{\zeta}_{\alpha}\})_p$ are collinear with $(PX'_0)_p$. This contradicts $\dim(S(TM) - \{\bar{\zeta}_{\alpha}\}) > 1$. This implies the result.

From the previous mentioned theorem, we have the following corollary.

Corollary 7. *There exist no totally contact umbilical lightlike real hypersurfaces of indefinite $\mathcal{S}$ -space form $\overline{M}(c)$ ($c \neq -3\epsilon$) with $\bar{\zeta}_{\alpha} \in TM$.*

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