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Research Article

The Measure-Theoretic Entropy and Topological Entropy of Actions over $\mathbb{Z}_m$

Chih-Hung Chang[1] and Yu-Wen Chen[2]

[1] *Department of Applied Mathematics, Feng Chia University, Taichung 40724, Taiwan*
[2] *Taipei Municipal Minsheng Junior High School, Taipei 10591, Taiwan*

Correspondence should be addressed to Chih-Hung Chang; chihhung@mail.fcu.edu.tw

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This paper studies the quantitative behavior of a class of one-dimensional cellular automata, named weakly permutive cellular automata, acting on the space of all doubly infinite sequences with values in a finite ring $\mathbb{Z}_m$, $m \geq 2$. We calculate the measure-theoretic entropy and the topological entropy of weakly permutive cellular automata with respect to any invariant measure on the space $\mathbb{Z}_m^{\mathbb{Z}}$. As an application, it is shown that the uniform Bernoulli measure is the unique maximal measure for linear cellular automata among the Markov measures.

1. Introduction

Cellular automata (CA for brevity), introduced by Ulam and von Neumann, have been systematically studied by Hedlund from purely mathematical point of view [1]. The study of such dynamics called CA has received remarkable attention in the last few years [1–3]. CA have been widely investigated in a lot of disciplines (e.g., mathematics, physics, and computer science). In [2], the dynamical behavior (ergodicity and topological transitivity) of n-dimensional linear CA (LCA) over the ring $\mathbb{Z}_m$ has been studied. Some open questions concerning the topological and ergodic dynamics of one-dimensional CA have been addressed in [4].

It is well known that there are several notions of entropy (i.e., measure-theoretical, topological, and directional) of measure-preserving transformation on probability space in ergodic theory. It is important to know how these notions are related with each other. In the last years, a lot of works are devoted to this subject (see, e.g., [5–13]). Recall that by the Variational Principle the topological entropy is the supremum of the entropies of invariant measures. In [8, 14, 15], the authors showed that the uniform Bernoulli measure is a measure of maximal entropy for some one-dimensional CA.

The notion of entropy has been extensively studied in many disciplines (e.g., computer science, mathematics, physics, chemistry, and information theory) with different purposes. This notion first arose in thermodynamics as a measure of the heat absorbed (or emitted), when external work is done on a system. In probability theory, it constitutes a measure of the uncertainty. The entropy has been interpreted as a measure of the chaotic character of a dynamical system by many authors (see [4, 16, 17]), the value $h_{\text{top}}(T)$ has been in general accepted as a measure of the complexity of the dynamics of T over the space X. Some authors have stated that the topological entropy of a map is a crude global measure of the exponential complexity of the structure of the orbits of the map (see [4, 17]). Badii and Politi [18] have studied the complexity exhibited by some CA with elementary rule by using both topological (graph-theoretical) and metric (thermodynamic) techniques. Lloyd and Pagels [19] have defined a measure complexity for the macroscopic states of physical systems. They have proved that the average complexity of a state must be proportional to Shannon entropy of the set of trajectories, $S = -\sum_i p_i \log p_i$.

In this paper we study the measure-theoretical entropy and topological entropy of one-dimensional *weakly permutive* CA F (defined later) acting on the space of all doubly infinite sequences with values in a finite ring $\mathbb{Z}_m$, $m \geq 2$. In [14], the author computed the measure-theoretical entropy with respect to the uniform Bernoulli measure for the case

where F is linear defined by the local rule $f(x_{-r},\dots,x_r) = \sum_{i=-r}^{r} x_i$. Recently, Ban et al. [15] studied the complexity of permutative CA (defined later) in thermodynamics and topological aspects, and they also gave the formulae to compute measure-theoretic and topological entropies. In this paper, for both measure-theoretic and topological entropies, we extend results obtained in [8, 14, 15, 20] to the case that F is a weakly permutive CA over the ring $\mathbb{Z}_m$ with respect to any invariant measure. We remark that LCA is a special case of weakly permutive CA. In addition, the formula of topological entropy extends D'amico et al.'s result [17] for the topological entropy of LCA. We also show that the uniform Bernoulli measure is the unique maximal measure for LCA, whenever we focus on the Markov measures.

We give an example herein. Suppose that the local rule is given by

$$f(x_{-2},x_{-1},x_0,x_1,x_2) = 18x_{-1} + 30(x_0+x_1) + 20x_2 \pmod{90} \tag{1}$$

(cf. Figure 1). The local rule is nonlinear. Applying Theorems 4 and 11, we conclude that the topological entropy of the prescribed weakly permutive CA is $4\log 3 + \log 5$ (see Example 10).

The rest of the elucidation is organized as follows. The upcoming section gives some definitions and results of permutive and weakly permutive CA. Section 3 demonstrates formulae of the measure-theoretic and topological entropies. The uniform Bernoulli measure is a measure of maximal entropy that is also demonstrated in Section 3.

2. Preliminary

Let $\mathbb{Z}_m = \{0,1,\dots,m-1\}$, and let $\Omega = \mathbb{Z}_m^{\mathbb{Z}}$ be the space of bi-infinite sequence $x = (x_n)_{-\infty}^{\infty}$. Hedlund examines CA in the viewpoint of symbolic dynamical systems [1]. He shows that $F:\Omega\to\Omega$ is a CA if and only if F can be represented as a sliding block code; that is, there exist $k\in\mathbb{Z}^+$ and a block map $f:\mathbb{Z}_m^{2k+1}\to\mathbb{Z}_m$ such that $F(x)_i = f(x_{i-k},\dots,x_{i+k})$ for $x\in\Omega$ and $i\in\mathbb{Z}$. Such f is called the local rule of F. The study of the local rule of a CA is essential for the understanding of this system.

A local rule $f:\mathbb{Z}_m^{2k+1}\to\mathbb{Z}_m$ is called *leftmost* (resp., *rightmost*) *permutive* if there exists an integer i, $-k\le i\le -1$ (resp., $1\le i\le k$), such that

(i) f is a permutation at x_i whenever the other variables are fixed;

(ii) f does not depend on x_j for $j<i$ (resp., $j>i$).

f is called *bipermutive* provided f is both left-most and right-most permutive. The family of *permutive cellular automata* consists of the following three types of local rules:

(1) f is leftmost permutive and does not depend on x_i for $i>0$;

(2) f is rightmost permutive and does not depend on x_i for $i<0$;

(3) f is bipermutive.

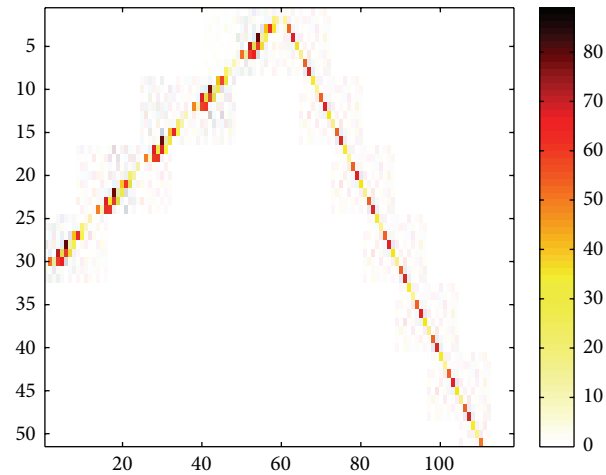

Figure 1: Spatial-temporal pattern determined by local rule $f(x_{-2},x_{-1},x_0,x_1,x_2) = 18x_{-1} + 30(x_0+x_1) + 20x_2 \pmod{90}$ in Example 10 with initial configuration $(\cdots\ 0\ 1\ 0\ \cdots)$. Theorems 4 and 11 indicate that the topological entropy is $4\log 3 + \log 5$ and the uniform Bernoulli measure is a maximal measure.

Without loss of generality, we may assume that $f(x_\ell,\dots,x_r)$ depends only on x_i for $\ell\le i\le r$ and the coefficients of x_ℓ and x_r are both nonzero in $\mathbb{Z}_m$. Let $m = p_1^{k_1}p_2^{k_2}\cdot p_n^{k_n}$ be factorized into the product of prime factors. Suppose that a local rule f is given, and set $f_i(x_\ell,\dots,x_r) = f(x_\ell,\dots,x_r) \bmod p_i^{k_i}$. f is called *weakly permutive* if, for $1\le i\le n$, there is $\tau_i\in\mathbb{N}$ such that $f_i^{\tau_i}$ is either constant, a multiple of an identity map, or permutive.

The present elucidation investigates the measure-theoretic and topological entropies of weakly permutive CA. For reader's convenience, we recall definitions of measure-theoretic entropy, topological entropy, and topological pressures. Reader may refer to [13] for more details.

Let μ be an invariant probability measure on (Ω,F), and let α and β be two finite measurable partitions of Ω. Define $\alpha\bigvee\beta$ and $H_\mu(\alpha)$ by

$$\alpha\bigvee\beta = \left\{A\bigcap B : A\in\alpha, B\in\beta\right\},$$
$$H_\mu(\alpha) = -\sum_{A\in\alpha}\mu(A)\log\mu(A), \tag{2}$$

respectively. The measure-theoretic entropy of F is defined by

$$h_\mu(F) = \sup\left\{\lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=0}^{n-1}F^{-i}\alpha\right)\right\}, \tag{3}$$

where the supremum is taken over all finite measurable partitions α.

Define $d:\Omega\times\Omega\to\mathbb{R}$ by

$$d(x,y) = \sum_{i=-\infty}^{\infty}\frac{|x_i-y_i|}{m^{|i|}},\quad x,y\in\Omega. \tag{4}$$

It is easy to verify that d is a metric and (Ω,d) is a compact metric space. Moreover, let ${}_a[s_a,\dots,s_b]_b = \{x\in\Omega : x_a = s_a,\dots,x_b = s_b\}$ be a cylinder in Ω, where $a\le b, a,b\in\mathbb{Z}$.

Then ${}_a[s_a,\ldots,s_b]_b$ is not only open but close in Ω. Let $\mathscr{P}$ be an open cover of Ω, denoted by

$$H(\mathscr{P}) = \inf\left\{\log \#\widehat{\mathscr{P}}\right\}, \tag{5}$$

where the infimum is taken over the set of finite subcovers $\widehat{\mathscr{P}}$ of $\mathscr{P}$ and $\#A$ denotes the cardinality of A. The topological entropy of F is defined by

$$h_{\text{top}}(F) = \sup\left\{\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}F^{-i}\mathscr{P}\right)\right\}, \tag{6}$$

where the supremum is taken over all open covers $\mathscr{P}$.

It is known that permutive possesses strongly mixing [3, 21]. Recall that $F:\Omega\to\Omega$ is strongly mixing if

$$\begin{aligned}&\lim_{n\to\infty}\mu\left(F^{-n}U\cap V\right)\\&\quad=\mu(U)\,\mu(V)\quad\text{for every measurable sets } U,V\subset\Omega.\end{aligned} \tag{7}$$

The following example demonstrates that permuittivity cannot be omitted.

Example 1. Let $f:\mathbb{Z}_4^3\to\mathbb{Z}_4$ be defined as

$$f(x_0,x_1,x_2) = 2x_0 + x_1 + \left[\frac{2}{5}x_2\right] + \left[\frac{3}{4}x_2\right] - x_2 \mod 4, \tag{8}$$

where $[\cdot]$ is Gauss function. It is seen that f is a permutation at x_1 but not permutive. Observe that x is Garden of Eden for $x\in\{y\in\mathbb{Z}_4^{\mathbb{Z}} : y_i = 0, y_{i+1} = 1 \text{ for some } i\in\mathbb{Z}\}$; namely, there exists no preimage of x. Then F is not strongly mixing.

3. Entropy of Weakly Permutive Cellular Automata

Let X,Y be two compact topological spaces, and let ν be a probability measure on X. If $\phi: X\to Y$ is onto, the push forward measure ν_Y of ν on Y is defined by $\nu_Y = \nu\circ\phi^{-1}$. It is wellknown that ν_Y is also a probability measure.

Lemma 2. *Let $m = pq$ for some relative prime factors p and q. Denote*

$$\begin{aligned}f_p(x_\ell,\ldots,x_r) &= f(x_\ell,\ldots,x_r) \mod p,\\ f_q(x_\ell,\ldots,x_r) &= f(x_\ell,\ldots,x_r) \mod q.\end{aligned} \tag{9}$$

Then $h_\mu(F) = h_{\mu_p}(F_p) + h_{\mu_q}(F_q)$ provided $\mu\cong\mu_p\times\mu_q$, where μ_p and μ_q are the push forward measures of μ on $\mathbb{Z}_p^{\mathbb{Z}}$ and $\mathbb{Z}_q^{\mathbb{Z}}$, respectively, and F_p and F_q are CA with local rules f_p and f_q, respectively.

Proof. Denote $\Omega_p = \mathbb{Z}_p^{\mathbb{Z}}$ and $\Omega_q = \mathbb{Z}_q^{\mathbb{Z}}$. Define $\Phi:\Omega\to\Omega_p\times\Omega_q$ by

$$(\Phi x)_i = (x_i \bmod p, x_i \bmod q), \quad i\in\mathbb{Z}. \tag{10}$$

Observe that $\mathbb{Z}_m\cong\mathbb{Z}_p\times\mathbb{Z}_q$ indicates that Φ is an isomorphism. Moreover,

$$\begin{aligned}&(\Phi\circ F)(x)_i\\&\quad=\left(f(x_{i+\ell},\ldots,x_{i+r}) \bmod p, f(x_{i+\ell},\ldots,x_{i+r}) \bmod q\right)\\&\quad=\left(f_p(x_{i+\ell},\ldots,x_{i+r}), f_q(x_{i+\ell},\ldots,x_{i+r})\right)\\&\quad=\left(f_p(x_{i+\ell},\ldots,x_{i+r} \bmod p), f_q(x_{i+\ell},\ldots,x_{i+r} \bmod q)\right)\\&\quad=\left(\left(F_p\times F_q\right)\circ\Phi\right)(x)_i,\end{aligned} \tag{11}$$

for $i\in\mathbb{Z}$. That is, the diagram

$$\begin{tikzcd}\Omega \arrow[r, "F"] \arrow[d, "\Phi"'] & \Omega \arrow[d, "\Phi"] \\ \Omega_p\times\Omega_q \arrow[r, "F_p\times F_q"'] & \Omega_p\times\Omega_q\end{tikzcd} \tag{12}$$

commutes. Thus F is topologically conjugated to $F_p\times F_q$. The isomorphism $\mu\cong\mu_p\times\mu_q$ indicates that $h_\mu(F) = h_{\mu_p\times\mu_q}(F_p\times F_q) = h_{\mu_p}(F_p) + h_{\mu_q}(F_q)$.

The proof is completed. □

Remark 3. Notably, the demonstration of Lemma 2 asserts that a CA F defined on $\mathbb{Z}_m^{\mathbb{Z}}$ is topologically conjugated to the direct product of the projection of F on $\mathbb{Z}_p^{\mathbb{Z}}$ and $\mathbb{Z}_q^{\mathbb{Z}}$ provided that $m = pq$ and p is relatively prime to q.

Theorem 4 comes straightforwardly from the definition of weakly permutive CA and Lemma 2, and hence the proof is omitted.

Theorem 4. *Let $m = p_1^{k_1}\cdot p_2^{k_2}\cdots p_n^{k_n}$ for some prime factors p_i and $k_i\in\mathbb{N}$. Suppose that F is a weakly permutive CA, μ is F-invariant, and $\mu\cong\mu_1\times\cdots\times\mu_n$, where μ_i is the push-forward measure of μ on $\mathbb{Z}_{p_i^{k_i}}^{\mathbb{Z}}$ for $i = 1,\ldots,n$. Then*

$$h_\mu(F) = \sum_{i=1}^{n} h_{\mu_i}(F_i). \tag{13}$$

An immediate application of Theorem 4 is computing the measure-theoretic entropy of a linear CA (LCA). Herein F is an LCA if its corresponding local rule is given by $f(x_\ell,\ldots,x_r) = \Sigma_{i=\ell}^{r} a_i x_i \bmod m$ for some $a_i\in\mathbb{Z}$, $i = \ell,\ldots,r$.

Theorem 5. *Let $m = p_1^{k_1}\cdot p_2^{k_2}\cdots p_n^{k_n}$ for some prime p_i and $k_i\in\mathbb{N}$. Suppose F is an LCA and μ is F-invariant such that $\mu\cong\mu_1\times\mu_2\times\cdots\times\mu_n$. Then*

$$h_\mu(F) = \sum_{i=1}^{n} h_{\mu_i}(F_i). \tag{14}$$

Moreover, suppose that μ is a Bernoulli measure. Denote

$$C_i = \left\{ j : \left(a_j, p_i\right) = 1 \right\} \bigcup \{0\}, \tag{15}$$

$$\overline{r}_i = \max C_i, \qquad \overline{\ell}_i = \min C_i, \tag{16}$$

for $1 \le i \le n$. Then

$$\begin{aligned} h_\mu(F) &= \sum_{i=1}^{n} h_{\mu_i}(F_i) \\ &= \sum_{i=1}^{n} \left(\overline{r}_i - \overline{\ell}_i\right) \sum_{j=0}^{p_i^{k_i}-1} - p_{i_j} \log p_{i_j}, \end{aligned} \tag{17}$$

where $p_{i_j} = \mu_i(j)$, $1 \le i \le n, 0 \le j \le p_i^{k_i} - 1$.

Remark 6. Theorem 5 only presents the explicit form of the measure-theoretic entropy of an LCA with respect to a Bernoulli measure. The exact formula of the measure-theoretic entropy of an LCA with respect to a Markov measure can also be obtained. Since the formula is much complicated, we omit the case.

Before demonstrating Theorem 5, we introduce the following lemma.

Lemma 7. *Let $m = p^k$ for some prime p, $k \in \mathbb{N}$. If F is an LCA, then F is weakly permutive.*

Proof. Denote by $\mathscr{L}$ the collection of linear local rules, and $\mathbb{Z}_m[x, x^{-1}] = \{\Sigma_{i=n_1}^{n_2} a_i x^i, n_1, n_2 \in \mathbb{Z}\}$. Define $\chi : \mathscr{L} \to \mathbb{Z}_m[x, x^{-1}]$ by

$$\chi\left(\sum_{i=n_1}^{n_2} \lambda_i x_i\right) = \sum_{i=n_1}^{n_2} \lambda_i x^{-i}. \tag{18}$$

It is easily seen that χ is bijective. Moreover, let $\mathbb{Z}_m[[x, x^{-1}]]$ denote the power series generated by $\{x, x^{-1}\}$ over $\mathbb{Z}_m$. Then $\widehat{\chi} : \Omega \to \mathbb{Z}_m[[x, x^{-1}]]$ defined by

$$\widehat{\chi}(\mathbf{b}) = \sum_{i=-\infty}^{\infty} b_i x^i, \qquad \text{where } \mathbf{b} = (b_i)_{i \in \mathbb{Z}} \in \Omega, \tag{19}$$

is also a bijection. Observe that, for each $\mathbf{b} = (b_i) \in \Omega$,

$$\begin{aligned} \widehat{\chi}\left(T_{f[-r,r]}(\mathbf{b})\right) &= \widehat{\chi}\left[\left(\sum_{n=-r+i}^{r+i} a_{n-i} b_n\right)_i\right] \\ &= \sum_{i=-\infty}^{\infty} \left(\sum_{n=-r+i}^{r+i} a_{n-i} b_n\right) x^i, \\ \mathbb{T}(\widehat{\chi}(\mathbf{b})) &= \mathbb{T}\left(\sum_{i=-\infty}^{\infty} b_i x^i\right) \\ &= \sum_{n=-r}^{r} a_n x^{-n} \left(\sum_{i=-\infty}^{\infty} b_i x^i\right) \\ &= \sum_{i=-\infty}^{\infty} \left(\sum_{n=-r+i}^{r+i} a_{n-i} b_n\right) x^i, \end{aligned} \tag{20}$$

where $\mathbb{T} \equiv \chi(f)$. This implements that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\quad F \quad} & \Omega \\ \hat{\chi} \downarrow & & \downarrow \hat{\chi} \\ \mathbb{Z}_m[[x, x^{-1}]] & \xrightarrow[\quad \mathbb{T} \quad]{} & \mathbb{Z}_m[[x, x^{-1}]] \end{array} \tag{21}$$

commutes. Moreover, yielding the Mathematical Induction, we have $f^n = \chi^{-1}(\mathbb{T}^n)$ for all $n \in \mathbb{N}$, where $f^n = f \circ f^{n-1}$.

Write $\mathbb{T}(x)$ as $\mathbb{T}(x) = \mathbb{T}_1(x) + p\mathbb{T}_2(x)$ such that $\mathbb{T}_1(x)$ is consisting of those monomials whose coefficients are coprime to p. We claim that, for all $i \in \mathbb{N}$,

$$\left(\mathbb{T}_1(x) + p\mathbb{T}_2(x)\right)^{p^i} \equiv \mathbb{T}_1^{p^i}(x) \pmod{p^{i+1}}. \tag{22}$$

It is seen that

$$\begin{aligned} &\left(\mathbb{T}_1(x) + p\mathbb{T}_2(x)\right)^p \\ &\quad = \sum_{j=0}^{p} \binom{p}{j} \left(\mathbb{T}_1(x)\right)^j \left(p\mathbb{T}_2(x)\right)^{p-j} \\ &\quad \equiv \sum_{j=p-1}^{p} \binom{p}{j} \left(\mathbb{T}_1(x)\right)^j \left(p\mathbb{T}_2(x)\right)^{p-j} \pmod{p^2} \\ &\quad = p\left(\mathbb{T}_1(x)\right)^{p-1} \left(p\mathbb{T}_2(x)\right) + \mathbb{T}_1^p(x) \\ &\quad \equiv \mathbb{T}_1^p(x) \pmod{p^2}. \end{aligned} \tag{23}$$

Suppose that

$$\left(\mathbb{T}_1(x) + p\mathbb{T}_2(x)\right)^{p^k} \equiv \mathbb{T}_1^{p^k}(x) \pmod{p^{k+1}} \tag{24}$$

In other words, $(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^k} = p^{k+1}Q(x) + \mathbb{T}_1^{p^k}(x)$ for some $Q(x)$. Therefore,

$$\begin{aligned} &\left(\mathbb{T}_1(x) + p\mathbb{T}_2(x)\right)^{p^{k+1}} \\ &\quad = \left[\left(\mathbb{T}_1(x) + p\mathbb{T}_2(x)\right)^{p^k}\right]^p \\ &\quad = \left[p^{k+1}Q(x) + \mathbb{T}_1^{p^k}(x)\right]^p \\ &\quad = \sum_{j=0}^{p} \binom{p}{j} \left(p^{k+1}Q(x)\right)^j \left[\mathbb{T}_1^{p^k}(x)\right]^{p-j} \\ &\quad \equiv \sum_{j=0}^{1} \binom{p}{j} \left(p^{k+1}Q(x)\right)^j \left[\mathbb{T}_1^{p^k}(x)\right]^{p-j} \pmod{p^{k+2}} \\ &\quad \equiv \mathbb{T}_1^{p^{k+1}}(x) \pmod{p^{k+2}}. \end{aligned} \tag{25}$$

This demonstrates our claim by the mathematical induction.

Let $n = p^{k-1}$; then $\mathbb{T}^n(x) \equiv \mathbb{T}_1^n(x) \pmod{p^k}$, and $f^n = \chi^{-1}(\mathbb{T}^n)$ is permutive. The proof is completed. □

Theorem 5 is obtained by combining Lemma 7 and the following result.

Theorem 8 (see [15]). *Suppose that the local rule of a CA F is given by $f(x_i,\dots,x_j)$, where $i \le j$, $i,j \in \mathbb{Z}$ and $\mu = (p_0,\dots,p_{m-1})$ is an F-invariant Bernoulli measure. Denote $L = \min\{i,0\}$ and $R = \max\{j,0\}$. One has the following results.*

(i) *If f is left permutive, then $h_\mu(F) = L\sum_{k=0}^{m-1} p_k \log p_k$;*

(ii) *If f is right permutive, then $h_\mu(F) = -R\sum_{k=0}^{m-1} p_k \times \log p_k$;*

(iii) *If f is bipermutive, then $h_\mu(F) = -(R-L)\sum_{k=0}^{m-1} p_k \times \log p_k$.*

In general, Lemma 7 does not hold. For instance, let $f : \mathbb{Z}_6^3 \to \mathbb{Z}_6$ be defined by $f(x_0,x_1,x_2) = x_0 + 2x_1 \mod 6$. Then f is weakly rightmost permutive. However, $\chi_f^n = (1+2x^{-1})^n$ whose coefficient of x^{-n} is either 2 or 4 mod 6 for all $n \in \mathbb{N}$. Thus f^n cannot be rightmost permutive for $n \in \mathbb{N}$.

Example 9. Let $m = 12$, and let μ be the $(p_0,p_1,\dots,p_{11})$-Bernoulli measure. Then $\Omega \cong \Omega_1 \times \Omega_2$, where $\Omega_1 = \mathbb{Z}_4^{\mathbb{Z}}$ and $\Omega_2 = \mathbb{Z}_3^{\mathbb{Z}}$. And the push forward measure of μ on Ω_1 and Ω_2 is

$$\begin{aligned}\mu_1 &= (p_0+p_4+p_8, p_1+p_5+p_9,\\ &\qquad p_2+p_6+p_{10}, p_3+p_7+p_{11}),\\ \mu_2 &= (p_0+p_3+p_6+p_9, p_1+p_4+p_7+p_{10},\\ &\qquad p_2+p_5+p_8+p_{11}),\end{aligned} \tag{26}$$

respectively. Set $f(x_{-1},x_0,x_1,x_2) = 2x_{-1}+x_0+2x_1+3x_2$; then

$$\begin{aligned}&f_1(x_{-1},x_0,x_1,x_2) = 2x_{-1}+x_0+2x_1+3x_2 \mod 4,\\ &f_2(x_{-1},x_0,x_1,x_2) = 2x_{-1}+x_0+2x_1 \mod 3,\\ &\bar{r}_1 = 2,\quad \bar{\ell}_1 = 0,\quad \bar{r}_2 = 1,\quad \bar{\ell}_2 = -1.\end{aligned} \tag{27}$$

Theorem 5 indicates that

$$\begin{aligned}&h_\mu(F)\\ &\quad = h_{\mu_1}(F_1) + h_{\mu_2}(F_2)\\ &\quad = -2\big((p_0+p_4+p_8)\log(p_0+p_4+p_8)\\ &\qquad + (p_1+p_5+p_9)\log(p_1+p_5+p_9)\\ &\qquad + (p_2+p_6+p_{10})\log(p_2+p_6+p_{10})\\ &\qquad + (p_3+p_7+p_{11})\log(p_3+p_7+p_{11})\\ &\qquad + (p_0+p_3+p_6+p_9)\log(p_0+p_3+p_6+p_9)\\ &\qquad + (p_1+p_4+p_7+p_{10})\log(p_1+p_4+p_7+p_{10})\\ &\qquad + (p_2+p_5+p_8+p_{11})\log(p_2+p_5+p_8+p_{11})\big).\end{aligned} \tag{28}$$

Example 10. Suppose $m = 90 = 2\cdot 3^2\cdot 5$. Let $f : \mathbb{Z}_{90}^4 \to \mathbb{Z}_{90}$ be given by $f(x_{-1},x_0,x_1,x_2) = 18x_{-1} + 10x_1(x_0+x_1) + 20x_2 \mod 90$. Then $\Omega \cong \Omega_1\times\Omega_2\times\Omega_3$ with $\Omega_1 = \mathbb{Z}_2^{\mathbb{Z}}$, $\Omega_2 = \mathbb{Z}_9^{\mathbb{Z}}$, and $\Omega_3 = \mathbb{Z}_5^{\mathbb{Z}}$. It is seen that $f_1 = 0 \mod 2$ is a constant map, and $f_2 = x_1(x_0+x_1)+2x_2 \mod 9$ and $f_3 = 3x_{-1} \mod 5$ are both permutive. Hence f is weakly permutive but not linear. Theorems 4 and 8 assert that

$$\begin{aligned}h_\mu(F) &= h_{\mu_2}(F_2) + h_{\mu_3}(F_3)\\ &= -2\sum_{i=0}^{8} p_{2_i}\log p_{2_i} - \sum_{i=0}^{4} p_{3_i}\log p_{3_i}\end{aligned} \tag{29}$$

provided $\mu \cong \mu_1\times\mu_2\times\mu_3$, where p_{j_i} is defined in Theorem 5.

Similar to the discussion of the formula of the measure-theoretic entropy of weakly permutive CA, the topological entropy of weakly permutive CA can be obtained analogously.

Theorem 11. *Let $m = p_1^{k_1}p_2^{k_2}\cdot p_n^{k_n}$ be factorized into the product of prime factors. Suppose that F is a weakly permutive CA and F_i is the projection of F on $\mathbb{Z}_{p_i^{k_i}}^{\mathbb{Z}}$ for $i = 1,\dots,n$. Then*

$$h_{\text{top}}(F) = \sum_{i=1}^{n} h_{\text{top}}(F_i). \tag{30}$$

In [17], the authors demonstrated the formula of the topological entropy of LCA. Lemma 7 indicates that LCA is a proper subset of weakly permutive CA. That makes Theorem 11 an extension of Theorem 12. More precisely, (31) holds for the topological entropy of weakly permutive CA.

Theorem 12. *Let $m = p_1^{k_1}p_2^{k_2}\cdot p_n^{k_n}$ be factorized into the product of prime factors. Suppose that F is a LCA. Then*

$$h_{\text{top}}(F) = \sum_{i=1}^{n} k_i\left(\bar{r}_i - \bar{\ell}_i\right)\log p_i, \tag{31}$$

where $\bar{r}_i$ and $\bar{\ell}_i$ are defined in (16).

Remark 13. The variational principle in thermodynamic formalism indicates that the topological entropy of a compact system is obtained by the supermum of its measure-theoretic entropies among all invariant measures, and a measure that attains the supremum is called a maximal measure. It comes immediately from Theorems 5 and 12 that, if we only consider invariant Markov measures, the uniform Bernoulli measure is the unique maximal measure for LCA.

Example 14. Let m and f be the same as in Example 9, and let μ be the uniform Bernoulli measure; then

$$\begin{aligned}h_\mu(F) &= -2\left(\log\frac{1}{4} + \log\frac{1}{3}\right)\\ &= 4\log 2 + 2\log 3 = h_{\text{top}}(F).\end{aligned} \tag{32}$$

Example 15. Let m and f be the same as in Example 10, and let μ be the uniform Bernoulli measure; then

$$\begin{aligned} h_\mu(F) &= -2\log\frac{1}{9} - \log\frac{1}{5} \\ &= 4\log 3 + \log 5 = h_{\text{top}}(F). \end{aligned} \tag{33}$$

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